Answers to three not quite straightforward questions in structural stability

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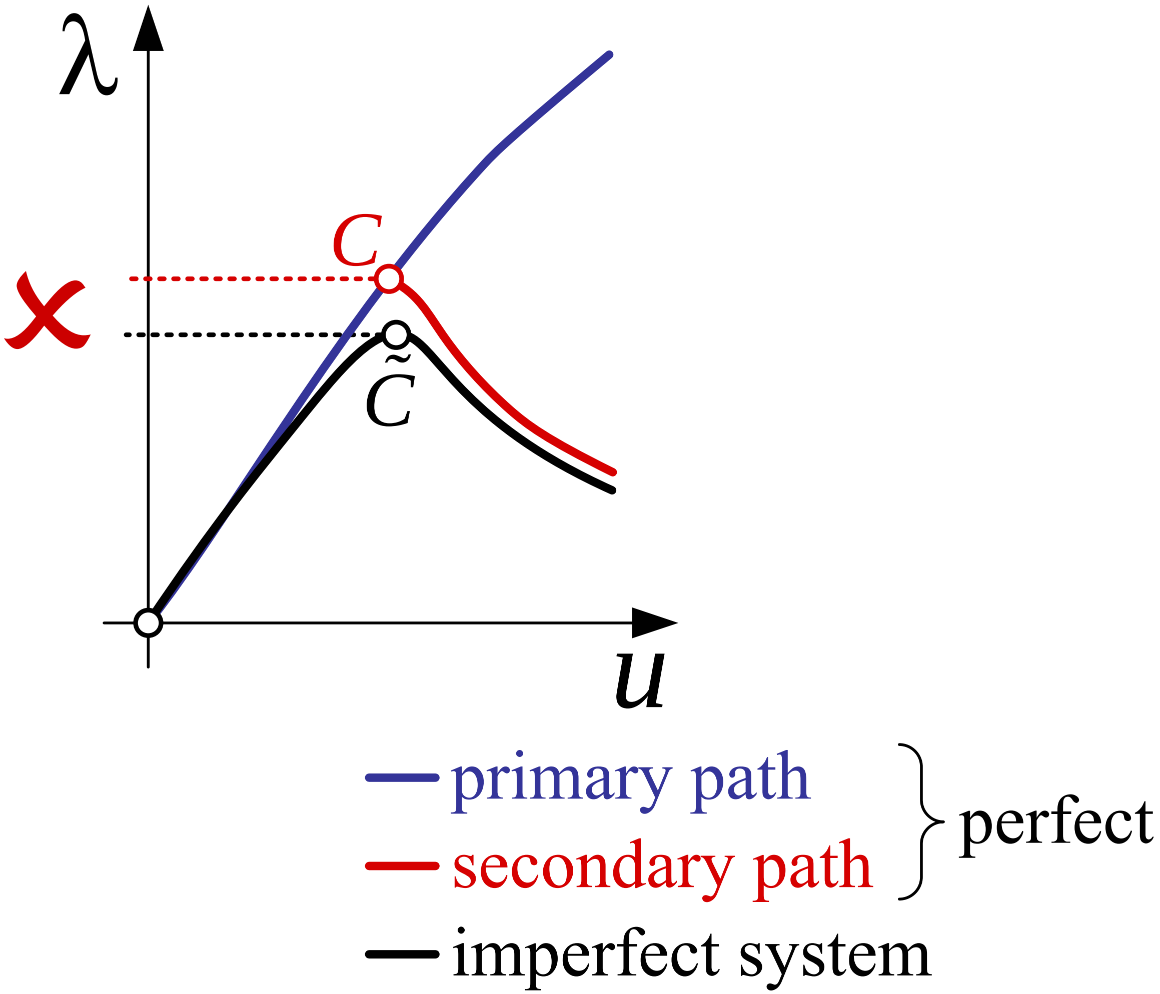
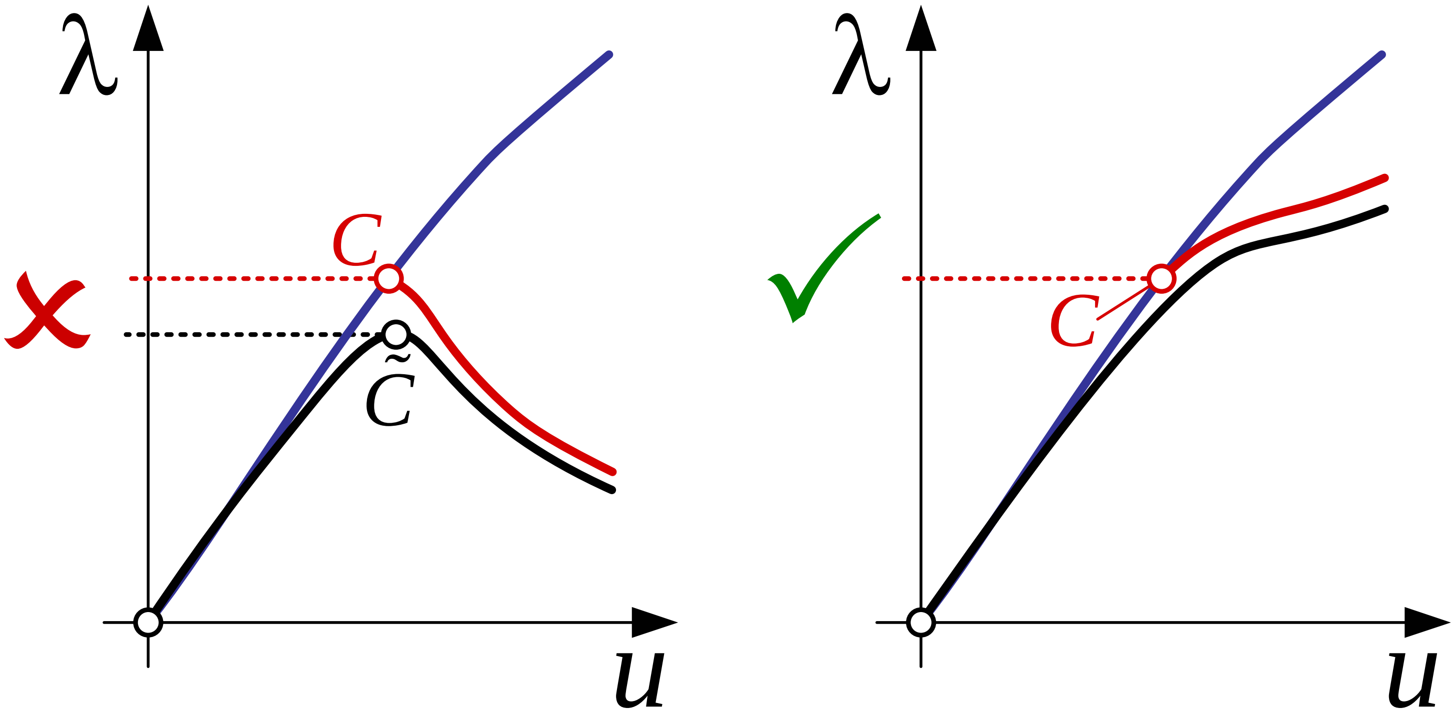
Motivation

modification of an imperfection-sensitive structure
such that it becomes imperfection-insensitive



Motivation

modification of an imperfection-sensitive structure such that it becomes imperfection-insensitive



- primary path
- secondary path
- imperfect system

} perfect

Question I

Are linear prebuckling paths and linear stability problems mutually conditional?

Literature

Steinboeck A, Mang HA. Are linear prebuckling paths and linear stability problems mutually conditional? In *Computational Mechanics* 2008, in press, doi: 10.1007/s00466-008-0257-3

Question II

Does the conversion from imperfection sensitivity into imperfection insensitivity require a symmetric postbuckling path?

Literature

Steinboeck A, Jia X, Hoefinger G, Mang HA. Conditions for symmetric, antisymmetric, and zero-stiffness bifurcation in view of imperfection sensitivity and insensitivity. In *Computer Methods in Applied Mechanics and Engineering* 2008, in press, doi: 10.1016/j.cma.2008.02.016

Question III

Is hilltop buckling necessarily imperfection sensitive?

Literature

Steinboeck A, Jia X, Hoefinger G, Rubin H, Mang HA. Remarkable postbuckling paths analyzed by means of the consistently linearized eigenproblem. In *International Journal for Numerical Methods in Engineering* 2008, in press, doi: 10.1002/nme.2317

Are linear prebuckling paths and linear stability problems mutually conditional?

- Motivation

Are linear prebuckling paths and linear stability problems mutually conditional?

Linear stability problem

$$\det(\tilde{\boldsymbol{K}}_T(\underbrace{\lambda}_{\text{load factor}})) = \det(\underbrace{\boldsymbol{K}_0}_{\text{small displacement stiffness}} + \lambda \underbrace{\boldsymbol{K}_1}_{\text{geometric stiffness}}) = 0$$

- no prebuckling rotations allowed
- simplifies computation of critical load λ_C

Belytschko T, Liu WK, Moran B. *Nonlinear finite elements for continua and structures*. Wiley, Chichester, 2000.

Wriggers P. *Nichtlineare Finite-Elemente-Methoden* (German, Nonlinear finite element methods). Springer, Berlin, 2001.

Zienkiewicz OC, Taylor RL. *The finite element method, volume 2, solid mechanics*. Butterworth-Heinemann: Oxford, England, 2000.

Are linear prebuckling paths and linear stability problems mutually conditional?

Linear stability problem

$$\det(\tilde{\boldsymbol{K}}_T(\lambda)) = \det(\boldsymbol{K}_0 + \lambda \boldsymbol{K}_1) = 0$$

load factor

small displacement stiffness

geometric stiffness

- no prebuckling rotations allowed
- simplifies computation of the critical load λ_C

⇕ linked ?

Linear prebuckling path

$$\tilde{u}_{,\lambda} = \boldsymbol{k} = \text{const.} \;\rightarrow\; \tilde{\boldsymbol{u}}(\lambda) = \boldsymbol{u}_0 + \lambda \boldsymbol{k}$$

Are linear prebuckling paths and linear stability problems mutually conditional?

Conditions for a linear stability problem
in the prebuckling domain

- negligible change of
  material tangent stiffness matrix
- small displacements
- linear stress-load relation
- loads do not depend on the displacements

Sources of nonlinearity

- geometric nonlinearity
- material behavior
- boundary conditions

Are linear prebuckling paths and linear stability problems mutually conditional?

Theory

potential energy V

displacements $\boldsymbol{u}$

out-of-balance-force $\boldsymbol{G}$

$$\boxed{\boldsymbol{G} := \frac{\partial V}{\partial \boldsymbol{u}} = \boldsymbol{F}^{I}(\boldsymbol{u}) - \lambda \boldsymbol{P}}$$

internal forces

reference load

load factor

$V_{,\boldsymbol{u}} = \boldsymbol{G} = \boldsymbol{0}$ … equilibrium condition

$V_{,\boldsymbol{uu}} = \boldsymbol{K}_T$ … tangent stiffness matrix

Are linear prebuckling paths and linear stability problems mutually conditional?

$$V_{,\boldsymbol{u}} = \boldsymbol{G} = \boldsymbol{F}^{I}(\boldsymbol{u}) - \lambda\,\boldsymbol{P} = \boldsymbol{0}$$ … equilibrium condition

$$\Big\downarrow \; \frac{d}{d\lambda}$$ along primary path

$$\tilde{\boldsymbol{K}}_T \cdot \tilde{\boldsymbol{u}}_{,\lambda} - \boldsymbol{P} = \boldsymbol{0}$$

$$\Big\downarrow \; \frac{d}{d\lambda}$$ along primary path

$$\tilde{\boldsymbol{K}}_{T,\lambda} \cdot \tilde{\boldsymbol{u}}_{,\lambda} + \tilde{\boldsymbol{K}}_T \cdot \tilde{\boldsymbol{u}}_{,\lambda\lambda} = \boldsymbol{0}$$

$$\Big\downarrow \; \tilde{\boldsymbol{u}}_{,\lambda} = \boldsymbol{k}$$ ... linear prebuckling path

$$\tilde{\boldsymbol{K}}_{T,\lambda} \cdot \boldsymbol{k} = \boldsymbol{0}$$

not sufficient for a linear stability problem $\tilde{\boldsymbol{K}}_T(\lambda) = \boldsymbol{K}_0 + \lambda\,\boldsymbol{K}_1$

Are linear prebuckling paths and linear stability problems mutually conditional?

$$V_{,\boldsymbol{u}} = \boldsymbol{G} = \boldsymbol{F}^{I}(\boldsymbol{u}) - \lambda\, \boldsymbol{P} = \boldsymbol{0}$$ … equilibrium condition

$$\Big\downarrow \frac{d}{d\lambda}$$ along primary path

$$\tilde{\boldsymbol{K}}_T \cdot \tilde{\boldsymbol{u}}_{,\lambda} - \boldsymbol{P} = \boldsymbol{0}$$

$$\Big\downarrow \tilde{\boldsymbol{K}}_T(\lambda) = \boldsymbol{K}_0 + \lambda\, \boldsymbol{K}_1 \; ...$$ linear stability problem

$$(\boldsymbol{K}_0 + \lambda\, \boldsymbol{K}_1) \cdot \tilde{\boldsymbol{u}}_{,\lambda} - \boldsymbol{P} = \boldsymbol{0}$$

not sufficient for a linear prebuckling path $\tilde{\boldsymbol{u}}(\lambda) = \boldsymbol{u}_0 + \lambda\, \boldsymbol{k}$

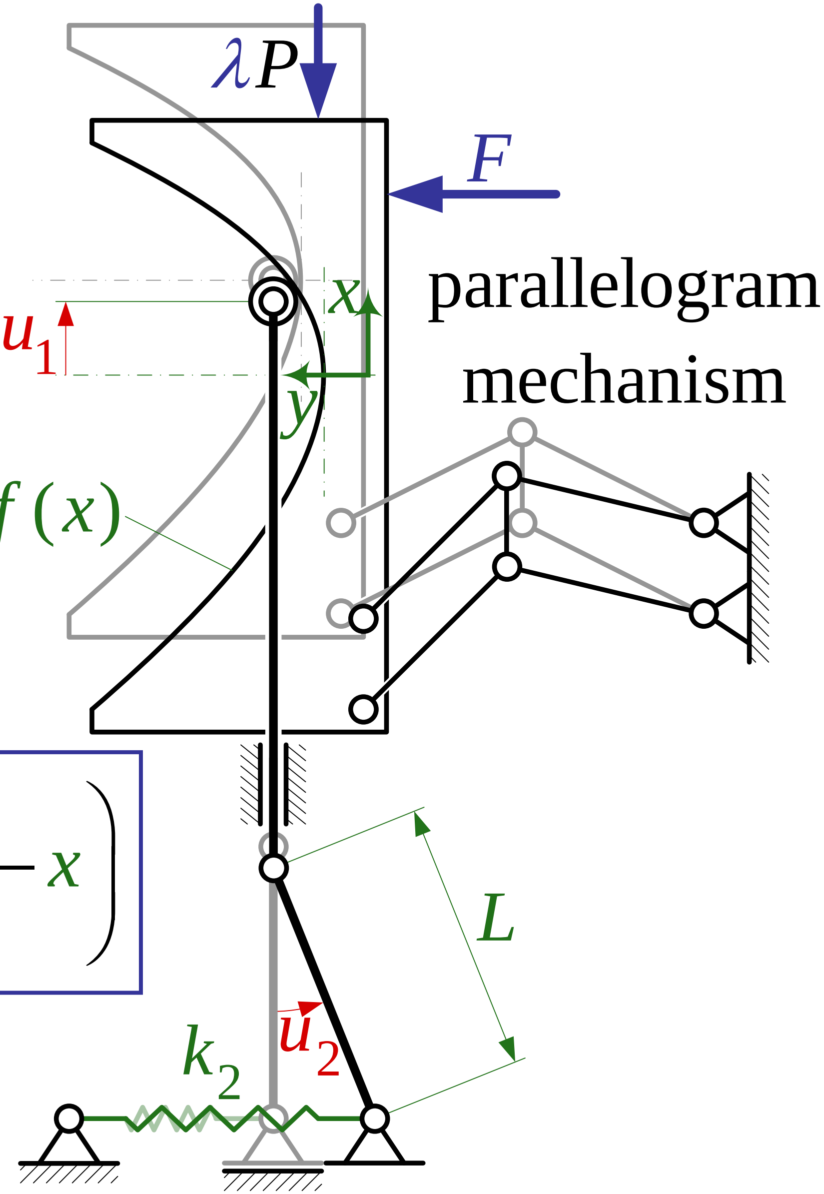
Are linear prebuckling paths and linear stability problems mutually conditional?

