Rigid Mechanics and Its Role in Nonlinear Structural Analysis

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Basic idea

- Divide and conquer: Make use of the property of each stage
- All we need for structural nonlinear analysis is a *linearized theory* plus *rigid mechanics*.

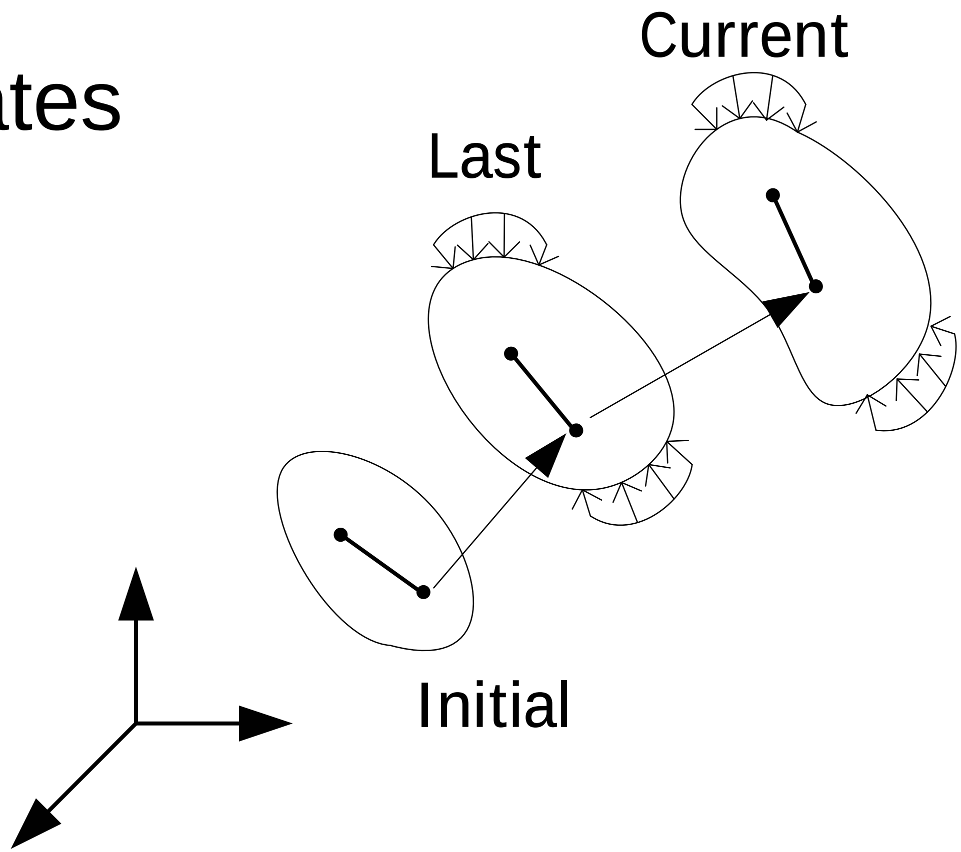
Incremental nonlinear analysis

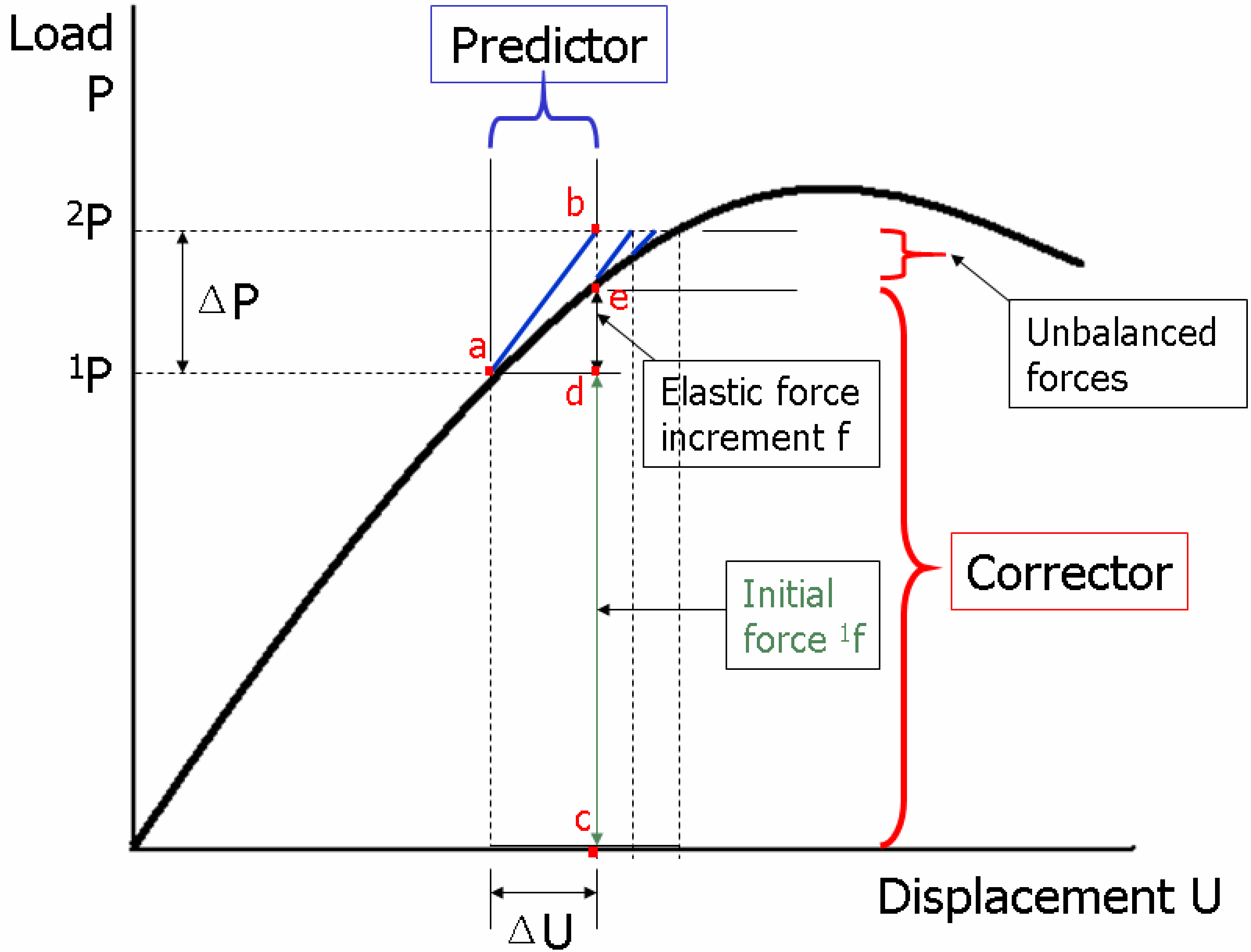
- *Predictor* vs. *corrector* stages
- *Rigid displacements* vs. *natural deformations*

- Use the *rigid body rule* whenever applicable

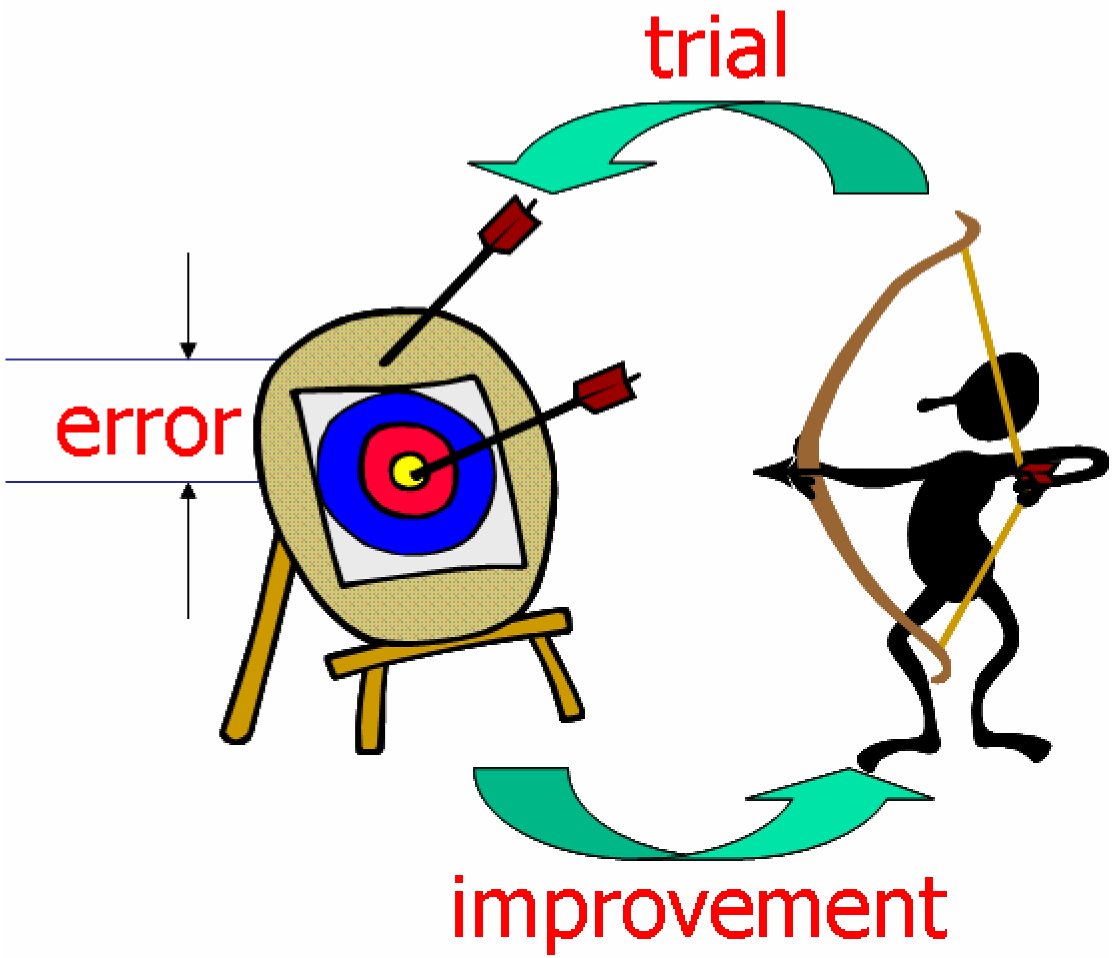
Incremental nonlinear analysis

- C_0, C_1, C_2 states
- Updated Lagrangian formulation




Load
P
Predictor
b
2P
ΔP
e
a
1P
d
Elastic force
increment f
Unbalanced
forces
Initial
force 1f
Corrector
c
ΔU
Displacement U

How to hit the target?



Nonlinear formulation

$$\left(\left[K_e\right]+\left[K_g\right]+h.o.t.\right)\left\{U\right\}=\left\{{}_1^2P\right\}-\left\{{}_1^1P\right\}$$

- $[K_e]$ = elastic stiffness
- $[K_g]$ = geometric stiffness
- h.o.t. = higher order terms
- {U} = displacements to be solved
- $\{^2P\}-\{^1P\}$ = load increments (given)

General procedure

- Given the load increments $\{^2P\}-\{^1P\}$
- *Predictor:* Solve for the structural displacements $\{U\}$

  → element displacements $\{u\}$
- *Corrector:* Compute the element forces $\{^2f\}$
- Compute *unbalanced forces* $\{R\}$
- Go for next iteration

Predictor stage

- Structural equation:

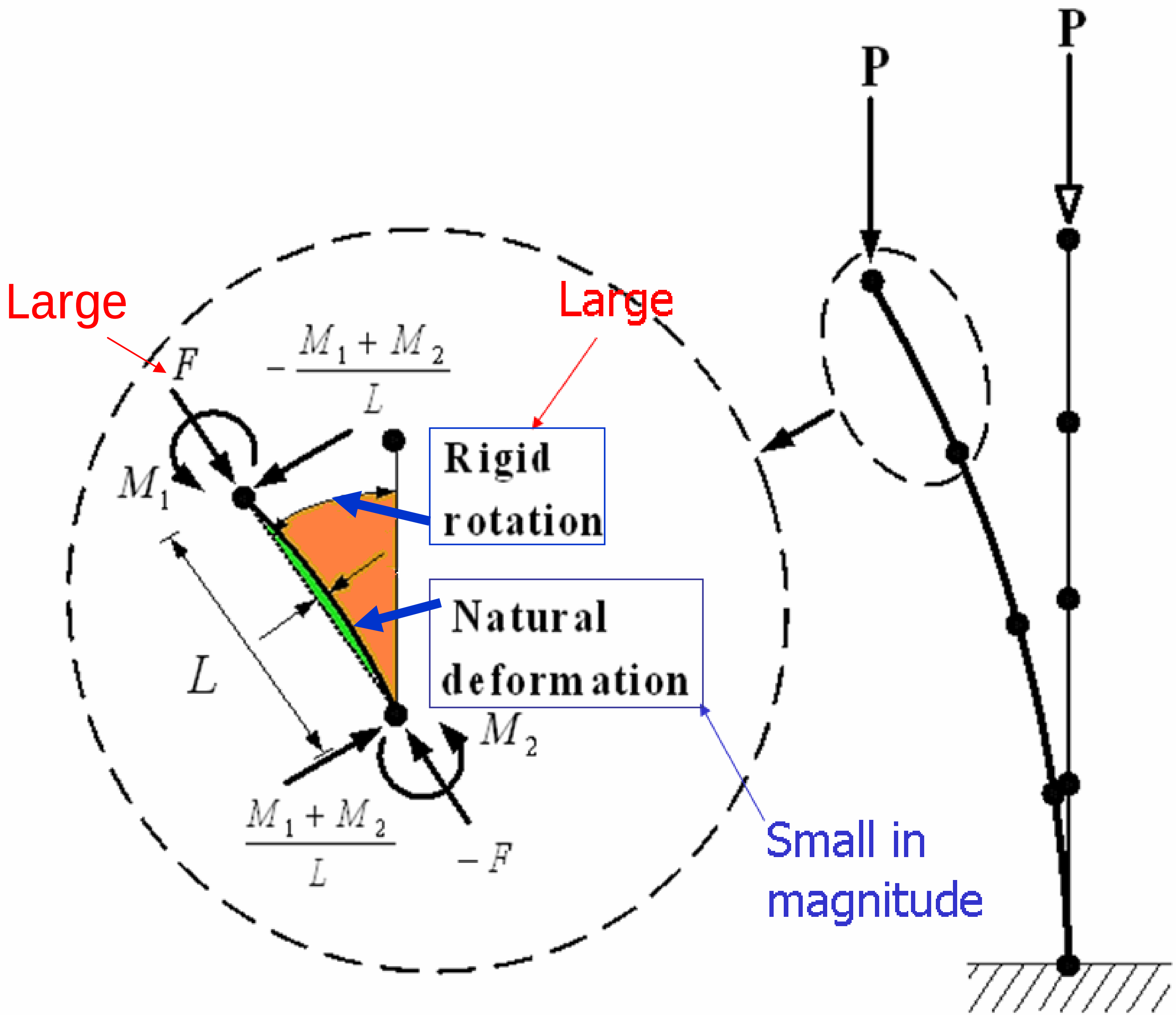
  $[K]\{U\} = \{^2P\} - \{^1P\}$

- [K] is allowed to be *approximate*
  - Difficult to derive an exact [K]
  - Little is gained by making [K] exact
- Iterations are always required
  - [K] affects the direction and number of iterations
  - Using an exact [K] matrix does not help much

Corrector stage

- Initial forces $\{^1f\}$ (large in magnitude)
  - Existing on the element
  - Rigid body rule → simple but exact
- Displacement increments {u} (small)
  - Force increments {f}
  - Only elastic actions need to be considered

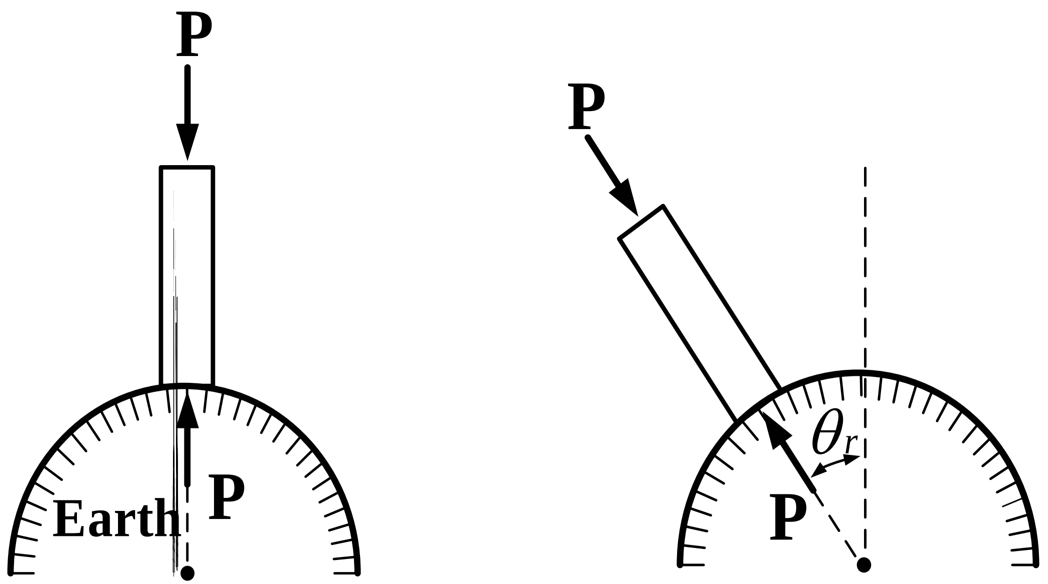
Rigid displacements
vs.
natural deformations


Large
Large
F
$-\frac{M_1+M_2}{L}$
M_1
Rigid rotation
Natural deformation
L
M_2
$\frac{M_1+M_2}{L}$
$-F$
Small in magnitude
P
P

Observations

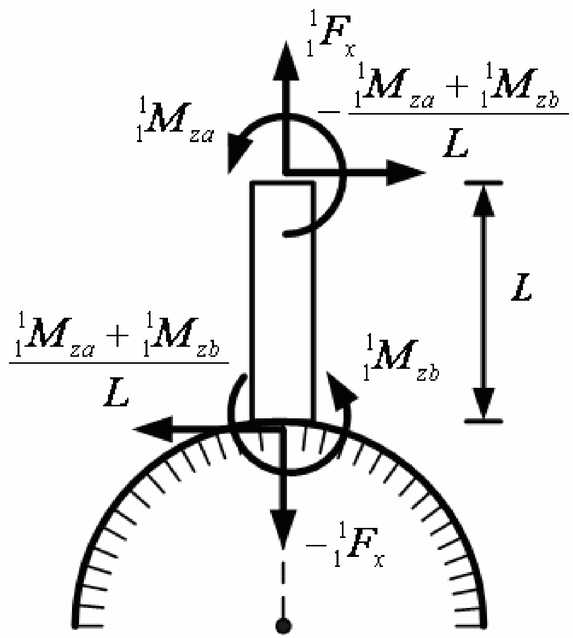
- Rigid displacements: large part
  - Predictor: Geometric stiffness $[k_g]$
    - *Rigid element*
  - Corrector: Updating of initial forces $\{^1f\}$
    - *Use the rigid body rule (exact)*
- Natural deformations: small part
  - Predictor: Elastic stiffness $[k_e]$
  - Corrector: Elastic actions $\{f\} = [k_e]\{u\}$

Rigid body rule

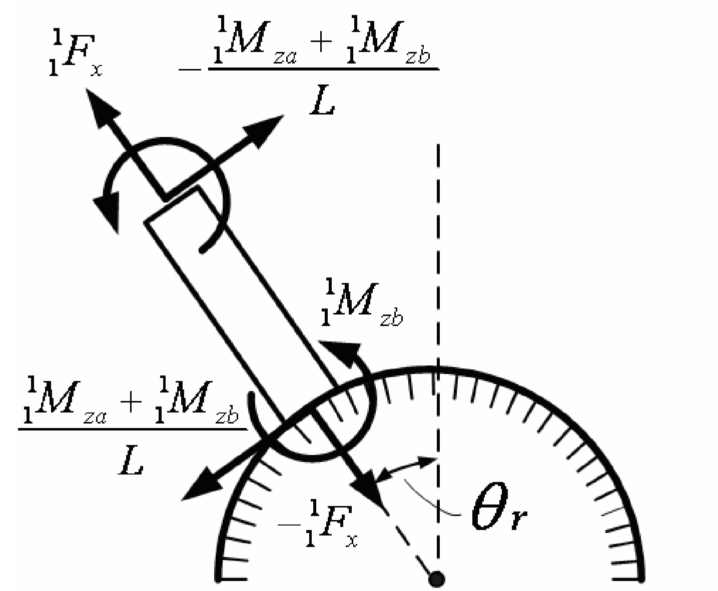


- Equilibrium
- Magnitudes of acting forces

2D beam element



C1 Configuration



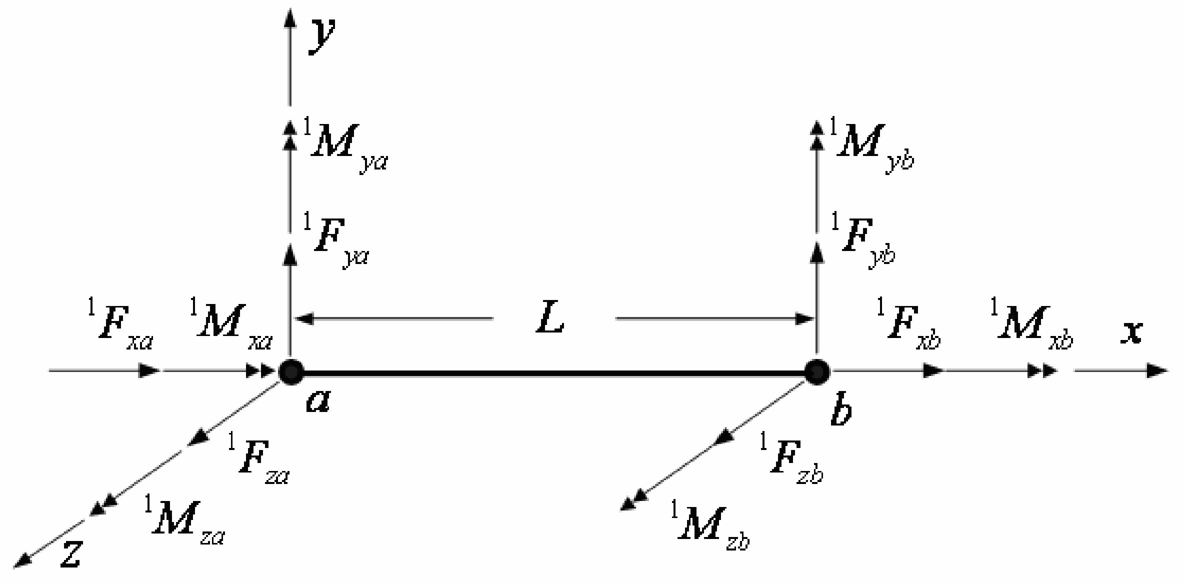
C2 Configuration

- Equilibrium
- Magnitude of acting forces.

Rigid body rule

- *Minimum criterion* for legitimacy of nonlinear elements
  - *Rigid-body qualified* geometric stiffness $[k_g]$ can be derived
- Rule for *updating* initial nodal forces existing on elements.