Example I

A linear stability problem

2 DOF, static, conservative, contour (a nonlinear spring)

$$f(x) = \frac{k_0 P}{k_1 F}\left(\frac{P}{k_1}\left(e^{x k_1 / P} - 1\right) - x\right)$$



Are linear prebuckling paths and linear stability problems mutually conditional?

Example I **A linear stability problem**

- potential energy

$$V = -\lambda P\left(u_1 + L(1-\cos(u_2))\right) + F f(u_1) + \frac{k_2}{2} L^2 \sin^2(u_2)$$

- nonlinear prebuckling path, load-displacement relation

$$\lambda = \frac{k_0}{k_1}\left(e^{u_1 k_1/P} - 1\right), \quad u_2 = 0$$

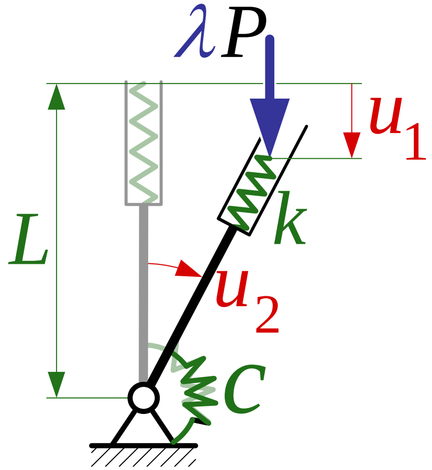
- linear stability problem, tangent stiffness matrix

$$\tilde{\boldsymbol{K}}_T = \begin{bmatrix} k_0 + k_1\lambda & 0 \\ 0 & L(k_2 L - \lambda P) \end{bmatrix}$$

Are linear prebuckling paths and linear stability problems mutually conditional?

Example II

A linear prebuckling path

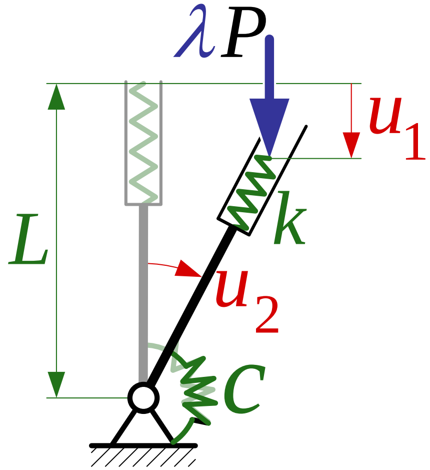


2 DOF, static, conservative

Are linear prebuckling paths and linear stability problems mutually conditional?

Example II

A linear prebuckling path



2 DOF, static, conservative

- potential energy

$$V = -\lambda P u_1 + \frac{1}{2} c u_2^2 + \frac{1}{2} k \left(L - \frac{L - u_1}{\cos(u_2)} \right)^2$$

- linear prebuckling path, load-displacement relation

$$\lambda = u_1 k / P, \quad u_2 = 0$$

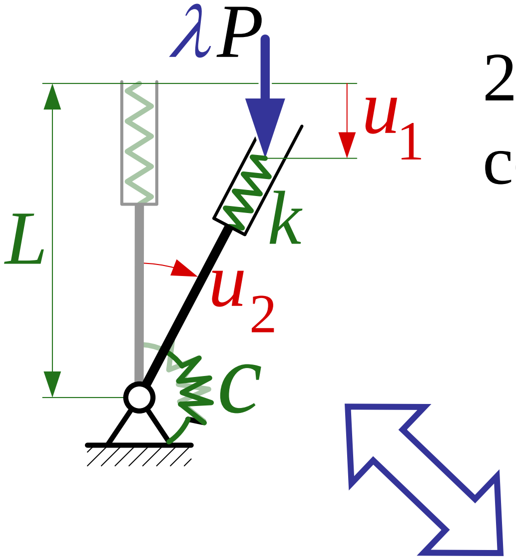
- nonlinear stability problem, tangent stiffness matrix

$$\tilde{\boldsymbol{K}}_T = \begin{bmatrix} k & 0 \\ 0 & c - \lambda P (L - \lambda P / k) \end{bmatrix}$$

Are linear prebuckling paths and linear stability problems mutually conditional?

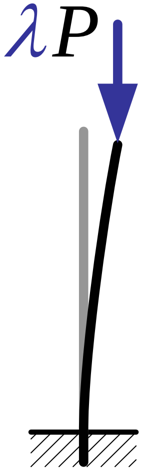
Example II

A linear prebuckling path



2 DOF, static, conservative

- effect is of higher order
- usually negligible
- e.g. Euler buckling cases



Conclusions

A linear prebuckling path is

- neither necessary
- nor sufficient

for a stability problem to be linear.

Literature

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Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Question II

Does the conversion from imperfection sensitivity into imperfection insensitivity require a symmetric postbuckling path?

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Motivation

- general research interest: conversion from imperfection sensitivity structures into imperfection insensitive structures
- What are the conditions for this conversion?
- Is symmetry required?
- *ab initio* design for imperfection insensitivity
- existence of qualitative properties which influence imperfection insensitivity

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Koiter’s initial postbuckling analysis

Idea: expansion of the out-of-balance force

$$V_{,\boldsymbol{u}} = \boldsymbol{G} = \boldsymbol{F}^{I}(\boldsymbol{u}) - \lambda\, \boldsymbol{P}$$

into a Taylor series at the bifurcation point C

Koiter W. On the stability of elastic equilibrium, Translation of ‘Over de Stabiliteit van het Elastisch Evenwicht’ (1945), In *NASA TT F-10833*, Polytechnic Institute Delft, Amsterdam, 1967.

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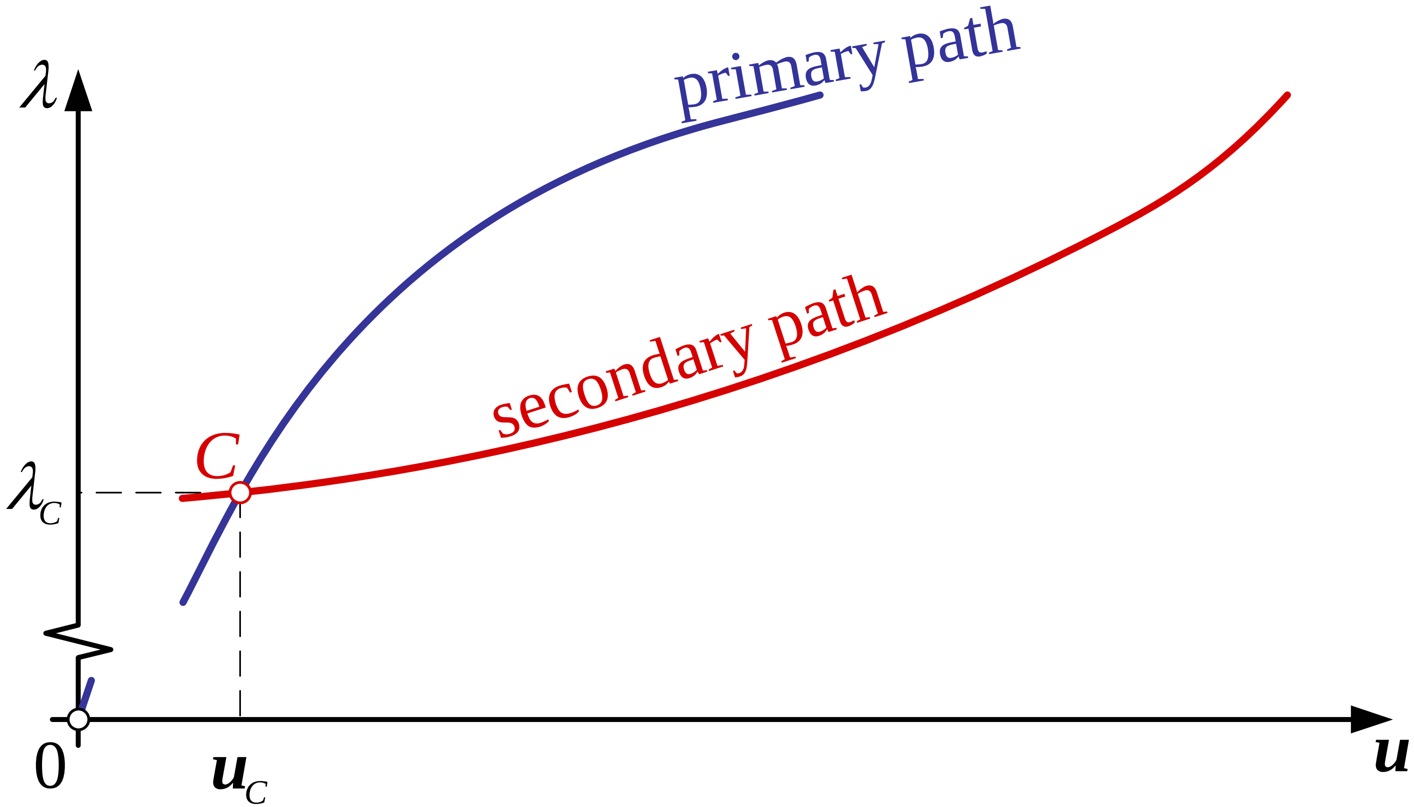

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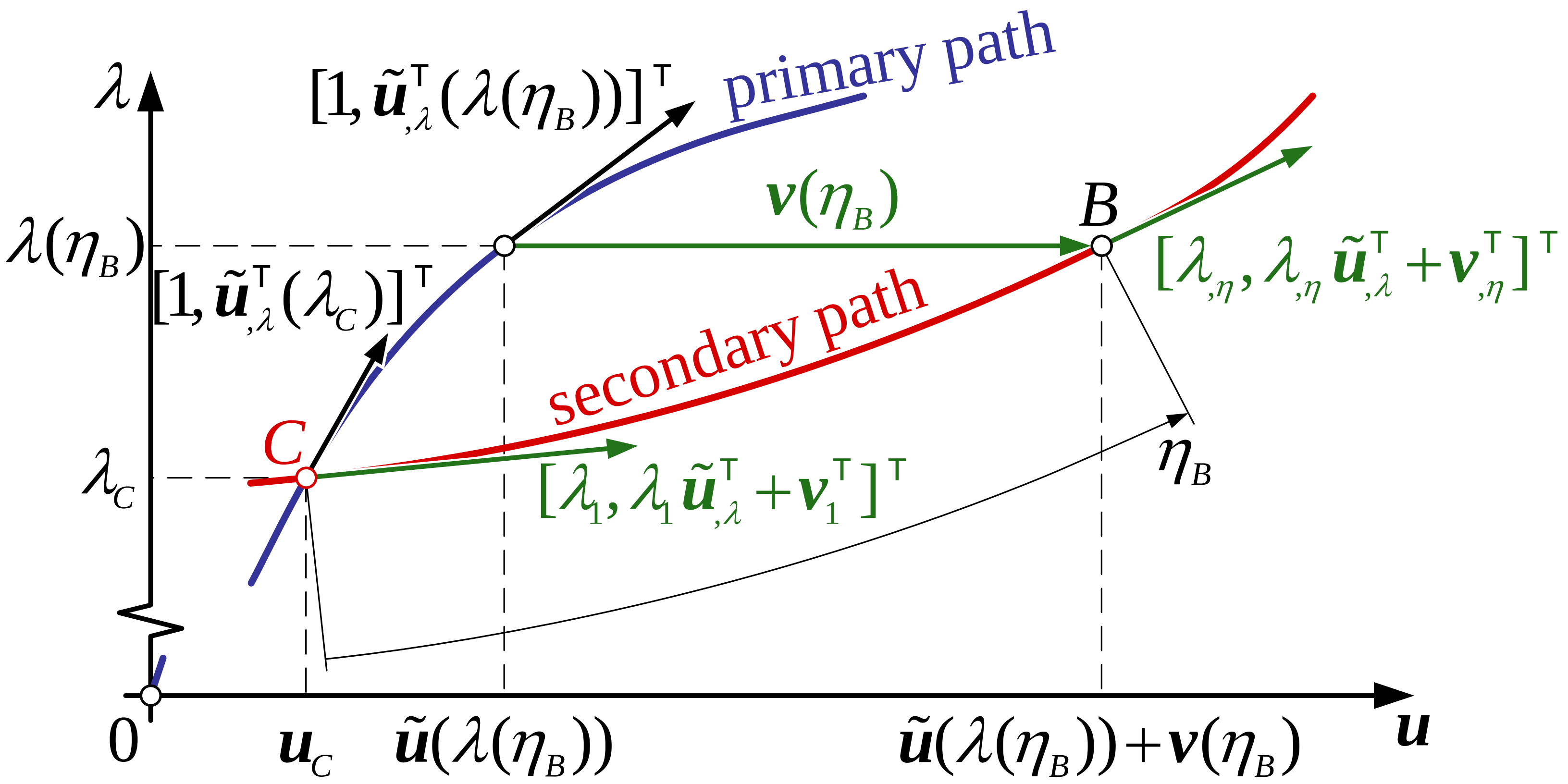