3D beam element



$$\{u\} = \langle u_a \quad v_a \quad w_a \quad \theta_{xa} \quad \theta_{ya} \quad \theta_{za} \quad u_b \quad v_b \quad w_b \quad \theta_{xb} \quad \theta_{yb} \quad \theta_{zb} \rangle^T$$

$$\{{}_1^1 f\} = \langle {}^1F_{xa} \quad {}^1F_{ya} \quad {}^1F_{za} \quad {}^1M_{xa} \quad {}^1M_{ya} \quad {}^1M_{za} \quad {}^1F_{xb} \quad {}^1F_{yb} \quad {}^1F_{zb} \quad {}^1M_{xb} \quad {}^1M_{yb} \quad {}^1M_{zb} \rangle^T$$

V.W. equation for 3D rigid beam

$$\frac{1}{2}\int_0^L \left[{}^1F_x \delta\left(u'^2 + v'^2 + w'^2\right) + {}^1F_x\left(\frac{I_y}{A}\delta w''^2 + \frac{I_z}{A}\delta v''^2\right) + {}^1F_x \frac{I_y + I_z}{A}\delta\theta_x'^2 \right] dx$$

$$+\int_0^L \left[-{}^1M_z \delta\left(w'\theta_x'\right) - {}^1M_y \delta\left(v'\theta_x'\right) - {}^1M_y \delta\left(u'w''\right) + {}^1M_z \delta\left(u'v''\right) \right] dx$$

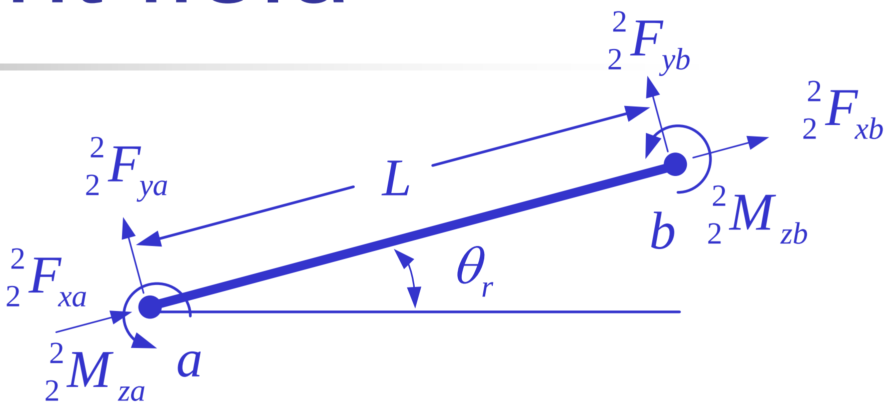
$$+\int_0^L \left[{}^1F_y \delta\left(w'\theta_x - u'v'\right) - {}^1F_z \delta\left(v'\theta_x + u'w'\right) \right] dx$$

$$+\frac{1}{2}\int_0^L \left[{}^1M_x \delta\left(v''w'\right) - {}^1M_x \delta\left(w''v'\right) \right] dx$$

$$= \left\{\delta u\right\}^T \left(\left\{{}_1^2 f\right\} - \left\{{}_1^1 f\right\}\right) + \left[\left({}^1M_z\theta_x - \frac{1}{2}{}^1M_x\theta_z\right)\delta\theta_y + \left(-{}^1M_y\theta_x + \frac{1}{2}{}^1M_x\theta_y\right)\delta\theta_z\right]_0^L$$

- Function of *member actions*

Rigid displacement field



- Rigid rotation (small)

$$u = (1 - x/L)u_a + (x/L)u_b$$

$$v = (1 - x/L)v_a + (x/L)v_b, \quad w = (1 - x/L)w_a + (x/L)w_b$$

$$\theta_x = (1 - x/L)\theta_{xa} + (x/L)\theta_{xb}$$

$$\theta_y = (1 - x/L)\theta_{ya} + (x/L)\theta_{yb}, \quad \theta_z = (1 - x/L)\theta_{za} + (x/L)\theta_{zb}$$

- Constraints

$$u_a = u_b, \quad \theta_{xa} = \theta_{xb}$$

$$\theta_{ya} = \theta_{yb}, \quad \theta_{za} = \theta_{zb}$$

Rigid rotation

- Just a *straight line*.
- Imagine *how many terms* are needed to represent a straight line using any series of functions?

3D rigid beam element

- Rigid element: $\left[k_g\right]\{u\} = \left\{{}^2_1 f\right\} - \left\{{}^1_1 f\right\}$
- Elastic element: $\left(\left[k_e\right] + \left[k_g\right]\right)\{u\} = \left\{{}^2_1 f\right\} - \left\{{}^1_1 f\right\}$
- Geometric stiffness matrix

$$[g] = \begin{bmatrix} 0 & -{}^1F_{yb}/L & -{}^1F_{zb}/L \\ -{}^1F_{yb}/L & {}^1F_{xb}/L & 0 \\ -{}^1F_{zb}/L & 0 & {}^1F_{xb}/L \end{bmatrix}$$

$$\left[k_g\right] = \begin{bmatrix} [g] & [h_a] & -[g] & [h_b] \\ [h_a]^T & [i_a] & -[h_a]^T & [0] \\ -[g] & -[h_a] & [g] & -[h_b] \\ [h_b]^T & [0] & -[h_b]^T & [i_b] \end{bmatrix}$$

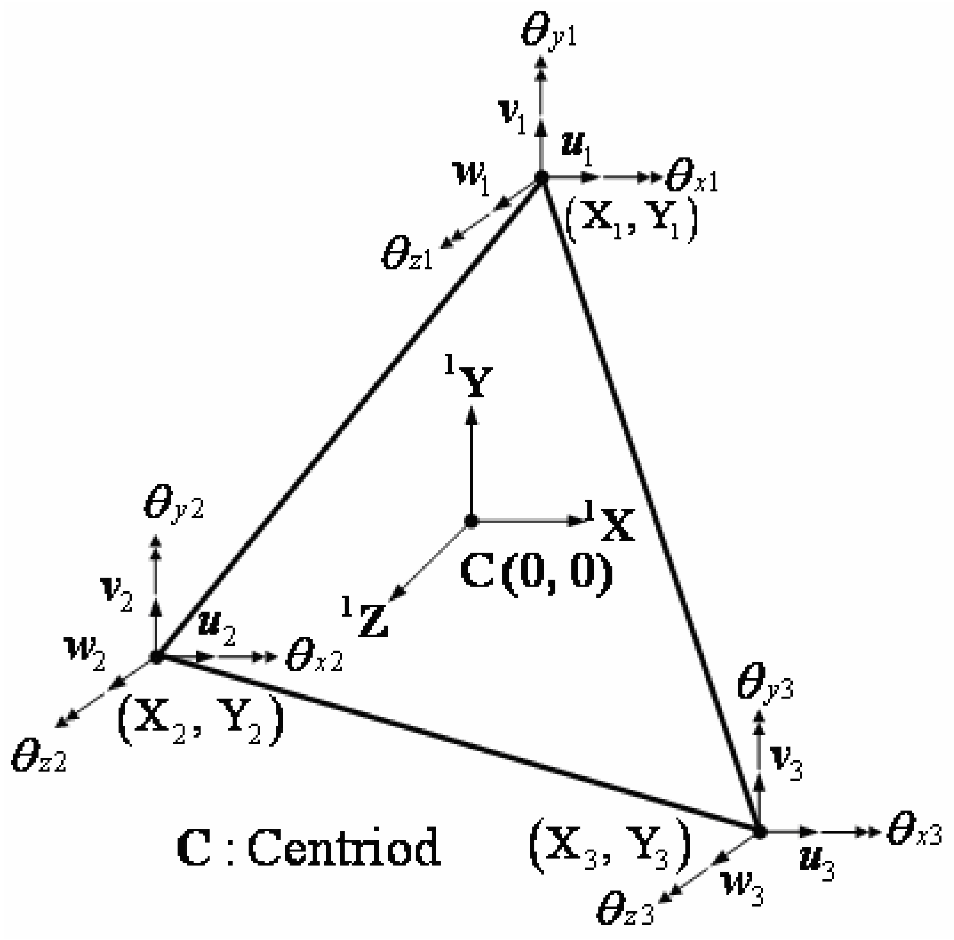
$$[h_b] = \begin{bmatrix} 0 & 0 & 0 \\ {}^1M_{yb}/L & -{}^1M_{xb}/2L & 0 \\ {}^1M_{zb}/L & 0 & -{}^1M_{xb}/2L \end{bmatrix}$$

$$[i_b] = \begin{bmatrix} 0 & 0 & 0 \\ -{}^1M_{zb} & 0 & {}^1M_{xb}/2 \\ {}^1M_{yb} & -{}^1M_{xb}/2 & 0 \end{bmatrix}$$

Geometric stiffness $[k_g]$ for rigid element

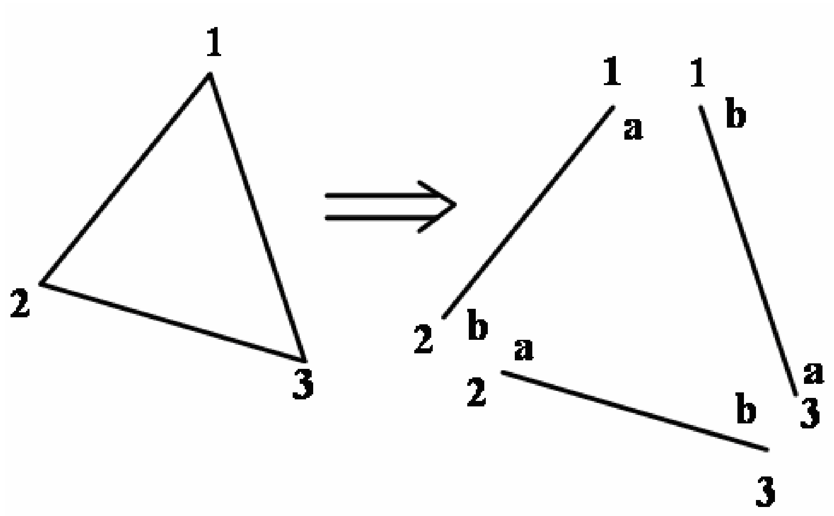
* Easy to derive
* No numerical integrations
* Explicit in expression

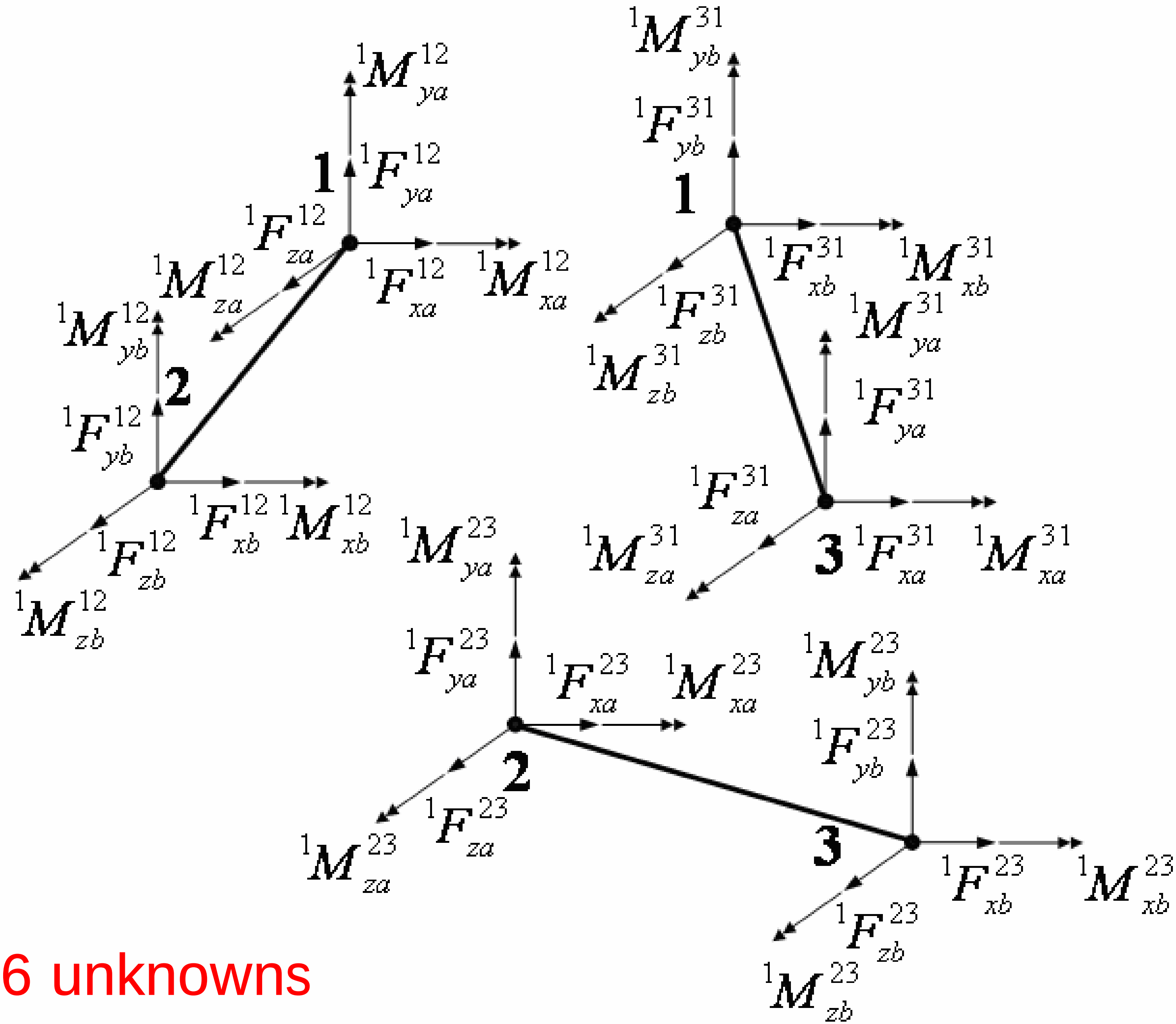
Triangular Plate Element (TPE)



Rigid TPE element

- Initial forces and external shape
- Irrelevant of elastic properties
- Regarded as 3 rigid beam elements




${}^1M^{12}_{ya}$
1
${}^1F^{12}_{ya}$
${}^1F^{12}_{za}$
${}^1M^{12}_{za}$
${}^1F^{12}_{xa}$
${}^1M^{12}_{xa}$
${}^1M^{12}_{yb}$
2
${}^1F^{12}_{yb}$
${}^1F^{12}_{xb}$
${}^1M^{12}_{xb}$
${}^1F^{12}_{zb}$
${}^1M^{12}_{zb}$
${}^1M^{31}_{yb}$
${}^1F^{31}_{yb}$
1
${}^1F^{31}_{xb}$
${}^1M^{31}_{xb}$
${}^1F^{31}_{zb}$
${}^1M^{31}_{ya}$
${}^1M^{31}_{zb}$
${}^1F^{31}_{ya}$
${}^1F^{31}_{za}$
${}^1M^{31}_{za}$
3
${}^1F^{31}_{xa}$
${}^1M^{31}_{xa}$
${}^1M^{23}_{ya}$
${}^1F^{23}_{ya}$
${}^1F^{23}_{xa}$
${}^1M^{23}_{xa}$
${}^1M^{23}_{yb}$
${}^1F^{23}_{yb}$
2
${}^1F^{23}_{za}$
${}^1M^{23}_{za}$
3
${}^1F^{23}_{xb}$
${}^1M^{23}_{xb}$
${}^1F^{23}_{zb}$
${}^1M^{23}_{zb}$

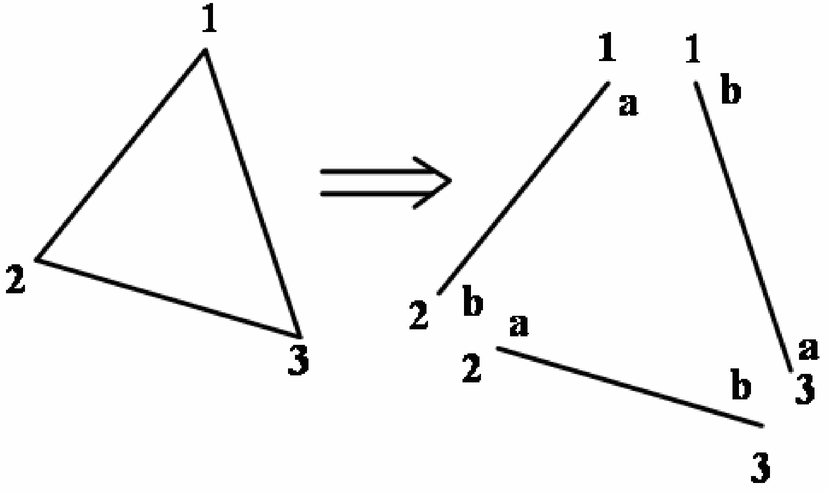
36 unknowns

Conditions used in calculation

- Eqns of equilibrium for 3 nodes of the TPE
  - 6 x 3 conditions
- Eqns of equilibrium of 3 beams
  - 6 x 3 conditions
- Eqns of equilibrium for the plate element
  - 6 conditions
- Total: 42 equations - 12 (dependent) = 30
- There are 36 unknowns
  - 6 are set to 0

Rigid TPE element

- Initial forces and external shape
- Irrelevant of elastic properties
- Regarded as 3 rigid beam elements



TPE element

- Assembly of 3 rigid beams

$$\left[k_g\right]^{TPE} = \sum_{ij=12,23,31} \left[T\right]^T \left[k_g\right]^{beam\_ij} \left[T\right]$$

- Transformation matrix

$$[T] = \begin{bmatrix} [R] & [0] & [0] & [0] \\ [0] & [R] & [0] & [0] \\ [0] & [0] & [R] & [0] \\ [0] & [0] & [0] & [R] \end{bmatrix}$$

$$[R] = \begin{bmatrix} -\frac{X_{ij}}{L_{ij}} & -\frac{Y_{ij}}{L_{ij}} & 0 \\ \frac{Y_{ij}}{L_{ij}} & -\frac{X_{ij}}{L_{ij}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inter-element continuity

- Equilibrium of joints at the *deformed state*
  - The *anti-symmetric* parts of the element matrices cancel out.
- The stiffness matrix is *symmetric* on the structure level

Elastic stiffness [k_e] for TPE

- Superposition of the following two elements:
  - Membrane actions: Cook (1987): plane hybrid element
  - Bending actions: Batoz et al. (1980): HSM element

Predictor stage

- By assembly:

$$\left(\left[K_e\right]+\left[K_g\right]\right)\left\{U\right\}=\left\{{}_1^2P\right\}-\left\{{}_1^1P\right\}$$

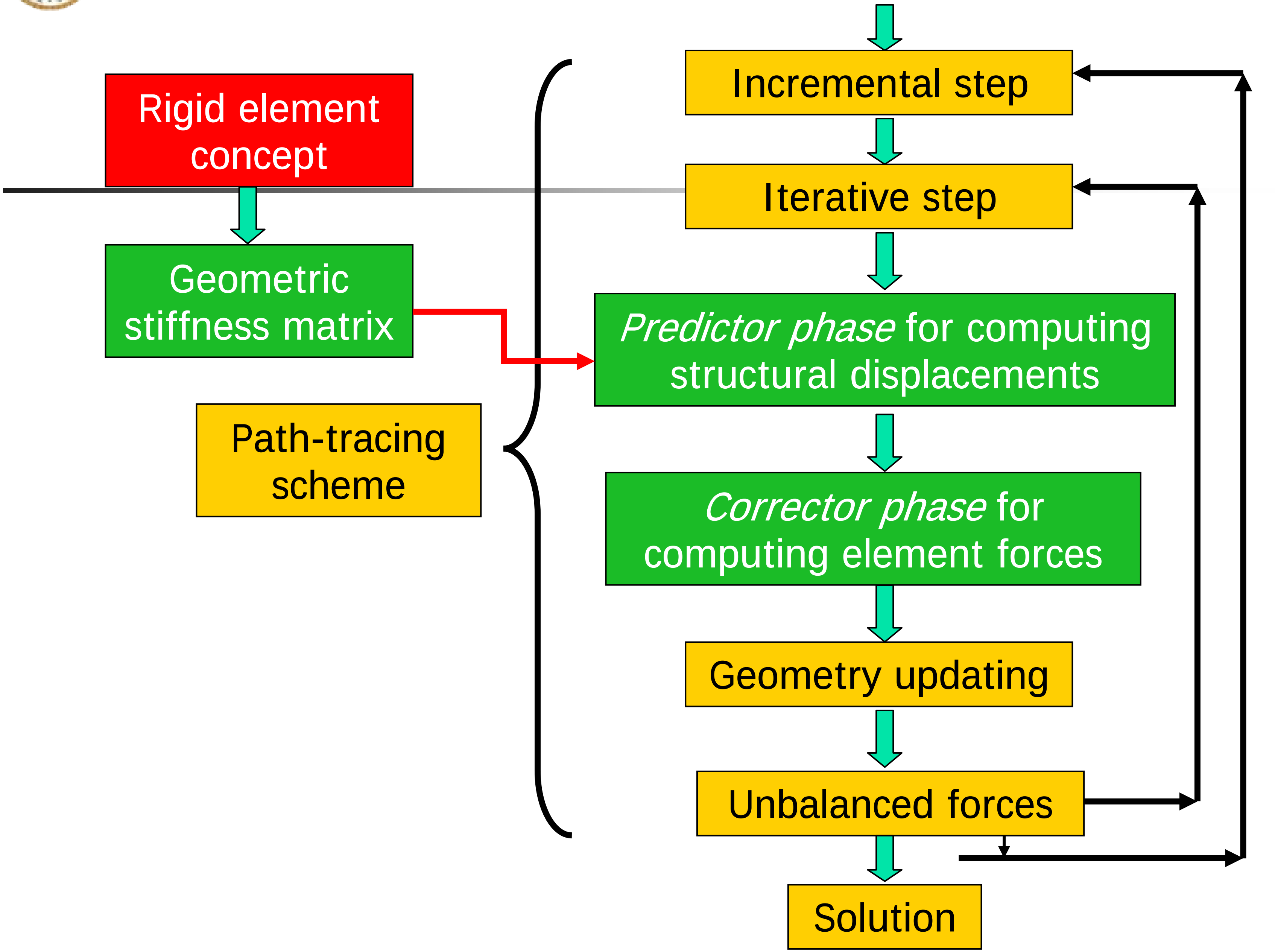
- Linearized
  - Approximate
- $[K_g]$ is the one derived for the rigid element
- Small load increments

Corrector stage

- Initial forces:
  - Treated as the ones acting at C_2 (by rigid body rule)
- Elastic actions
  - Computed using only $[k_e]$
- Total forces at

$$\left\{ {}_2^2 f \right\} = \left\{ {}_1^1 f \right\} + \left[k_e \right] \left\{ u \right\}$$

Rigid Elastic


Rigid element concept
Geometric stiffness matrix
Path-tracing scheme
Incremental step
Iterative step
Predictor phase for computing structural displacements
Corrector phase for computing element forces
Geometry updating
Unbalanced forces
Solution

Incremental-iterative scheme for postbuckling analysis

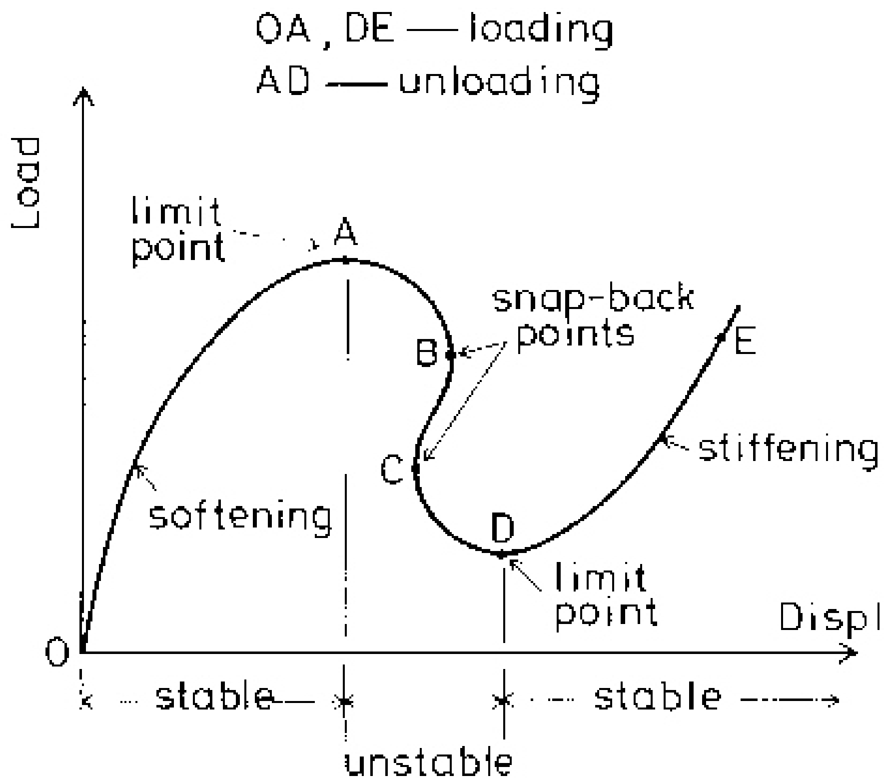


Fig. 1 General characteristics of a nonlinear system.

Good path-tracing scheme?