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Koiter’s initial postbuckling analysis

Idea: expansion of the out-of-balance force

$$V_{,\boldsymbol{u}} = \boldsymbol{G} = \boldsymbol{F}^{I}(\boldsymbol{u}) - \lambda\,\boldsymbol{P}$$

into a Taylor series at the bifurcation point C

- coordinate transformation for the secondary path

$$(\mathbf{v},\eta) \mapsto (\boldsymbol{u},\lambda) = (\tilde{\boldsymbol{u}}(\lambda(\eta)) + \mathbf{v}, \lambda(\eta))$$

- series expansion of coordinates

$$\mathbf{v}(\eta) = \mathbf{v}_1\,\eta + \mathbf{v}_2\,\eta^2 + \mathbf{v}_3\,\eta^3 + \ldots$$

$$\lambda(\eta) = \lambda_C + \lambda_1\,\eta + \lambda_2\,\eta^2 + \lambda_3\,\eta^3 + \ldots$$

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

- transformed out-of-balance force

$$\boldsymbol{G}(\boldsymbol{v},\eta) := \boldsymbol{G}(\tilde{\boldsymbol{u}}(\lambda(\eta)) + \boldsymbol{v}, \lambda(\eta))$$

- series expansion of the out-of-balance force at C

$$\boldsymbol{G}(\boldsymbol{v},\eta) = \boldsymbol{G}_{0C} + \boldsymbol{G}_{1C}\,\eta + \boldsymbol{G}_{2C}\,\eta^{2} + \boldsymbol{G}_{3C}\,\eta^{3} + \ldots = \boldsymbol{0}$$

$\Big\downarrow$ must hold for arbitrary values of η

$$\boldsymbol{G}_{nC} = \boldsymbol{0} \quad \forall n \in Q$$

$\Big\downarrow$ allows computation

$$(\lambda_1, \boldsymbol{v}_1),\ (\lambda_2, \boldsymbol{v}_2),\ (\lambda_3, \boldsymbol{v}_3),\ \ldots$$

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Necessary and sufficient condition
for **imperfection insensitivity**

$$\lambda(\eta) = \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \ldots + \lambda_n \eta^n + \ldots$$

the first non-vanishing coefficient must have an even subscript and must be positive

$$\lambda_1 = 0 \quad \ldots \quad \text{horizontal tangent is necessary}$$

Bochenek B. Problems of structural optimization for post-buckling behaviour. In *Structural and Multidisciplinary Optimization* 2003; **25**/5-6:423-435.

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Symmetric load-displacement paths

- linear mapping $\boldsymbol{T} : \mathrm{U}^N \rightarrow \mathrm{U}^N$ (symmetry group)

example: $$\boldsymbol{T} = \begin{bmatrix} 1 & & & \\ & \ddots & & \mathbf{0} \\ & \mathbf{0} & 1 & \\ & & & -1 \end{bmatrix}$$

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Symmetric load-displacement paths

- linear mapping $\boldsymbol{T} : \mathrm{U}^N \rightarrow \mathrm{U}^N$ (symmetry group)
- symmetry requires $V(\boldsymbol{u},\lambda) = V(\boldsymbol{T}\cdot\boldsymbol{u},\lambda)$
- mirror symmetry w.r.t. η

$$V(\tilde{\boldsymbol{u}}(\lambda(\eta))+\boldsymbol{v}(\eta),\lambda(\eta)) = \\ V(\tilde{\boldsymbol{u}}(\lambda(-\eta))+\boldsymbol{v}(-\eta),\lambda(-\eta)) \quad \forall \eta \in \mathrm{U}$$

- consequences:

uniqueness of the primary path requires

① $\tilde{\boldsymbol{u}}(\lambda) = \boldsymbol{T}\cdot\tilde{\boldsymbol{u}}(\lambda) \quad \forall \lambda \in \mathrm{U}$

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Symmetric load-displacement paths

- linear mapping $\boldsymbol{T} : \mathrm{U}^N \rightarrow \mathrm{U}^N$ (symmetry group)
- symmetry requires $V(\boldsymbol{u}, \lambda) = V(\boldsymbol{T} \cdot \boldsymbol{u}, \lambda)$
- mirror symmetry w.r.t. η

$$V(\tilde{\boldsymbol{u}}(\lambda(\eta)) + \boldsymbol{v}(\eta), \lambda(\eta)) =$$
$$V(\tilde{\boldsymbol{u}}(\lambda(-\eta)) + \boldsymbol{v}(-\eta), \lambda(-\eta)) \quad \forall \eta \in \mathrm{U}$$

- consequences:

secondary path

② $\boldsymbol{v}(\eta) = \boldsymbol{T} \cdot \boldsymbol{v}(-\eta) \quad \forall \eta \in \mathrm{U}$

③ $\lambda(\eta) = \lambda(-\eta) \quad \forall \eta \in \mathrm{U}$

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Imperfection insensitivity

$$\lambda(\eta) = \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \ldots + \lambda_n \eta^n + \ldots$$

the first non-vanishing coefficient must have an even subscript and must be positive

Symmetry

① $\tilde{\boldsymbol{u}}(\lambda) = \boldsymbol{T} \cdot \tilde{\boldsymbol{u}}(\lambda) \quad \forall \lambda \in \mathrm{U}$

② $\boldsymbol{v}(\eta) = \boldsymbol{T} \cdot \boldsymbol{v}(-\eta) \quad \forall \eta \in \mathrm{U}$

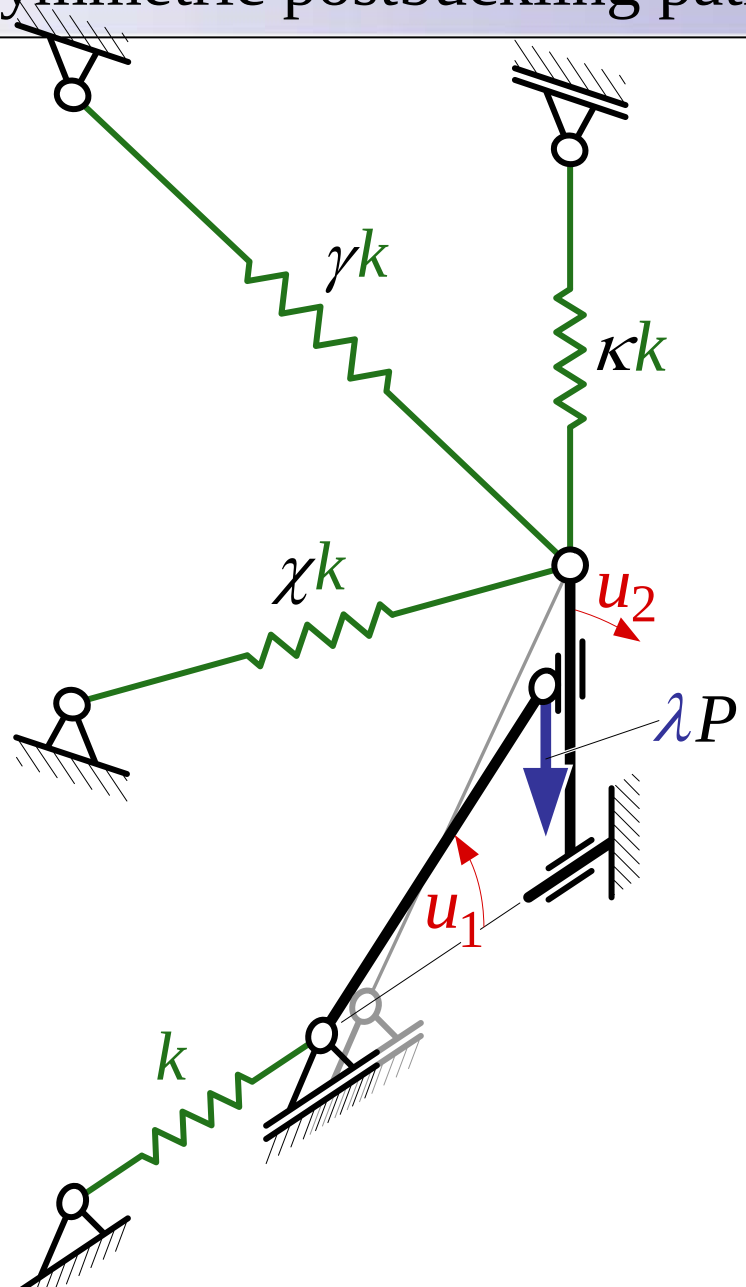
③ $\lambda(\eta) = \lambda(-\eta) \quad \forall \eta \in \mathrm{U}$ $\Rightarrow \quad \lambda_1 = \lambda_3 = \ldots = 0$

↓

symmetry is not necessary for the conversion from imperfection sensitivity into insensitivity

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Example



Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Question II

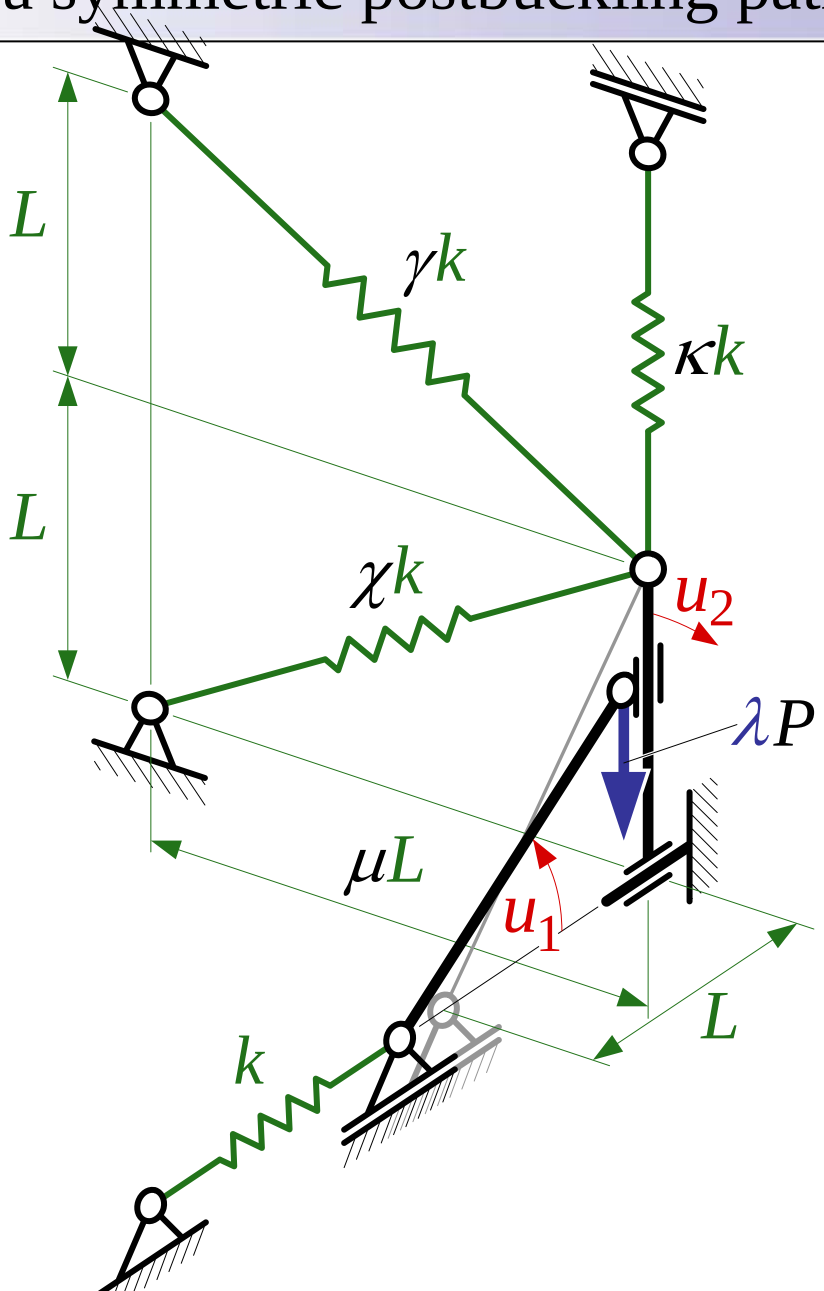
Motivation

Theory

Example

Conclusions

Example



2 DOF

static, conservative

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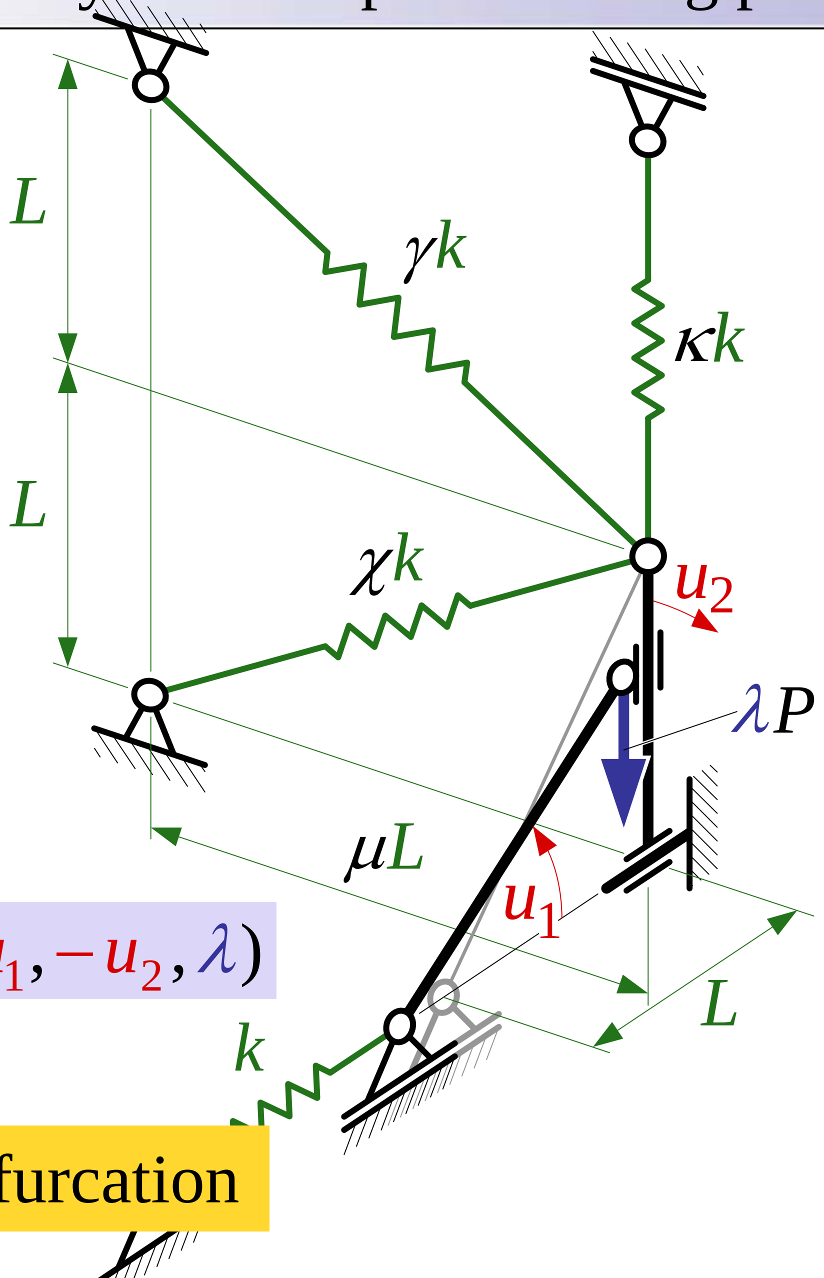