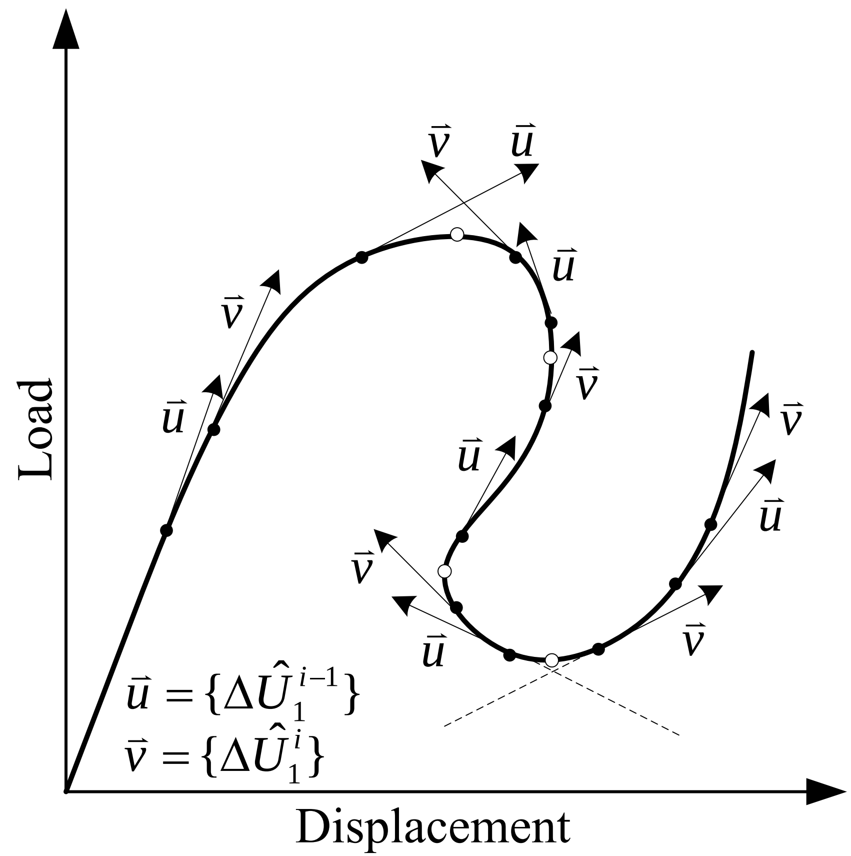
- To pass the *critical points*
  - Iterations should not be performed at constant loads
- To reflect the stiffness change using *variable* load increments
  - Reliable indicator is needed
- To reverse the *loading directions*
  - A detector is needed to trace the postbuckling response

Generalized displacement control (GDC) method (Yang & Shieh, *AIAA J.*, 1990)

- General stiffness parameter

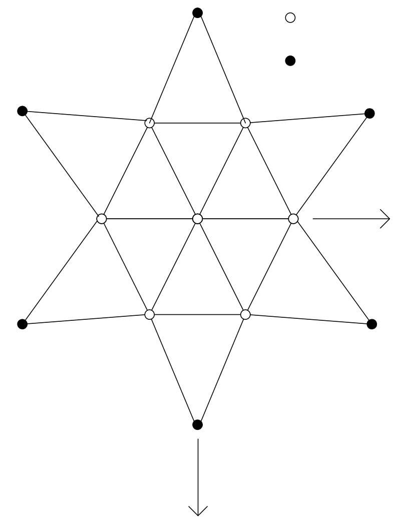
$$GSP = \frac{\{\Delta \hat{U}_1^1\}^T \{\Delta \hat{U}_1^1\}}{\{\Delta \hat{U}_1^{i-1}\}^T \{\Delta \hat{U}_1^i\}}$$

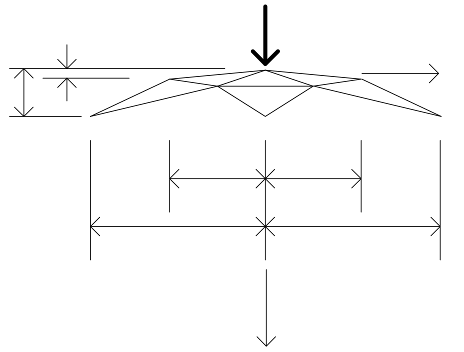
- Indicator for stiffness change
- Limit points
- Unloading

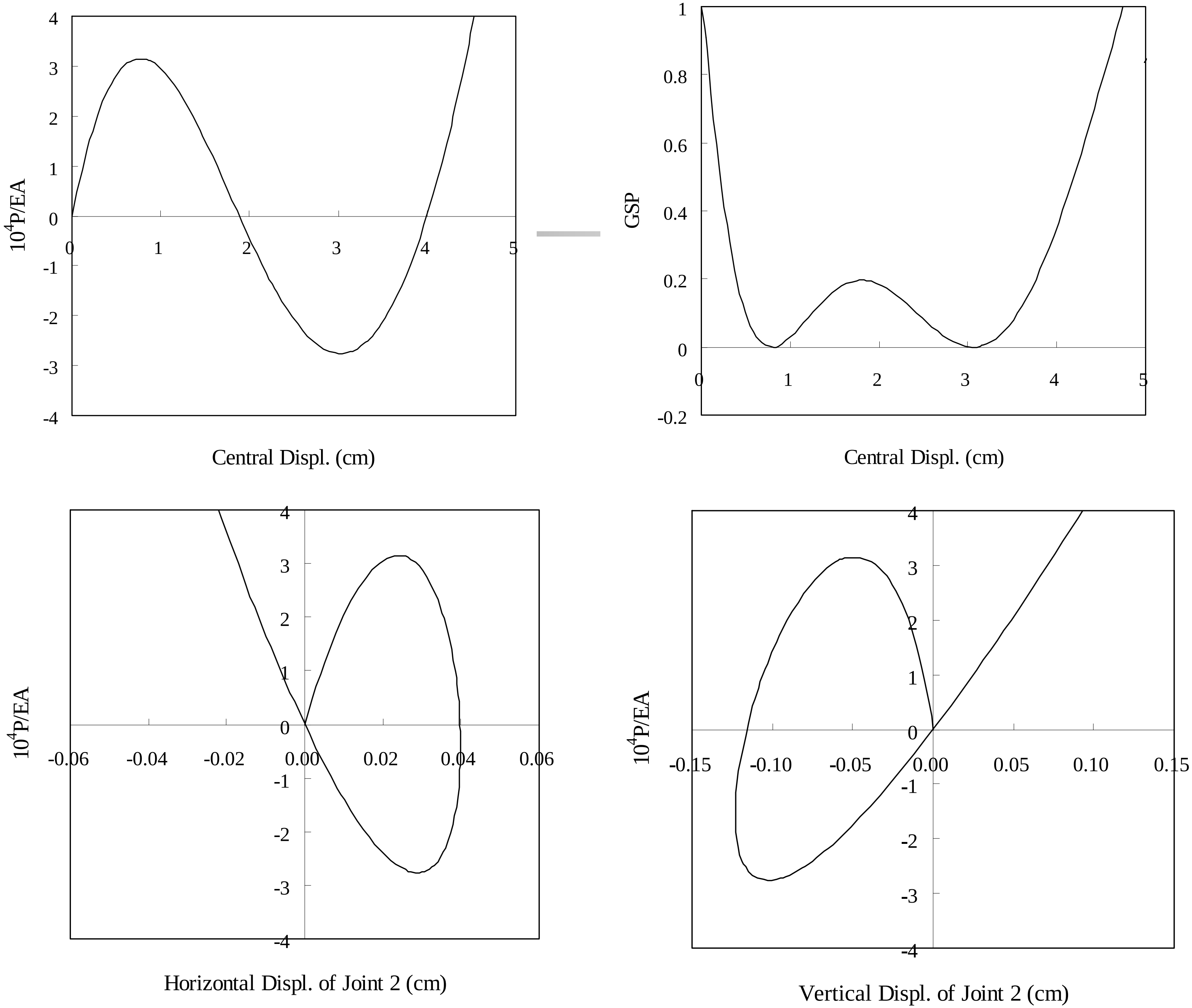


24-member shallow dome

- Hangai et al. (1971), Jagannathan et al. (1975), Holzer et al. (1980), Papadrakakis (1981)
- Zero load condition
  - Vert displ of joint 1 equals 2 cm + vert displ of joint 2



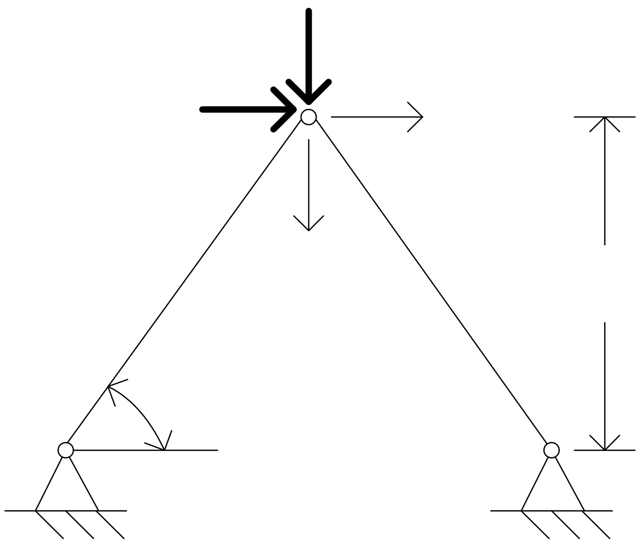




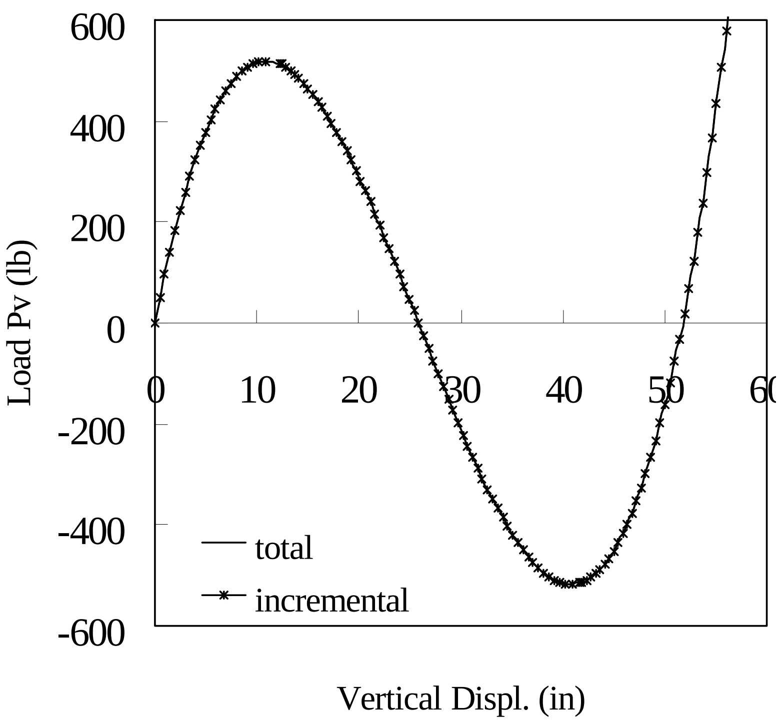
10^4P/EA
Central Displ. (cm)
GSP
Central Displ. (cm)
10^4P/EA
Horizontal Displ. of Joint 2 (cm)
10^4P/EA
Vertical Displ. of Joint 2 (cm)

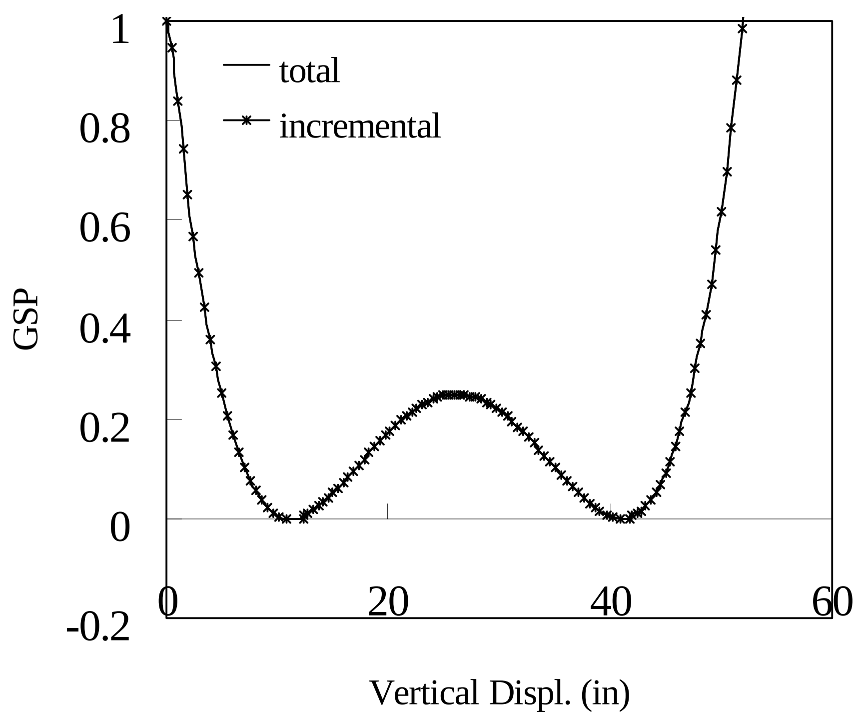
Two-member truss

- Pecknold et al. (1985)
- Adjacent equilibrium paths
- Bench mark for testing path-tracing schemes
- Perfect and imperfect loadings