2 DOF

static, conservative

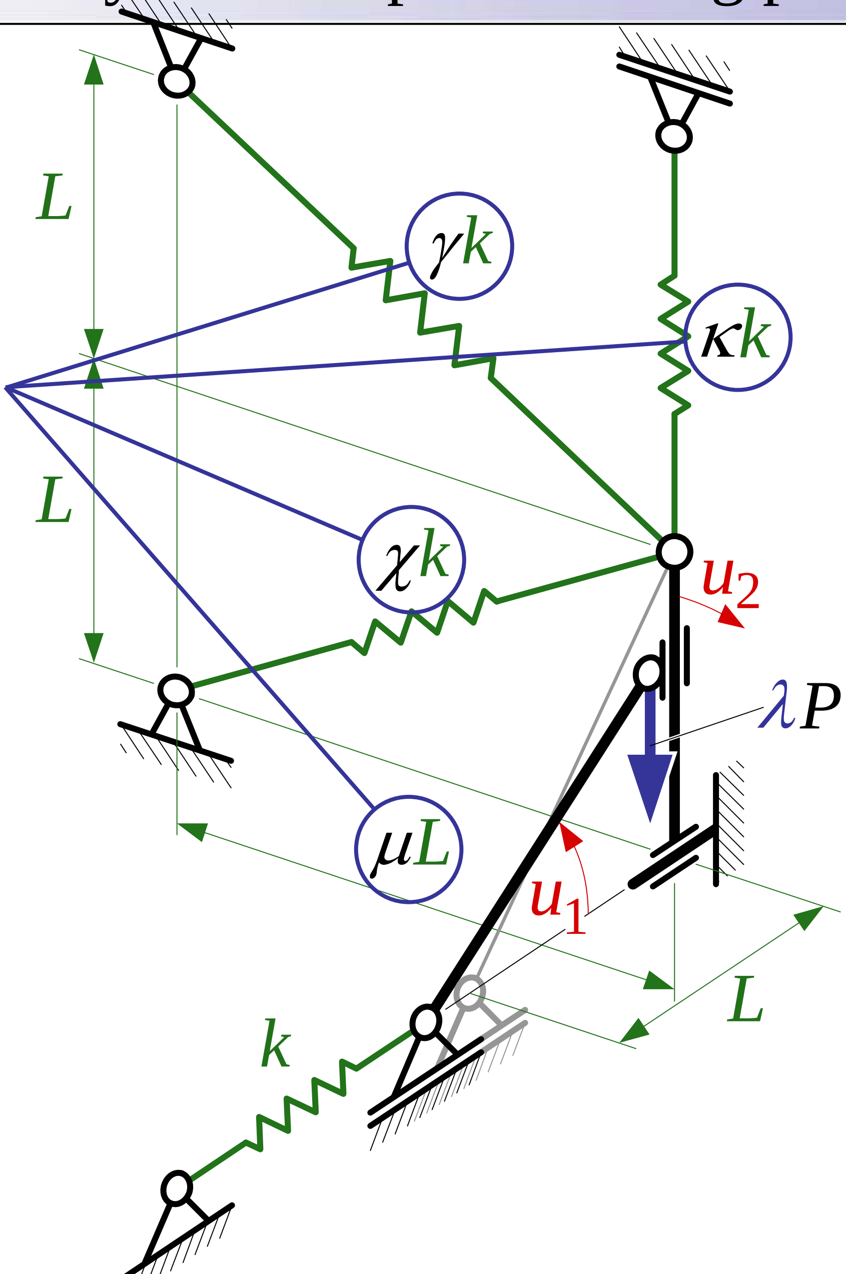
$$V(u_1, u_2, \lambda) \neq V(u_1, -u_2, \lambda)$$

$\downarrow$

non-symmetric bifurcation

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Example



design parameters

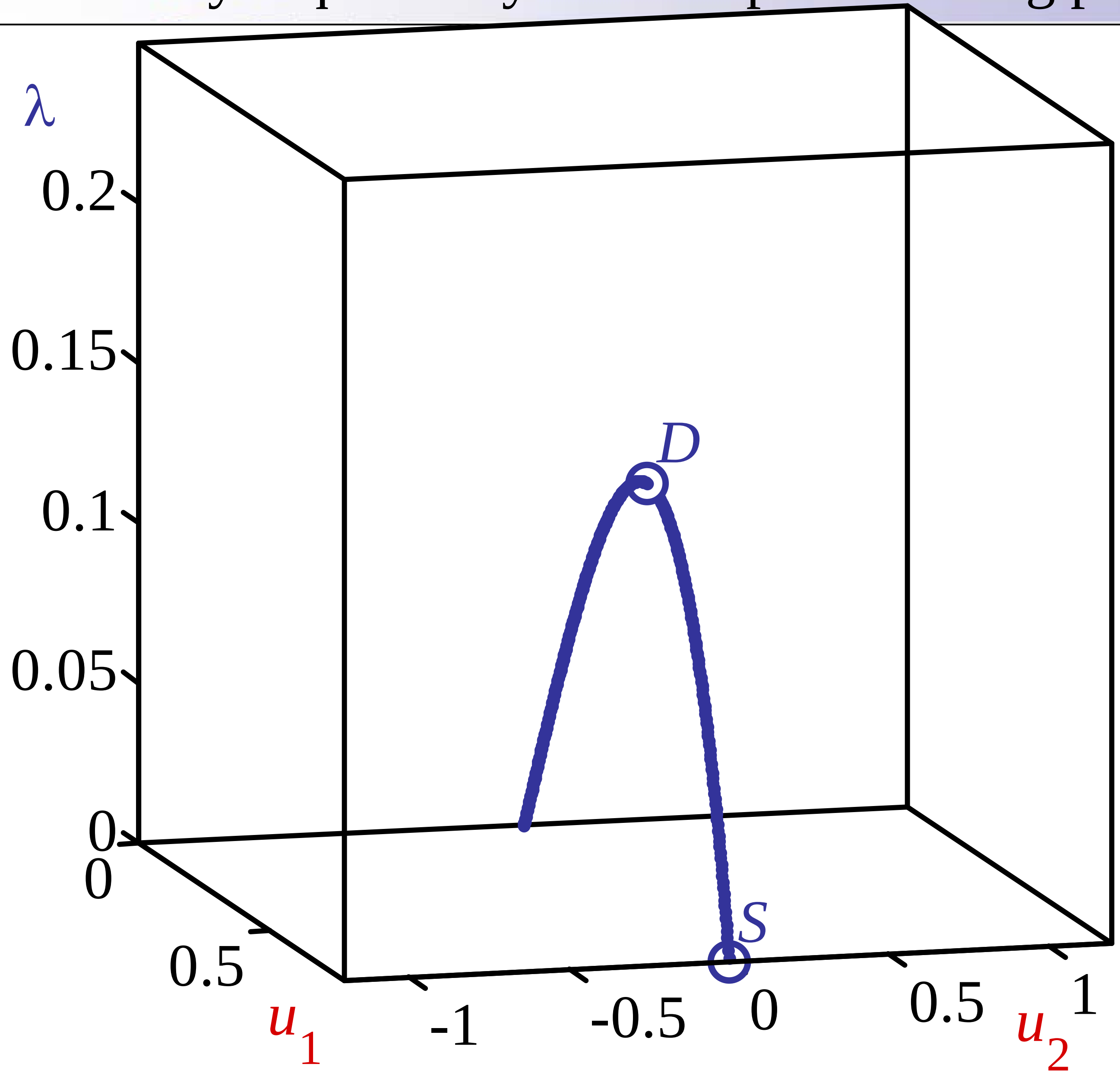
design parameters γ/χ and μ are chosen such that

$$\lambda_1 = 0 \ \wedge \ \lambda_3 = 0$$

but $\lambda_5 \neq 0$,

κ is modified

Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?



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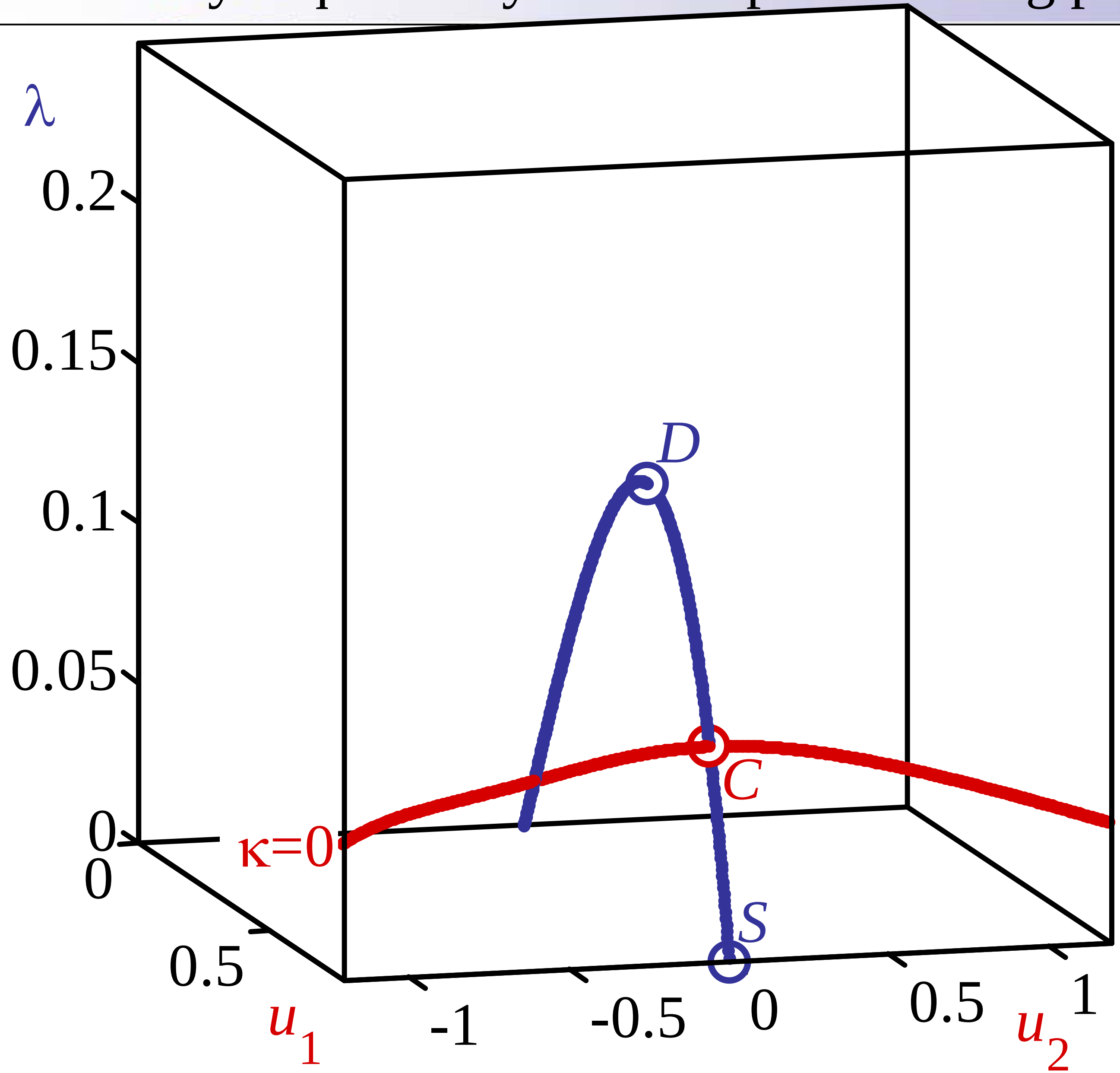

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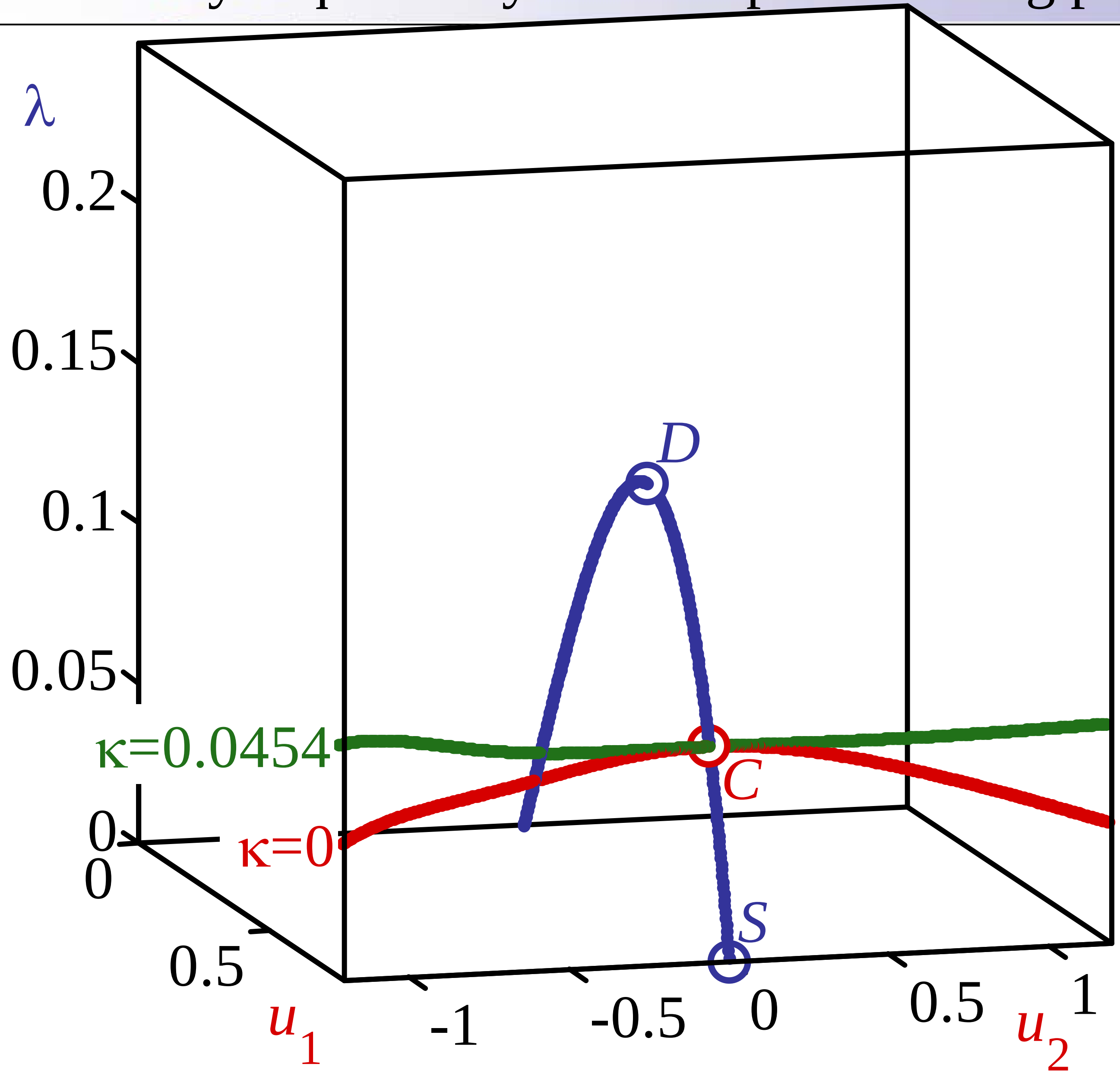

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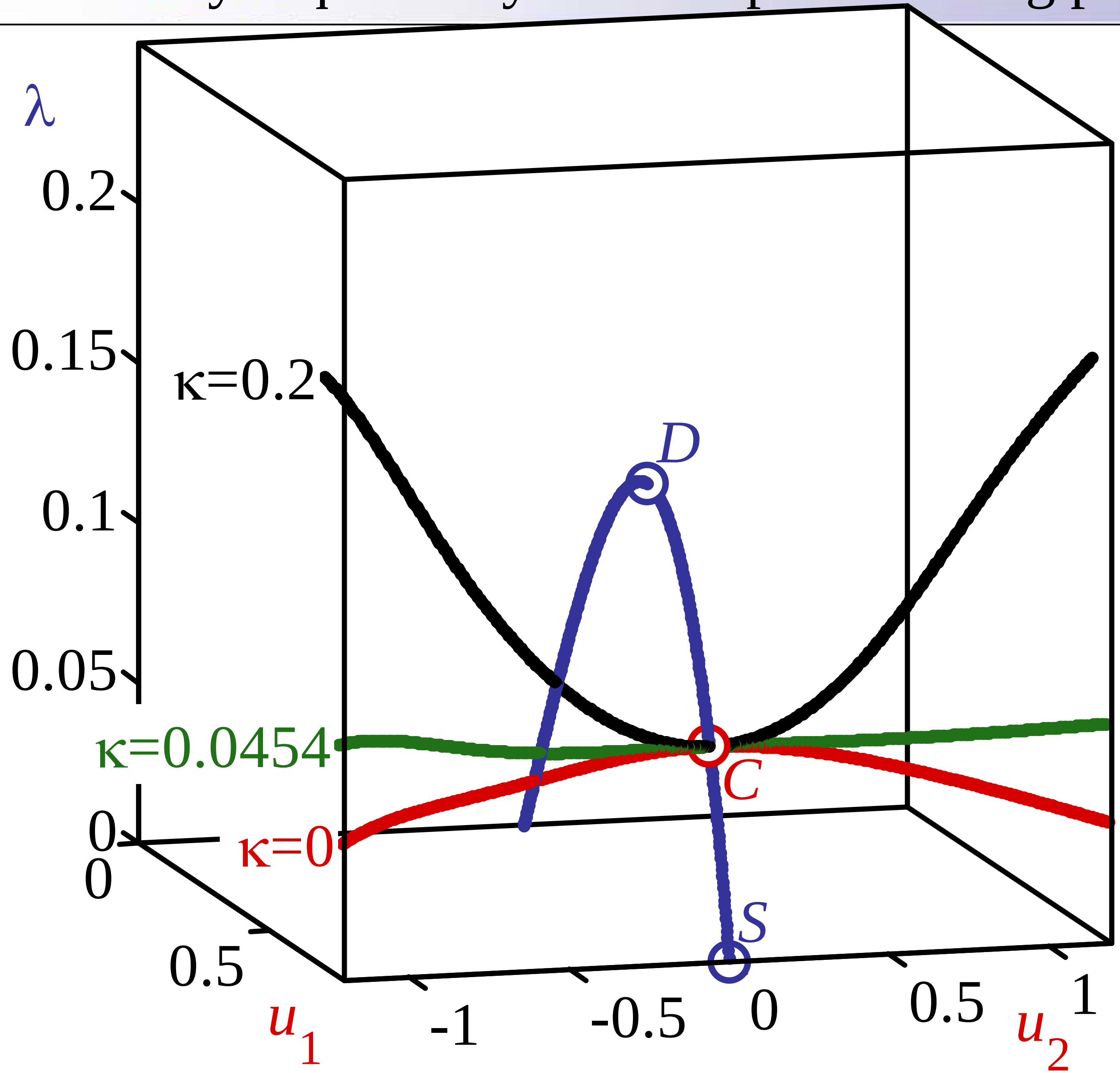

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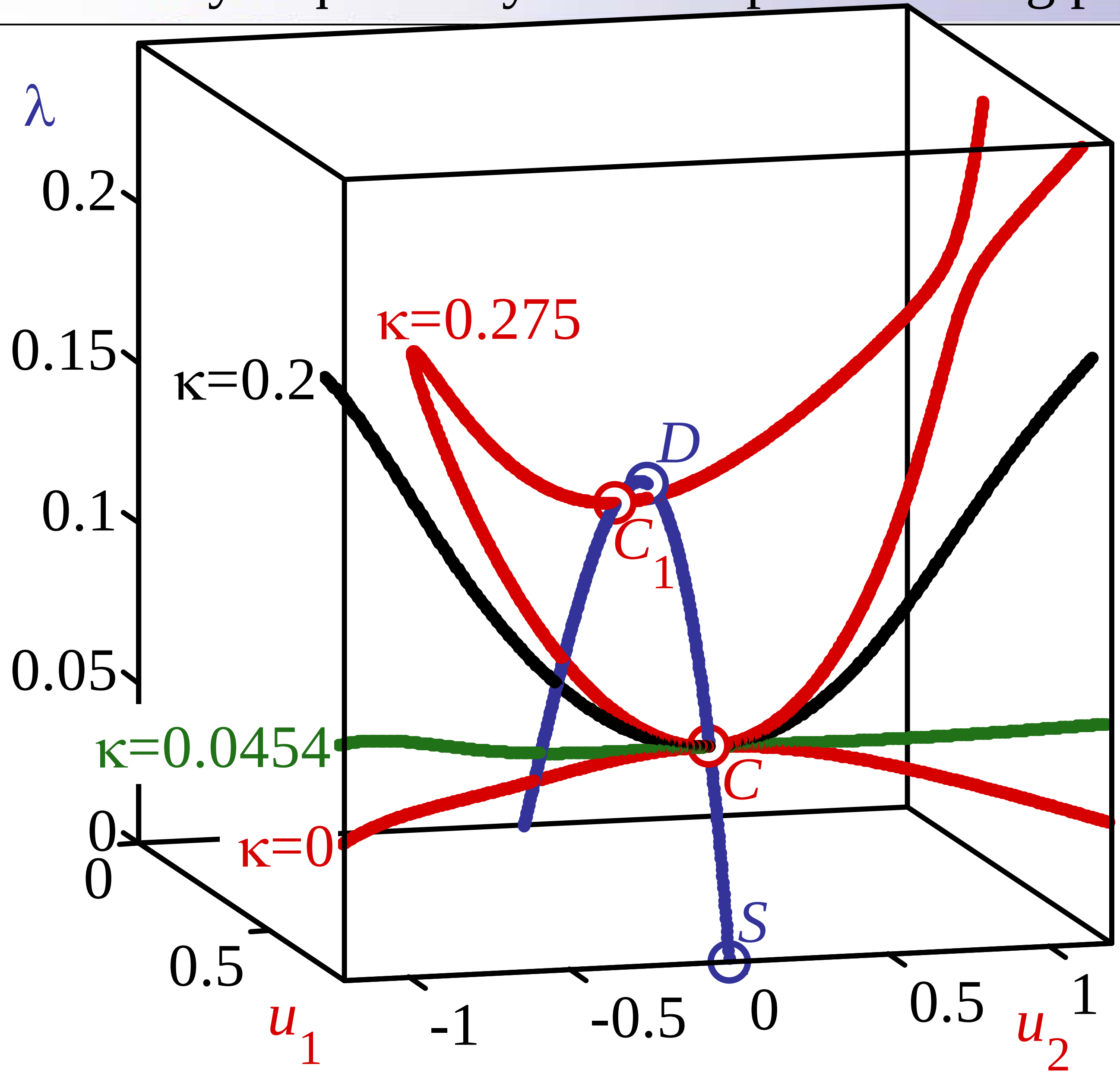

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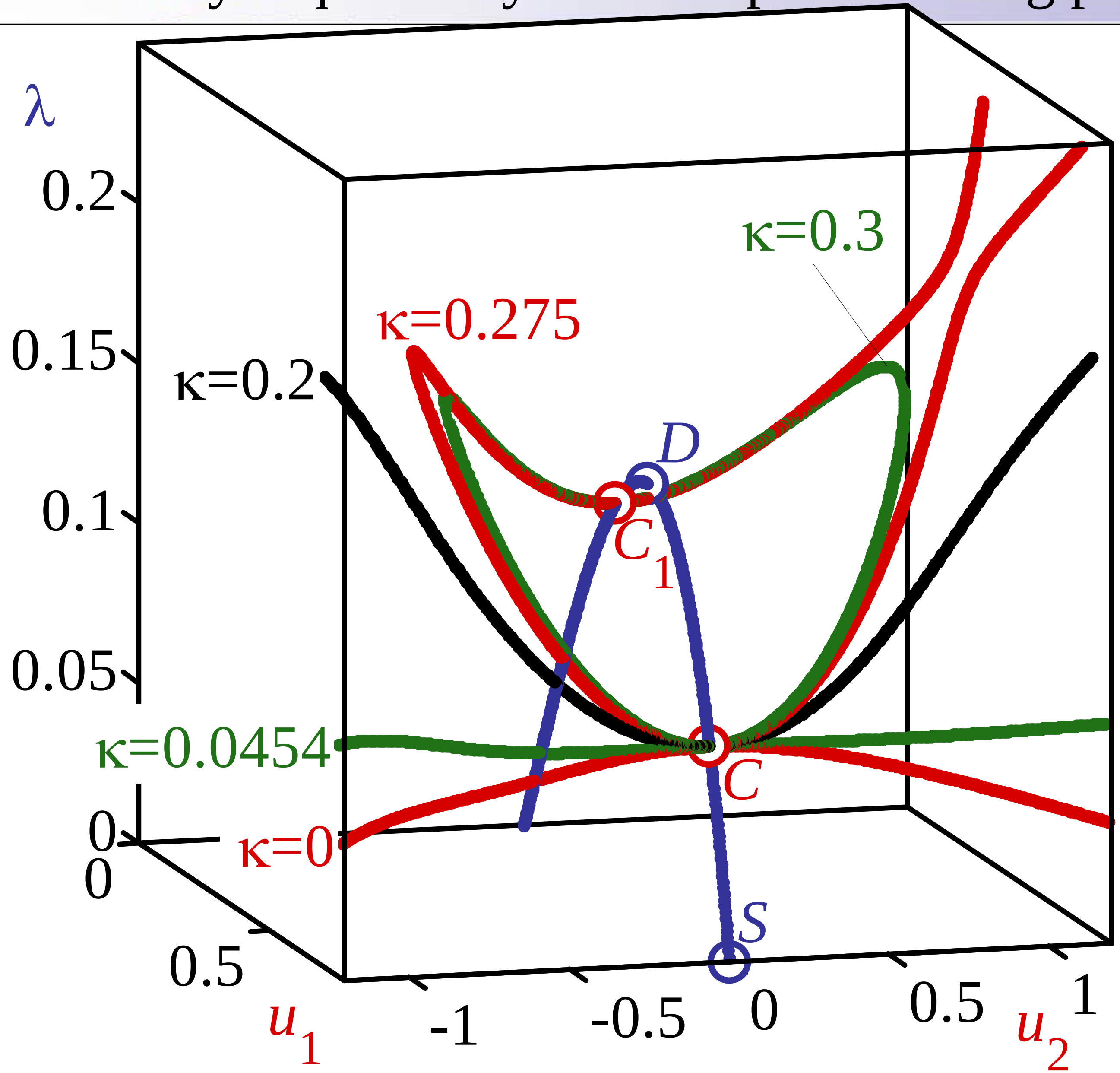