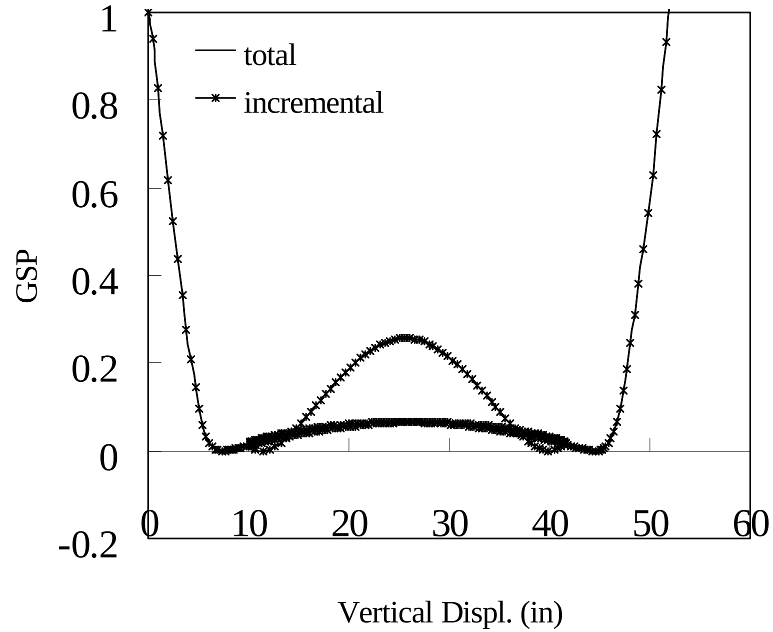


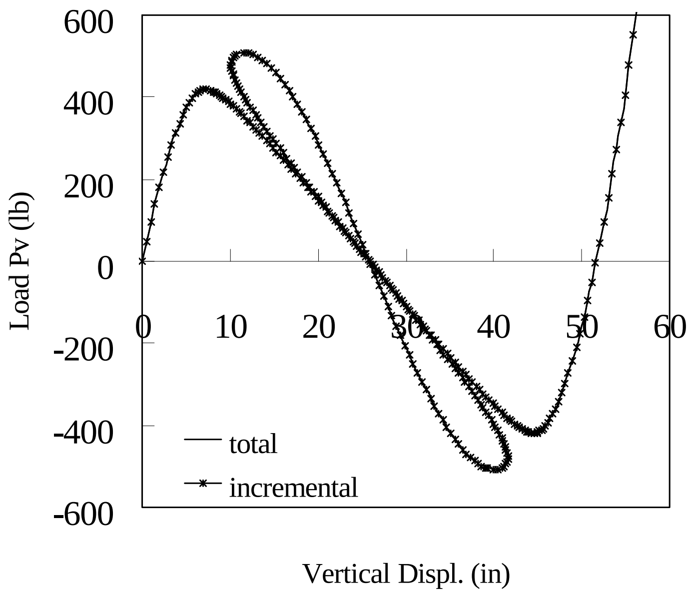
Perfect case

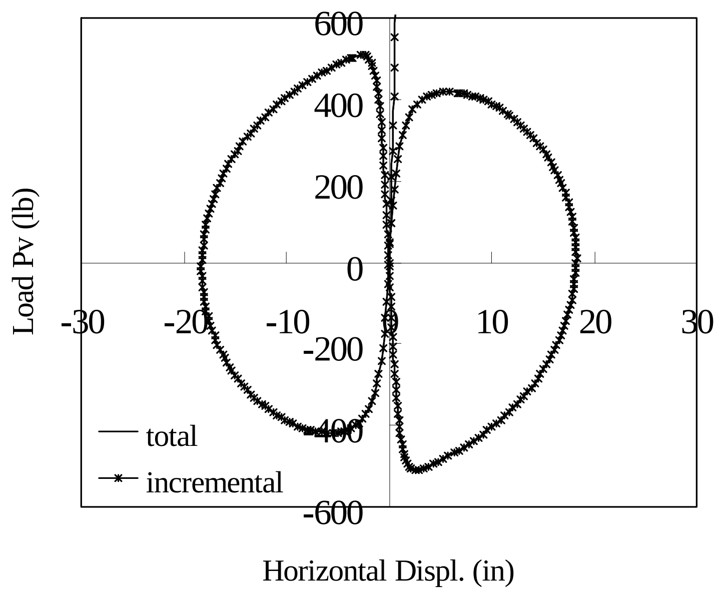




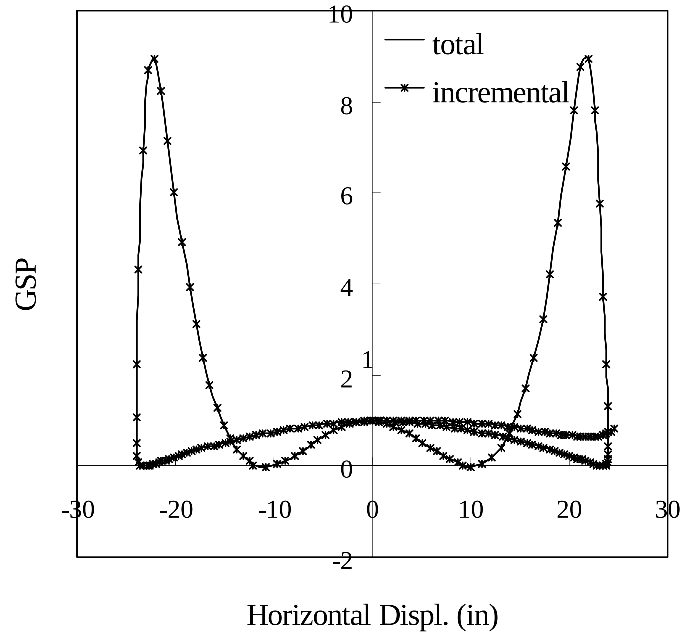
Imperfect case

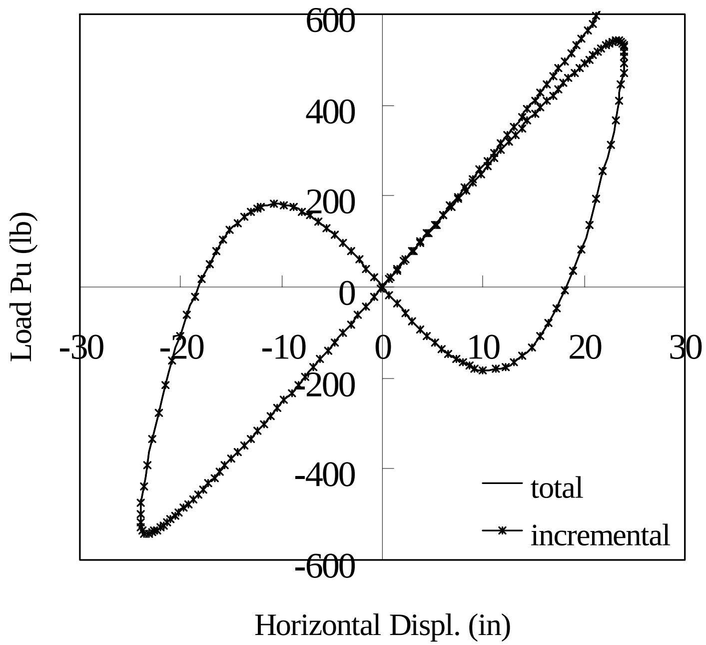


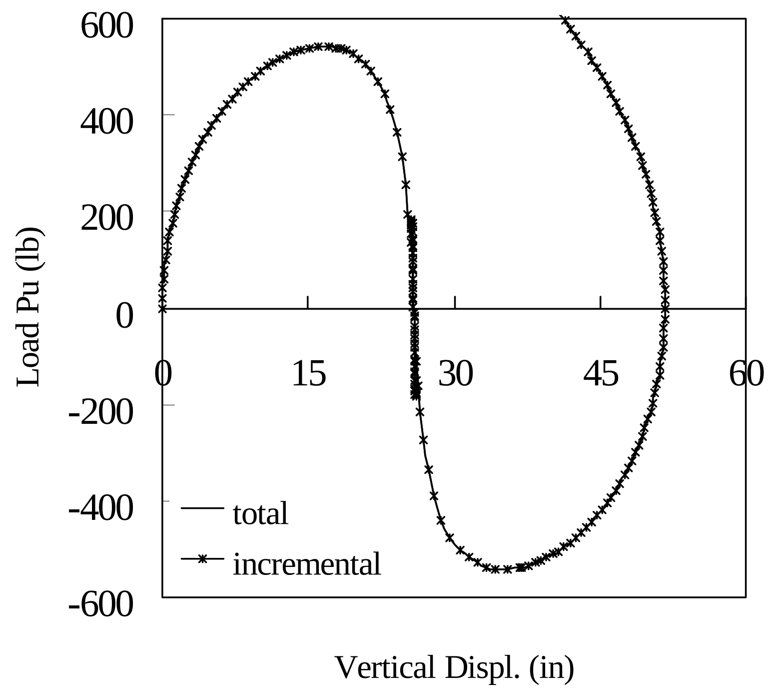




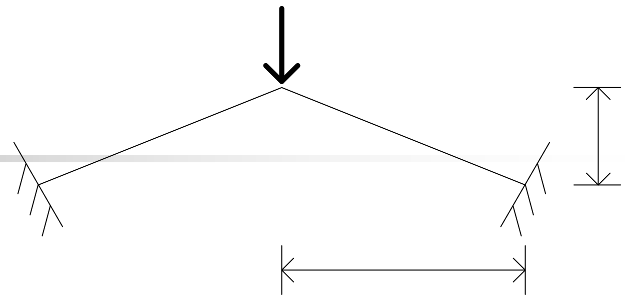
Horizontal load

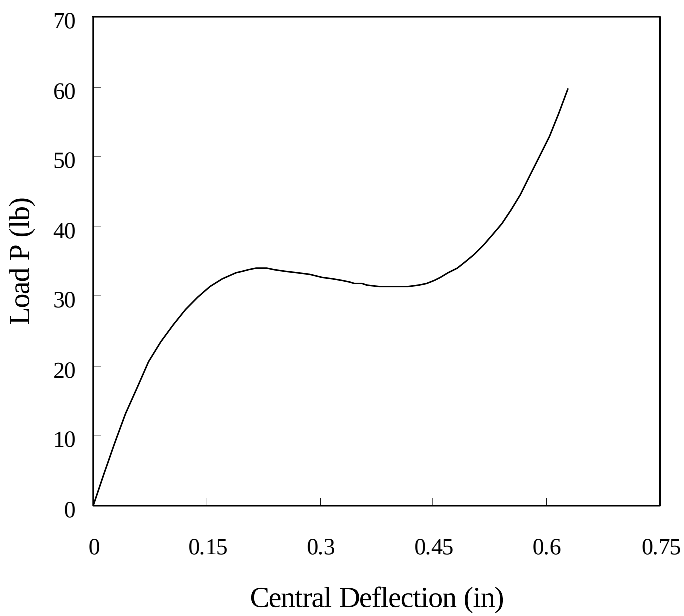


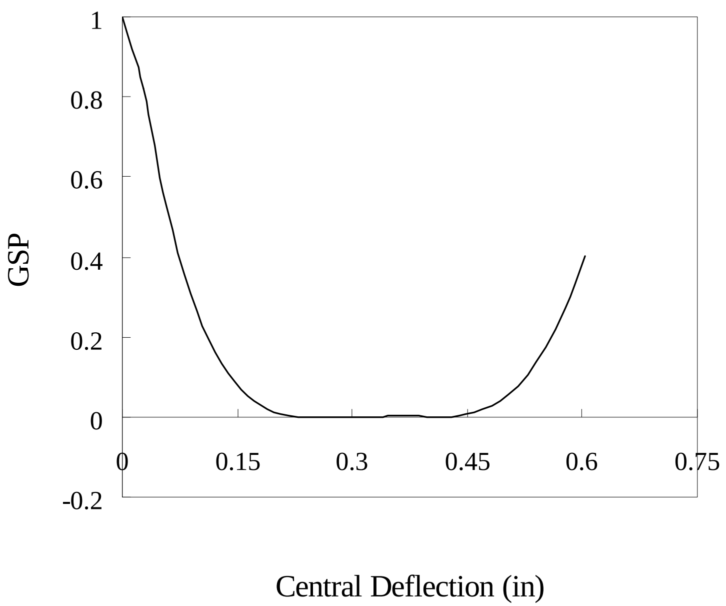




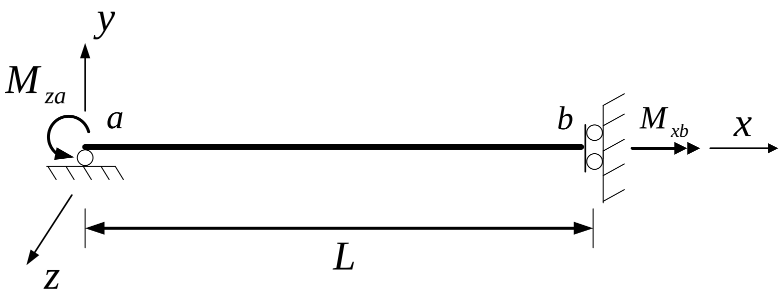
Williams' toggle

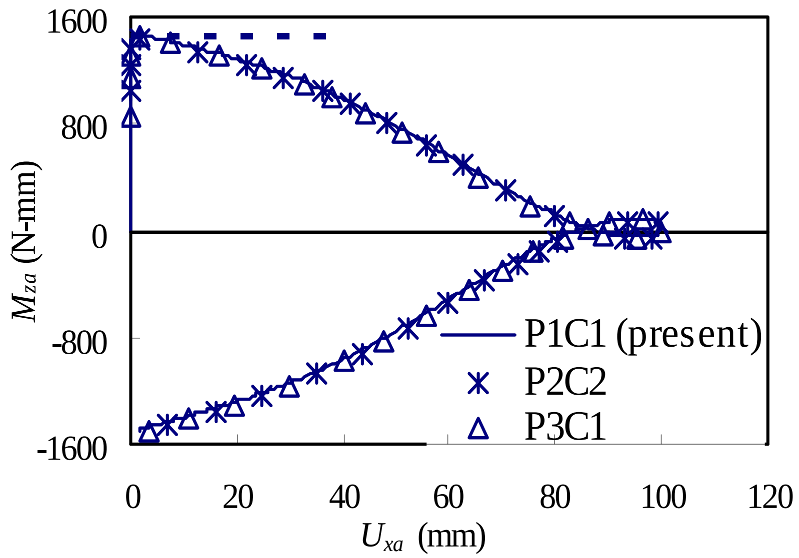


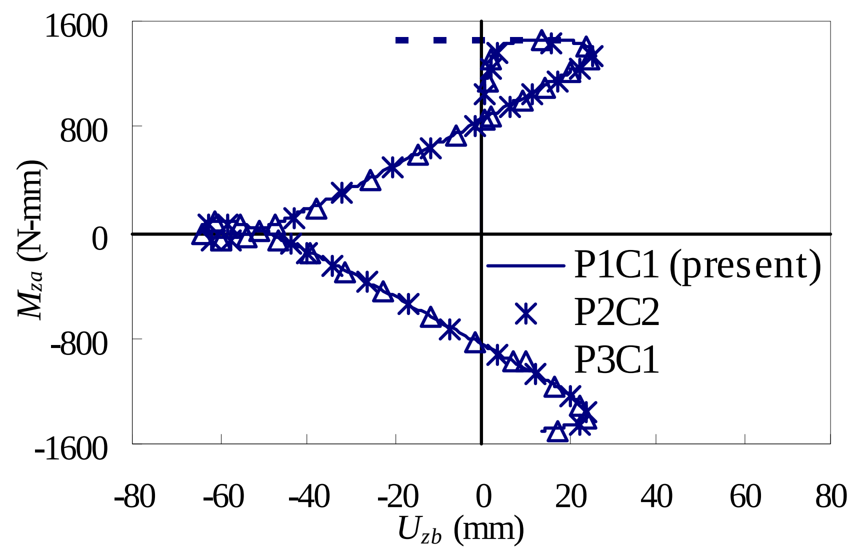




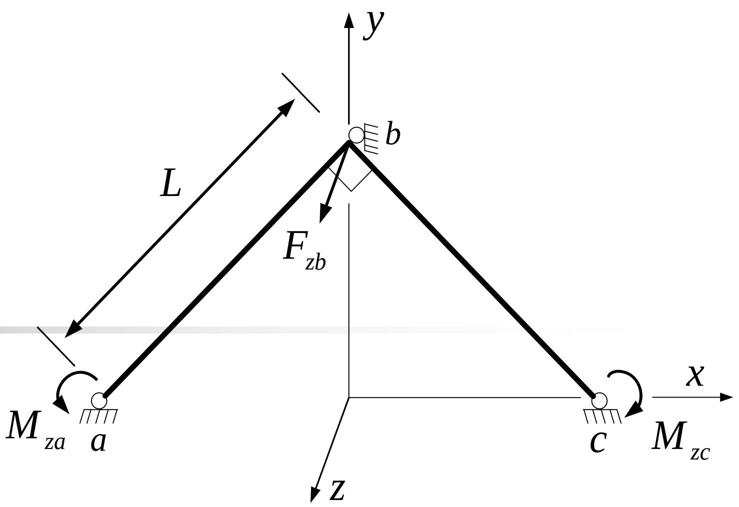
Beam under uniform bending

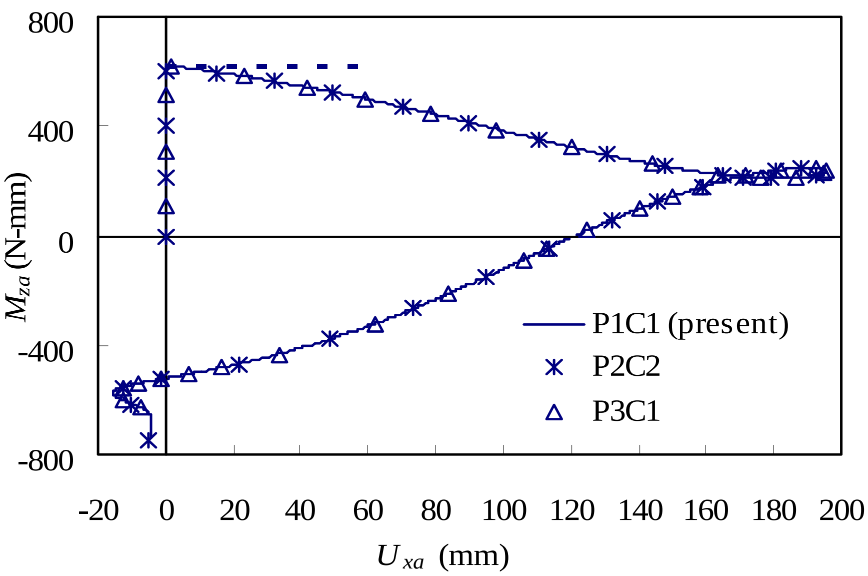


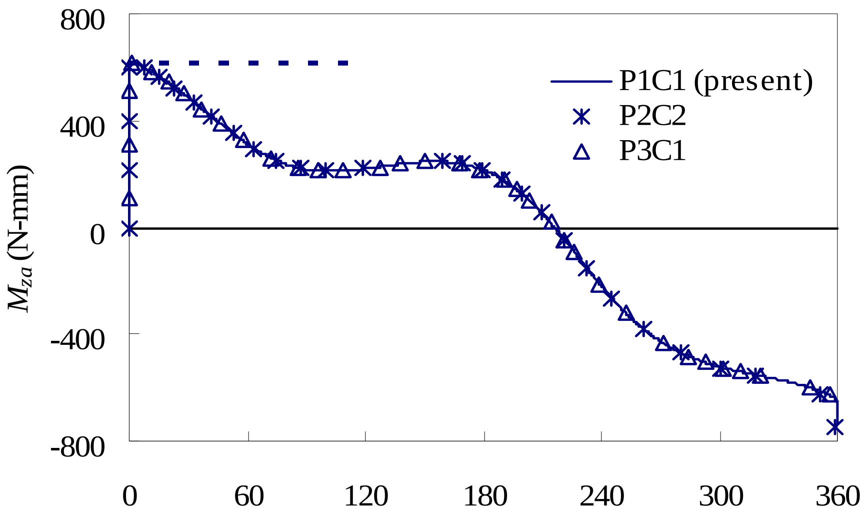