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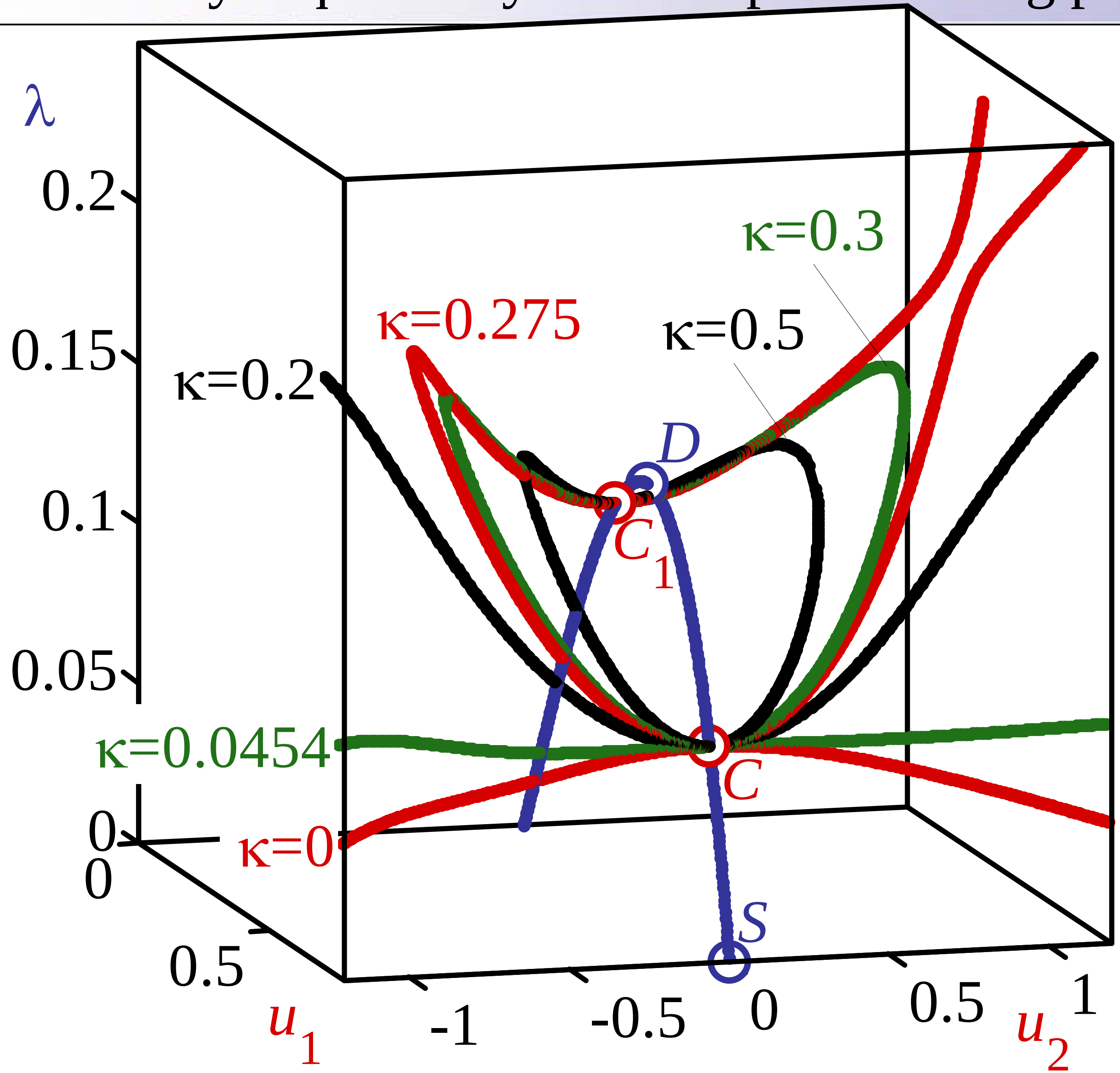

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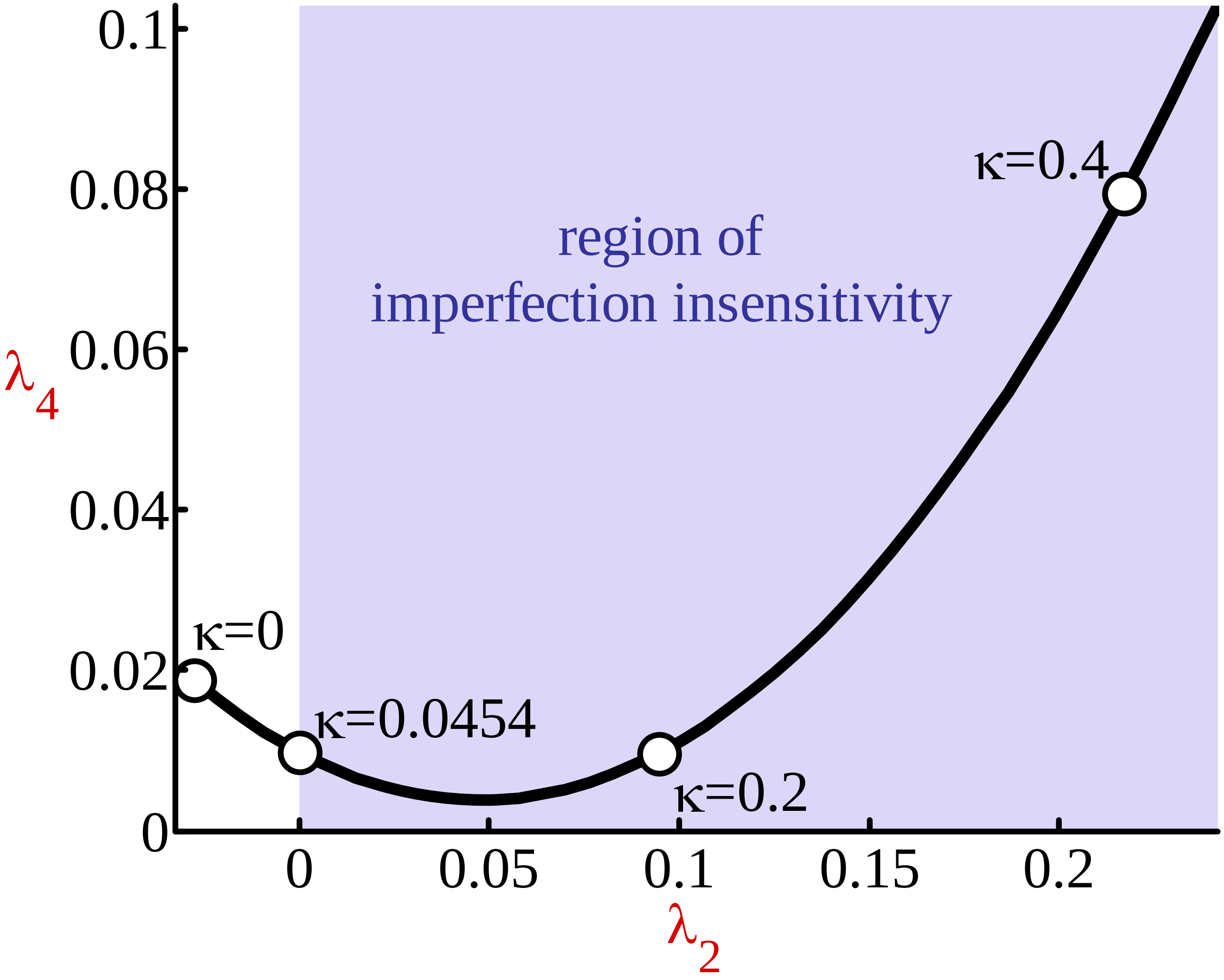


Does the conversion from imperfection sensitivity into insensitivity require a symmetric postbuckling path?

Conclusions

- Conversion into imperfection insensitivity requires $\lambda_1=0$, which holds automatically for symmetric bifurcation.
- Symmetric bifurcation is not necessary.
- Conversion is possible without change of the prebuckling behavior and without change of the buckling load.
- Increasing the stiffness
  - can lead to conversion into imperfection insensitivity
  - may result in qualitative changes of the secondary path

Is hilltop buckling necessarily imperfection sensitive?

Question III

Is hilltop buckling necessarily imperfection sensitive?

Is hilltop buckling necessarily imperfection sensitive?

Motivation

- divide and conquer -
  the category of hilltop buckling
- hilltop buckling as a starting point
  of sensitivity analysis
- Is hilltop buckling a “worst case“
  buckling scenario?

Is hilltop buckling necessarily imperfection sensitive?

Hilltop buckling

- coincidence of a bifurcation point and a snap-through point
- Assertion I: Hilltop buckling requires $\lambda_1=0$.
- Assertion II: Hilltop buckling is necessarily imperfection sensitive.

Fujii F, Noguchi H. Multiple hill-top branching. In *Proceedings of the 2nd International Conference on Structural Stability and Dynamics*, World Scientific, Singapore, 2002.

Ikeda K, Ohsaki M, Kanno Y. Imperfection sensitivity of hilltop branching points of sysytems with dihedral group symmetry. In *International Journal of Non-Linear Mechanics* 2005; **40**:755-774

Is hilltop buckling necessarily imperfection sensitive?

Proof

- path parameter ξ referring to the primary path
- at the snap-through point $\rightarrow \quad \lambda_{,\xi} = 0 \quad \wedge \quad \lambda_{,\xi\xi} < 0$ (local maximum of λ)
- coefficient a_1 occurs in some expressions $\boldsymbol{G}_{nC}$

$$a_1 = -\frac{1}{2}\frac{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\lambda\lambda} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\lambda} \cdot \boldsymbol{v}_1} = -\frac{1}{2\lambda_{,\xi}}\left(\frac{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi\xi} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1} - \frac{\lambda_{,\xi\xi}}{\lambda_{,\xi}}\right)$$

- $\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1 \neq 0$ is known from the consistently linearized eigenproblem

$$a_1 = \frac{1}{2}\frac{\lambda_{,\xi\xi}}{\lambda_{,\xi}^2} = -\infty \quad \ldots \quad \text{pole of } 2^{\text{nd}} \text{ order}$$

Is hilltop buckling necessarily imperfection sensitive?

Proof

- path parameter η referring to the secondary path
- $\lambda(\eta) = \lambda_C + \lambda_1 \eta + \lambda_2 \eta^2 + \ldots + \lambda_n \eta^n + \ldots$
- $\eta=0$ refers to the stability limit λ_C
- coefficient $a_1 = -\frac{1}{2\lambda_{,\eta}} \left(\frac{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta\eta} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta} \cdot \boldsymbol{v}_1} - \frac{\lambda_{,\eta\eta}}{\lambda_{,\eta}} \right) \Bigg|_{\eta=0}$

$$\lambda_{,\eta}\Big|_{\eta=0} = \lambda_1, \ \lambda_{,\eta\eta}\Big|_{\eta=0} = 2\lambda_2$$

$$\tilde{\boldsymbol{K}}_{T,\eta} / \lambda_{,\eta} = \tilde{\boldsymbol{K}}_{T,\xi} / \lambda_{,\xi}$$

$$a_1 = -\frac{1}{2\lambda_1^2} \left(\frac{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1} \lambda_{,\xi} - 2\lambda_2 \right)$$

Is hilltop buckling necessarily imperfection sensitive?

Proof

$$a_1 = -\frac{1}{2\lambda_1^2}\left(\frac{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1}\lambda_{,\xi} - 2\lambda_2\right)$$

no hilltop buckling

$$\lambda_1 = 0,\ \lambda_{,\xi} \neq 0$$

$$\downarrow$$

$$a_1 \neq 0$$

$$\downarrow$$

$$\frac{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1}\lambda_{,\xi} - 2\lambda_2 = 0$$

Is hilltop buckling necessarily imperfection sensitive?

Proof

$$a_1 = -\frac{1}{2\lambda_1^2}\left(\frac{\boldsymbol{v}_1^T\cdot\tilde{\boldsymbol{K}}_{T,\eta\eta}\big|_{\eta=0}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1^T\cdot\tilde{\boldsymbol{K}}_{T,\xi}\cdot\boldsymbol{v}_1}\lambda_{,\xi}-2\lambda_2\right)$$

no hilltop buckling

$$\lambda_1 = 0,\ \lambda_{,\xi} \neq 0$$

$$\downarrow$$

$$a_1 \neq 0$$

$$\downarrow$$

$$\frac{\boldsymbol{v}_1^T\cdot\tilde{\boldsymbol{K}}_{T,\eta\eta}\big|_{\eta=0}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1^T\cdot\tilde{\boldsymbol{K}}_{T,\xi}\cdot\boldsymbol{v}_1}\lambda_{,\xi}-2\lambda_2=0$$

hilltop buckling

$$\lambda_{,\xi} = 0,\ \boldsymbol{v}_1^T\cdot\tilde{\boldsymbol{K}}_{T,\xi}\cdot\boldsymbol{v}_1 \neq 0$$

$$\downarrow$$

$$a_1 = \frac{\lambda_2}{\lambda_1^2}$$

$$\downarrow\ \boxed{a_1 = -\infty}$$

$$\lambda_1 = 0,\ -\infty < \lambda_2 < 0$$

q.e.d.

Is hilltop buckling necessarily imperfection sensitive?

Proof

$$a_1 = -\frac{1}{2\lambda_1^2}\left(\frac{\mathbf{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta\eta}\big|_{\eta=0} \cdot \boldsymbol{v}_1}{\mathbf{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1}\lambda_{,\xi} - 2\lambda_2\right)$$

no hilltop buckling

$$\lambda_1 = 0,\ \lambda_{,\xi} \neq 0$$

$$\downarrow$$

$$a_1 \neq 0$$

$$\downarrow$$

$$\frac{\mathbf{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\eta\eta}\big|_{\eta=0} \cdot \boldsymbol{v}_1}{\mathbf{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1}\lambda_{,\xi} - 2\lambda_2 = 0$$

hilltop buckling

$$\lambda_{,\xi} = 0,\ \mathbf{v}_1^T \cdot \tilde{\boldsymbol{K}}_{T,\xi} \cdot \boldsymbol{v}_1 \neq 0$$

$$\downarrow$$

$$a_1 = \frac{\lambda_2}{\lambda_1^2}$$

$$\downarrow \quad \boxed{a_1 = -\infty}$$

$$\lambda_1 = 0,\ -\infty < \lambda_2 < 0$$

Hilltop buckling is necessarily imperfection sensitive.

Is hilltop buckling necessarily imperfection sensitive?

Theory (cont.)

- design parameter κ
- series expansion

$$\boldsymbol{G}(\boldsymbol{v},\eta) = \boldsymbol{G}_{0C} + \boldsymbol{G}_{1C}\,\eta + \boldsymbol{G}_{2C}\,\eta^2 + \ldots = \mathbf{0}$$

yields parameter-dependent equation

$$\lambda_4(\kappa) = a_1(\kappa)\,\lambda_2^2(\kappa) + b_2(\kappa)\,\lambda_2(\kappa) + d_3(\kappa)$$

$$\Big\downarrow \quad \text{“solution”}$$

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2\,a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2\,a_1(\kappa)}$$

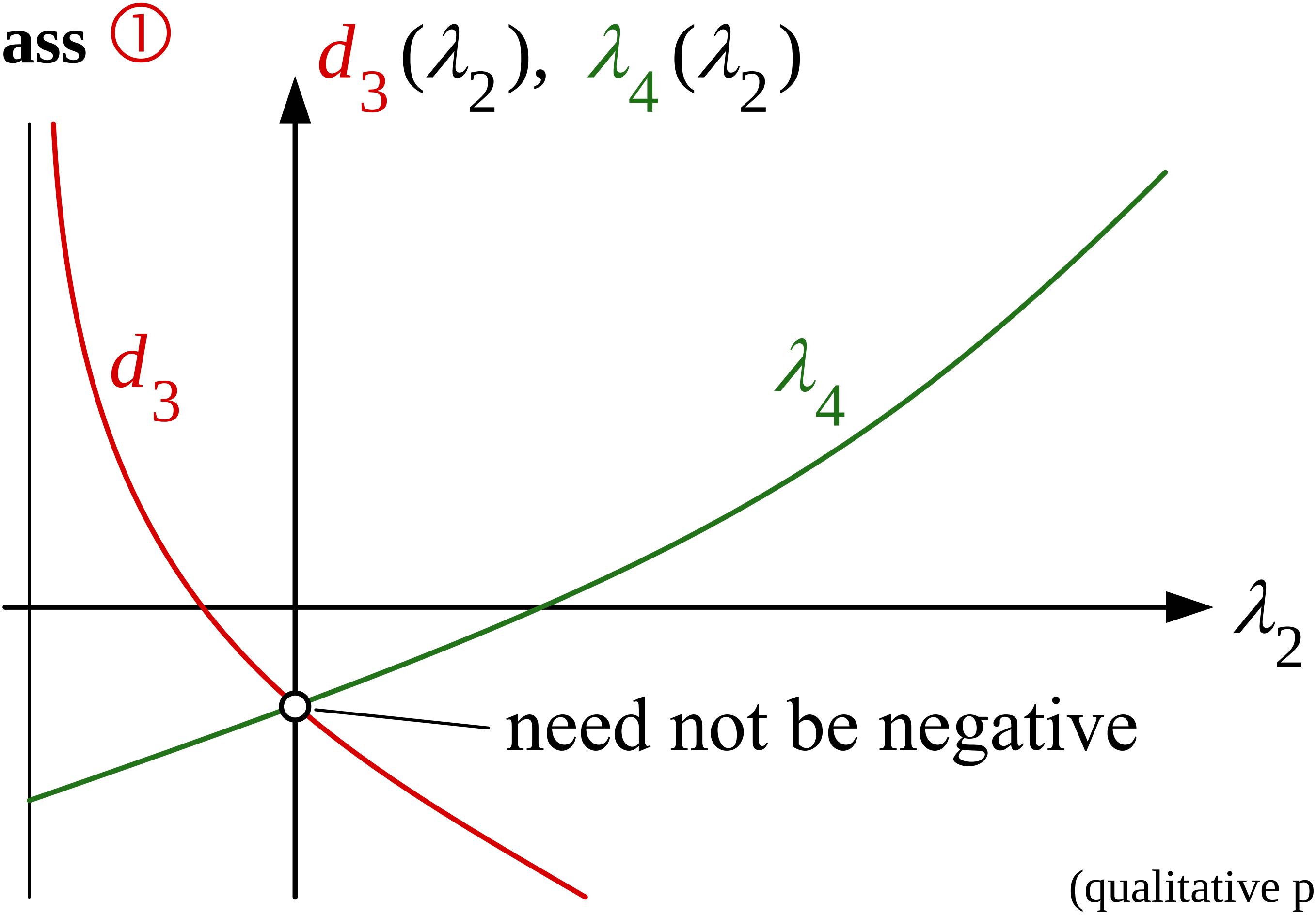
- allows differentiation between two characteristic classes of the relation $\lambda_4 = \lambda_4(\lambda_2(\kappa))$

imws

Is hilltop buckling necessarily imperfection sensitive?

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2\,a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2\,a_1(\kappa)}$$

Class ①



need not be negative

(qualitative plot)

Is hilltop buckling necessarily imperfection sensitive?

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2\,a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2\,a_1(\kappa)}$$

Class ①

- generally, $b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa)) > 0$
  → double roots are not possible
- at the hilltop buckling point

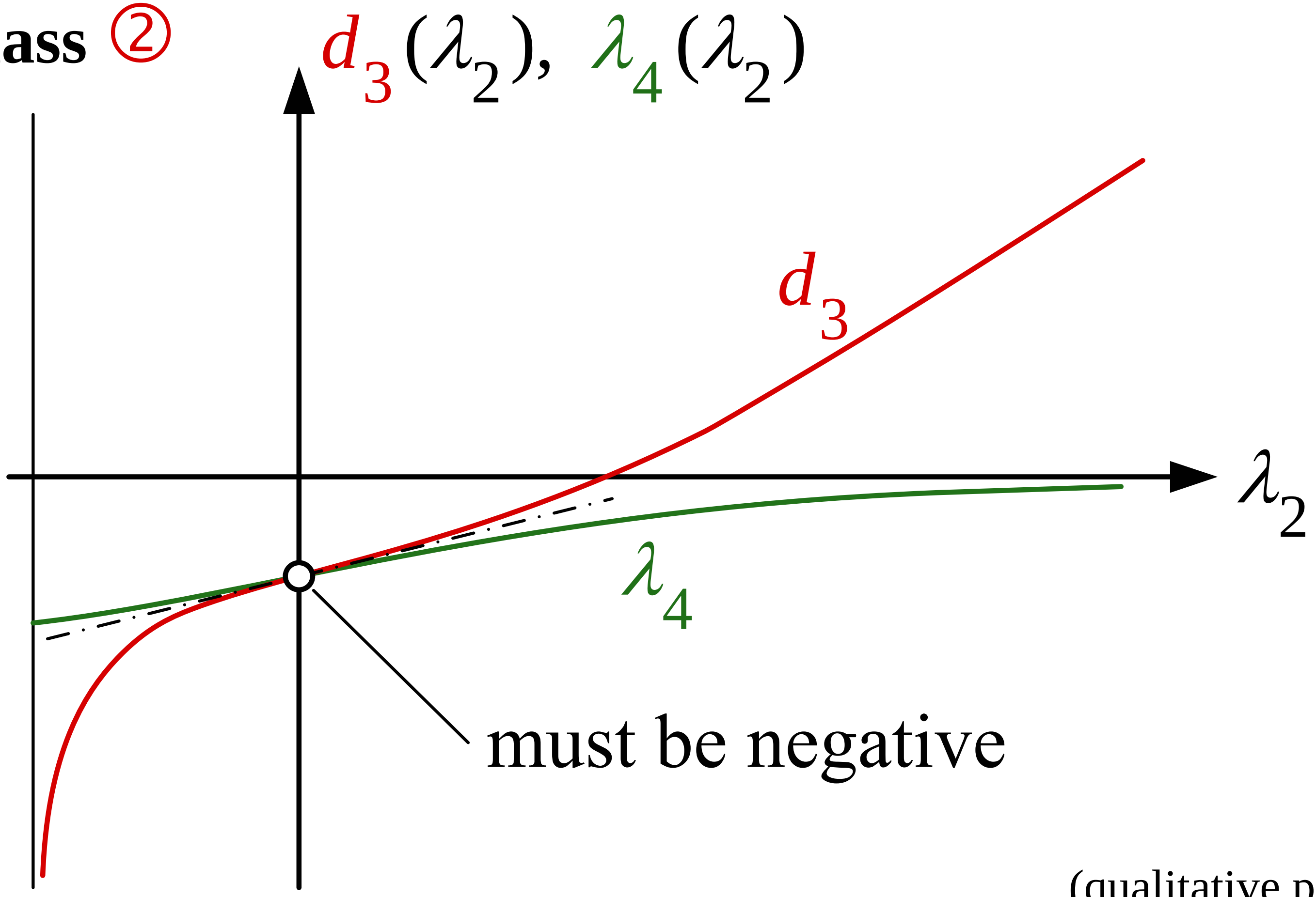
$$\left.\begin{array}{l} b_2 = +\infty \;\ldots\; \text{pole of } 1^{\text{st}} \text{ order} \\ a_1 = -\infty \;\ldots\; \text{pole of } 2^{\text{nd}} \text{ order} \end{array}\right. \longrightarrow \frac{b_2}{a_1} = 0$$

- $\lambda_2=0$ is not possible because it would be a double root
- $\lambda_2 < 0 \quad \rightarrow \quad \lambda_2(\kappa) = -\sqrt{(\lambda_4(\kappa) - d_3(\kappa)) / a_1(\kappa)}$

Is hilltop buckling necessarily imperfection sensitive?

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2\,a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2\,a_1(\kappa)}$$

Class ②



(qualitative plot)

imws

Is hilltop buckling necessarily imperfection sensitive?

$$\lambda_2(\kappa)_{1,2} = -\frac{b_2(\kappa)}{2\,a_1(\kappa)} \pm \frac{\sqrt{b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))}}{2\,a_1(\kappa)}$$

Class ②

- $b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa)) = 0 \quad \forall \kappa \in U \;\rightarrow$
  double root $\lambda_2(\kappa)_{1,2} = -\dfrac{b_2(\kappa)}{2\,a_1(\kappa)} = -\dfrac{2\,(d_3(\kappa) - \lambda_4(\kappa))}{b_2(\kappa)}$
- $2\,a_1(\kappa)\,\lambda_2(\kappa)_{1,2} + b_2(\kappa) = 0$
  $\lambda_2 = 0$ corresponds to $a_1 = 0 \rightarrow b_2 = 0 \;\wedge\; d_3 - \lambda_4 = 0$
- but hilltop buckling requires $a_1 = -\infty,\ \lambda_2 < 0$

$\downarrow$

$b_2 = -\infty$ … pole of 2^{nd} order

$d_3 - \lambda_4 = -\infty$ … pole of 2^{nd} order

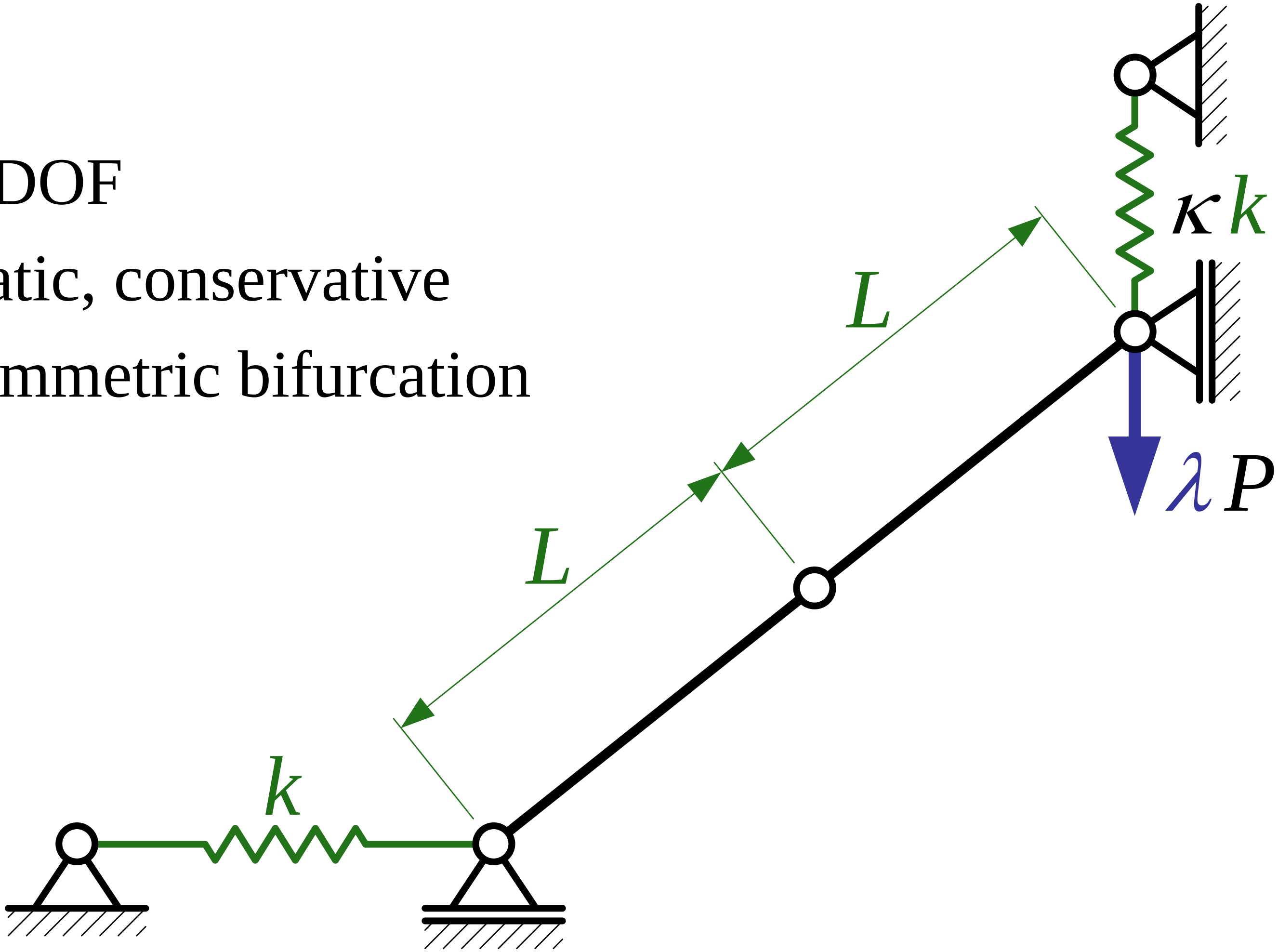
Is hilltop buckling necessarily imperfection sensitive?