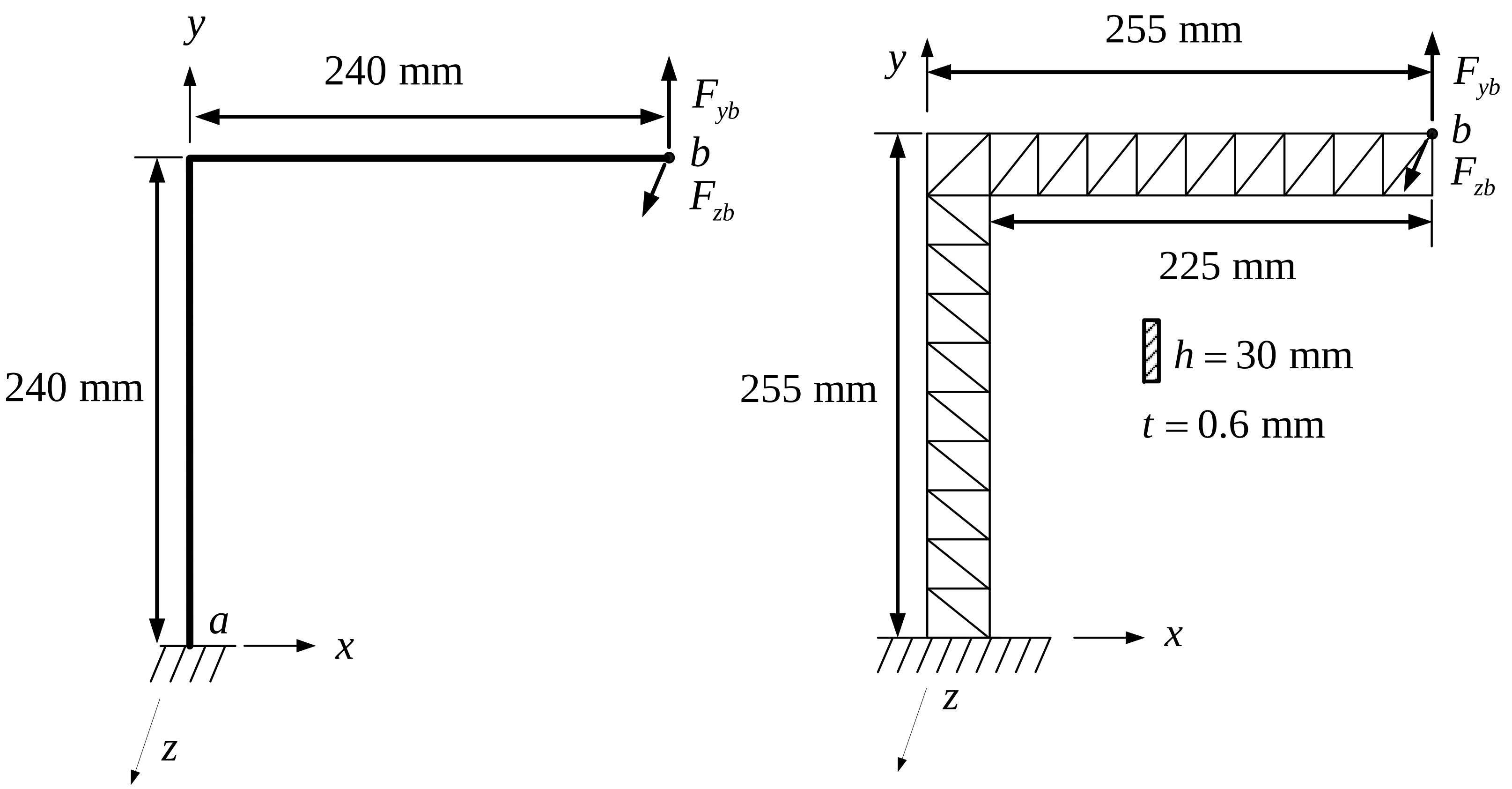
Angled frame

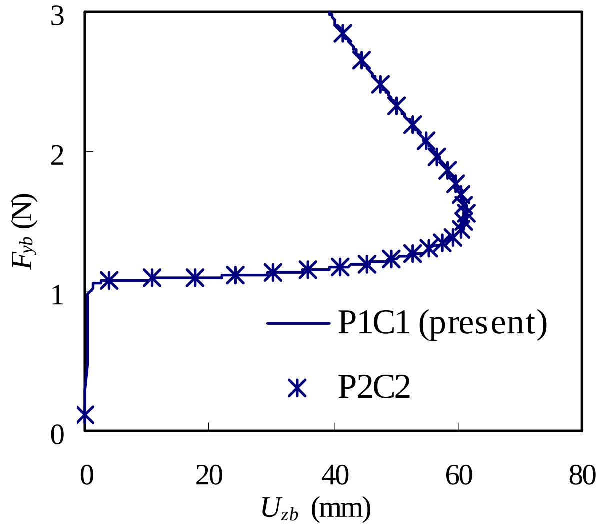




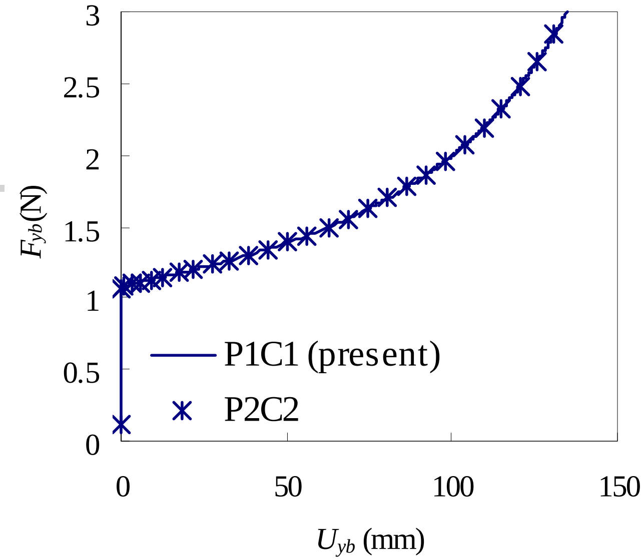


Right angled frame

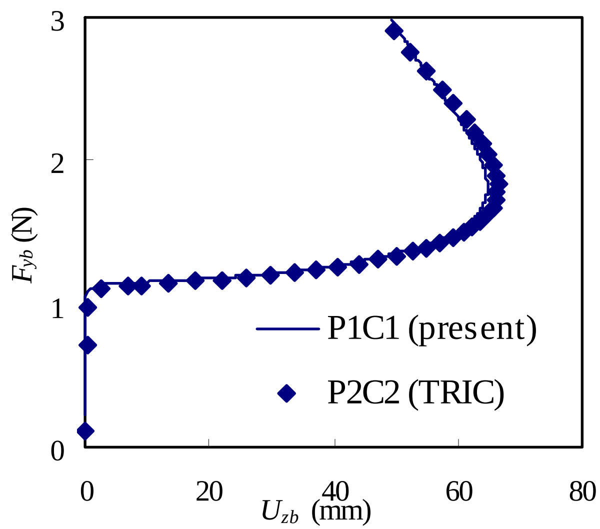




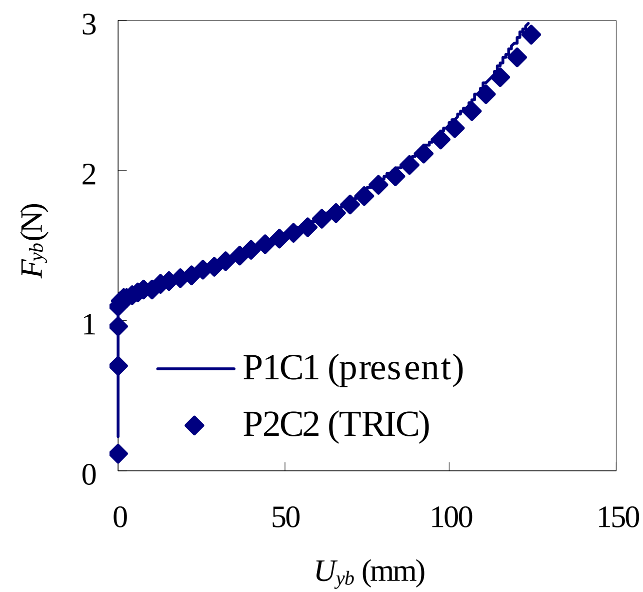
3
2
1
0
0
20
40
60
80
Fyb (N)
Uzb (mm)
P1C1 (present)
P2C2



3
2.5
2
1.5
1
0.5
0
0
50
100
150
Fyb(N)
Uyb (mm)
P1C1 (present)
P2C2

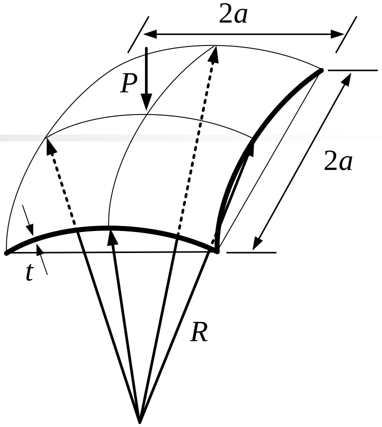


3
2
1
0
0
20
40
60
80
Fyb (N)
Uzb (mm)
P1C1 (present)
P2C2 (TRIC)



3
2
1
0
0
50
100
150
Fyb(N)
Uyb (mm)
P1C1 (present)
P2C2 (TRIC)

Circular dome



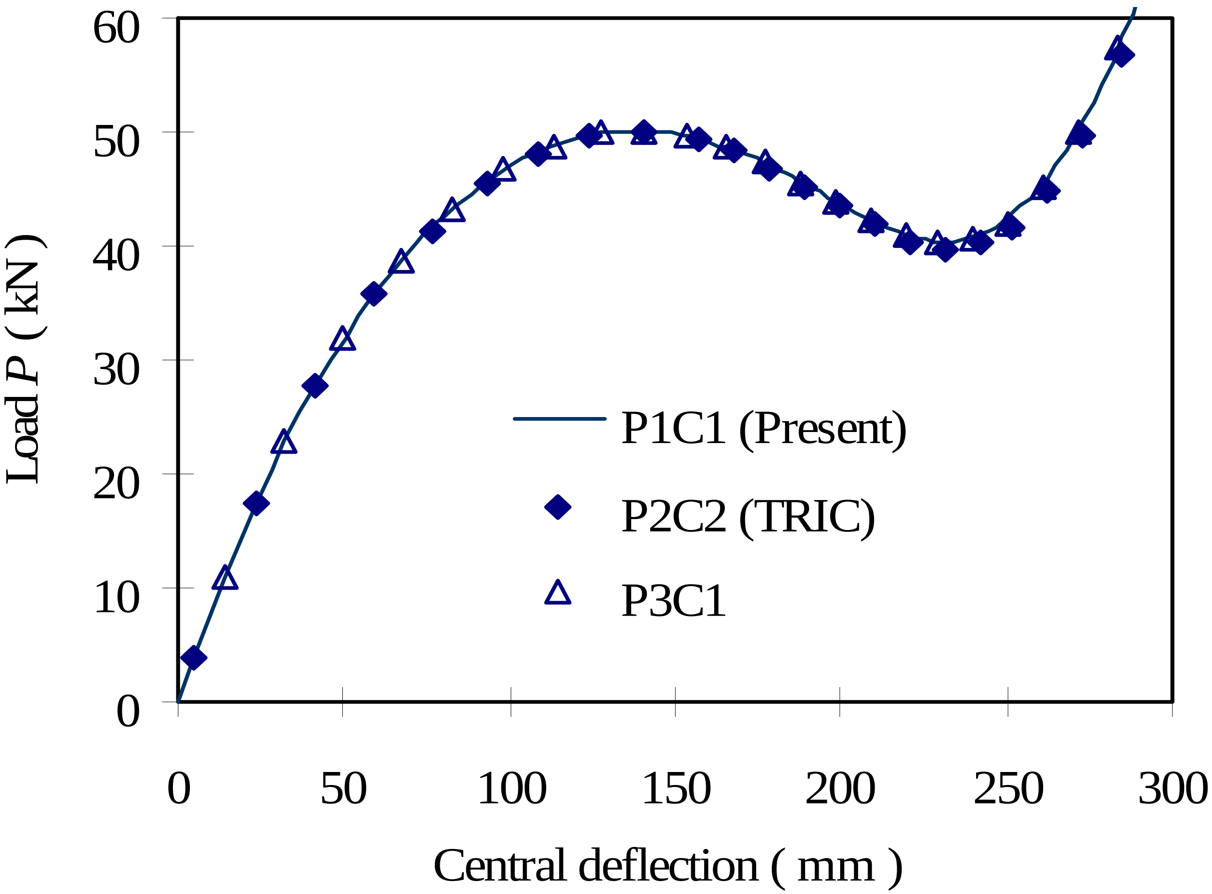
$R = 2540$ mm

$a = 784.90$ mm

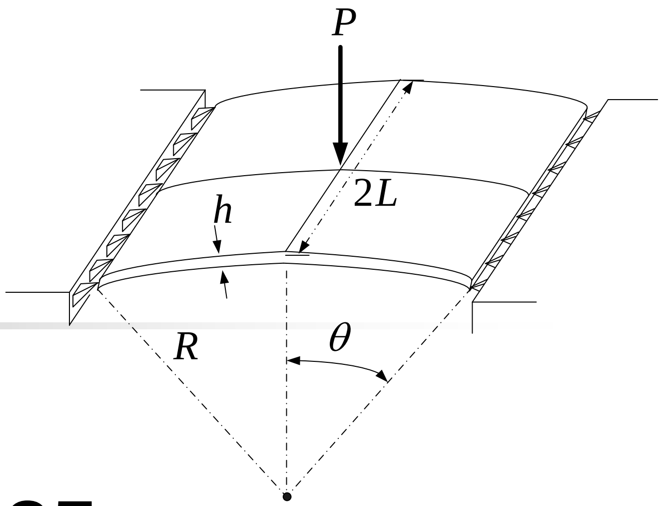
$t = 99.45$ mm

$E = 68.95$ N/mm^2

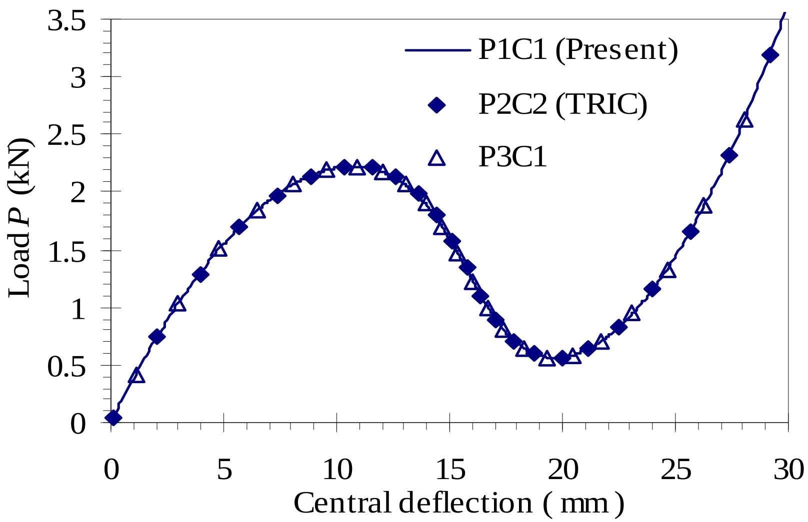
$\nu = 0.3$



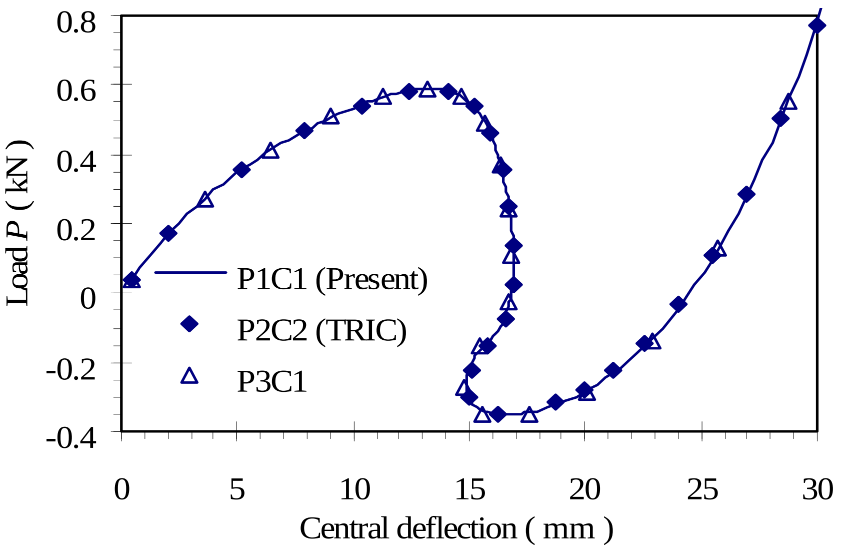
Shallow dome



- H = 12.7 mm h = 6.35 mm



thick



thin

Conclusions

- Effect of rigid displacements
  - Predictor: geometric stiffness $[k_g]$
  - Corrector: initial force updating
- Effect of natural deformations
  - Predictor: Elastic stiffness $[k_e]$
  - Corrector; Element force increments: Use $[k_e]$ matrix
- Rigid triangular plate element (TPE)
  - Assembled from 3 rigid beams
  - All explicit

Related publications

- Truss and beam structures
  - *Engineering Structures,* 29(6), 2007, 1189-1200.
- Plate and shell structures
  - *Comp. Meth. in Appl. Mech & Eng.*, 196(7), 2007, 1178-1192.

The End!