Example of class ①

2 DOF

static, conservative

symmetric bifurcation



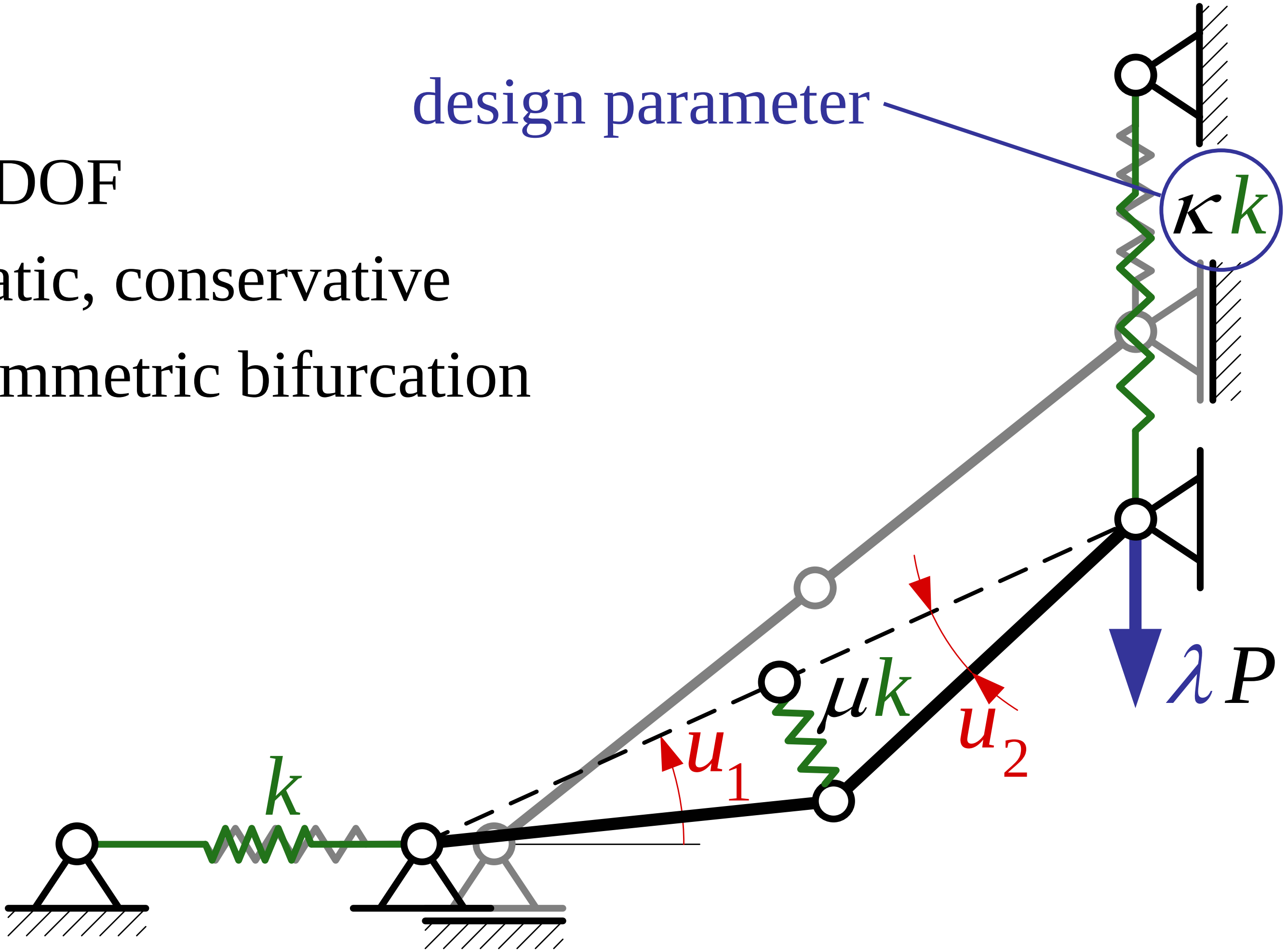
Is hilltop buckling necessarily imperfection sensitive?

Example of class ①

2 DOF

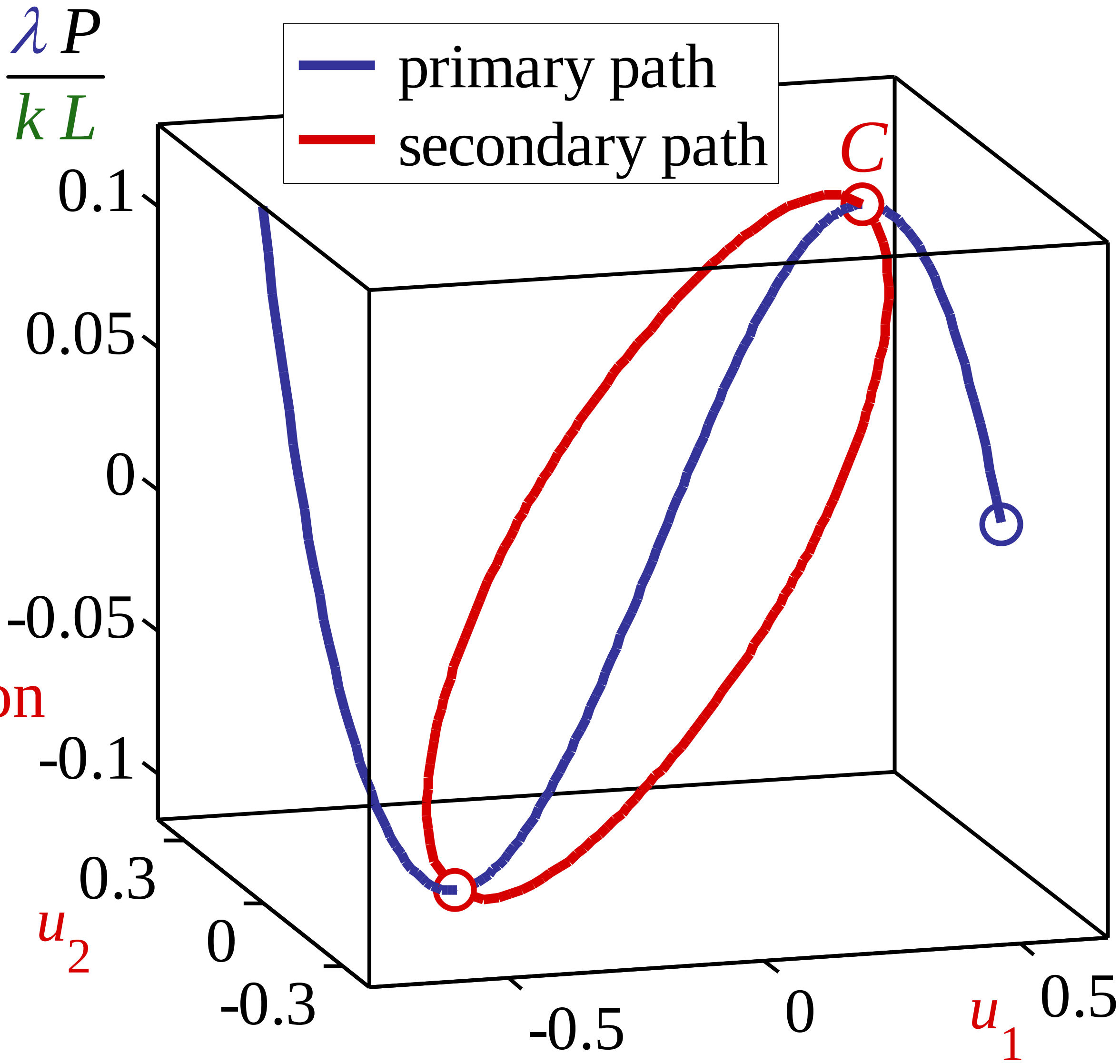
static, conservative

symmetric bifurcation



Is hilltop buckling necessarily imperfection sensitive?

Load-displacement path - hilltop buckling



$\kappa = 0$

$\lambda_2 < 0$

imperfection sensitive

Is hilltop buckling necessarily imperfection sensitive?

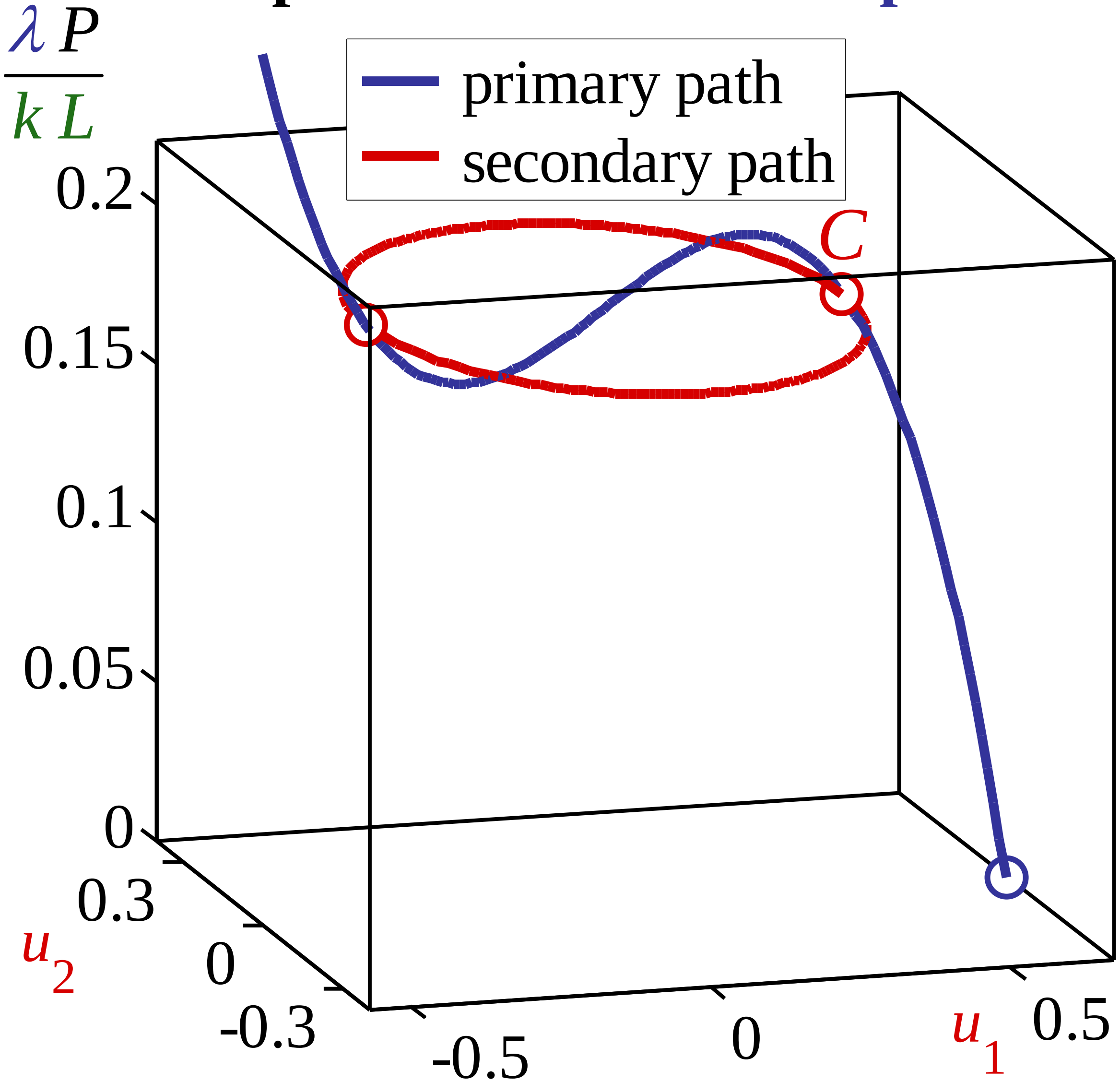
Load-displacement path - zero-stiffness postbuckling

$$\kappa = \frac{\mu}{4}$$

$$\lambda_2 = 0$$

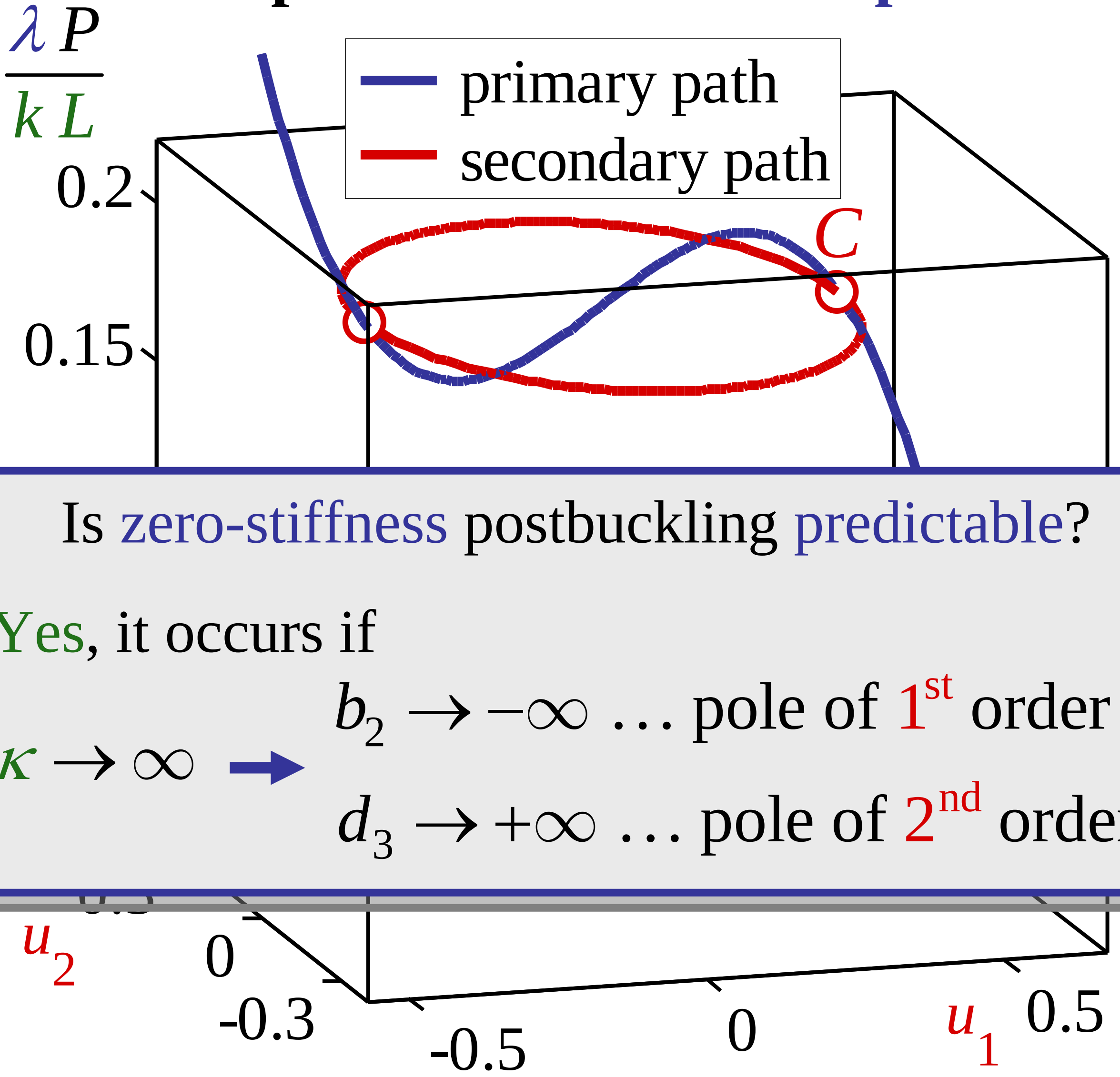
$$\lambda_4 = 0$$

$$\vdots$$



Is hilltop buckling necessarily imperfection sensitive?

Load-displacement path - zero-stiffness postbuckling



$$\kappa = \frac{\mu}{4}$$

$$\lambda_2 = 0$$

$$\lambda_4 = 0$$

$$\vdots$$

Is zero-stiffness postbuckling predictable?

Yes, it occurs if

$$\kappa \to \infty \quad \Rightarrow \quad \begin{array}{l} b_2 \to -\infty \ \ldots \text{ pole of 1st order} \\ d_3 \to +\infty \ \ldots \text{ pole of 2nd order} \end{array}$$

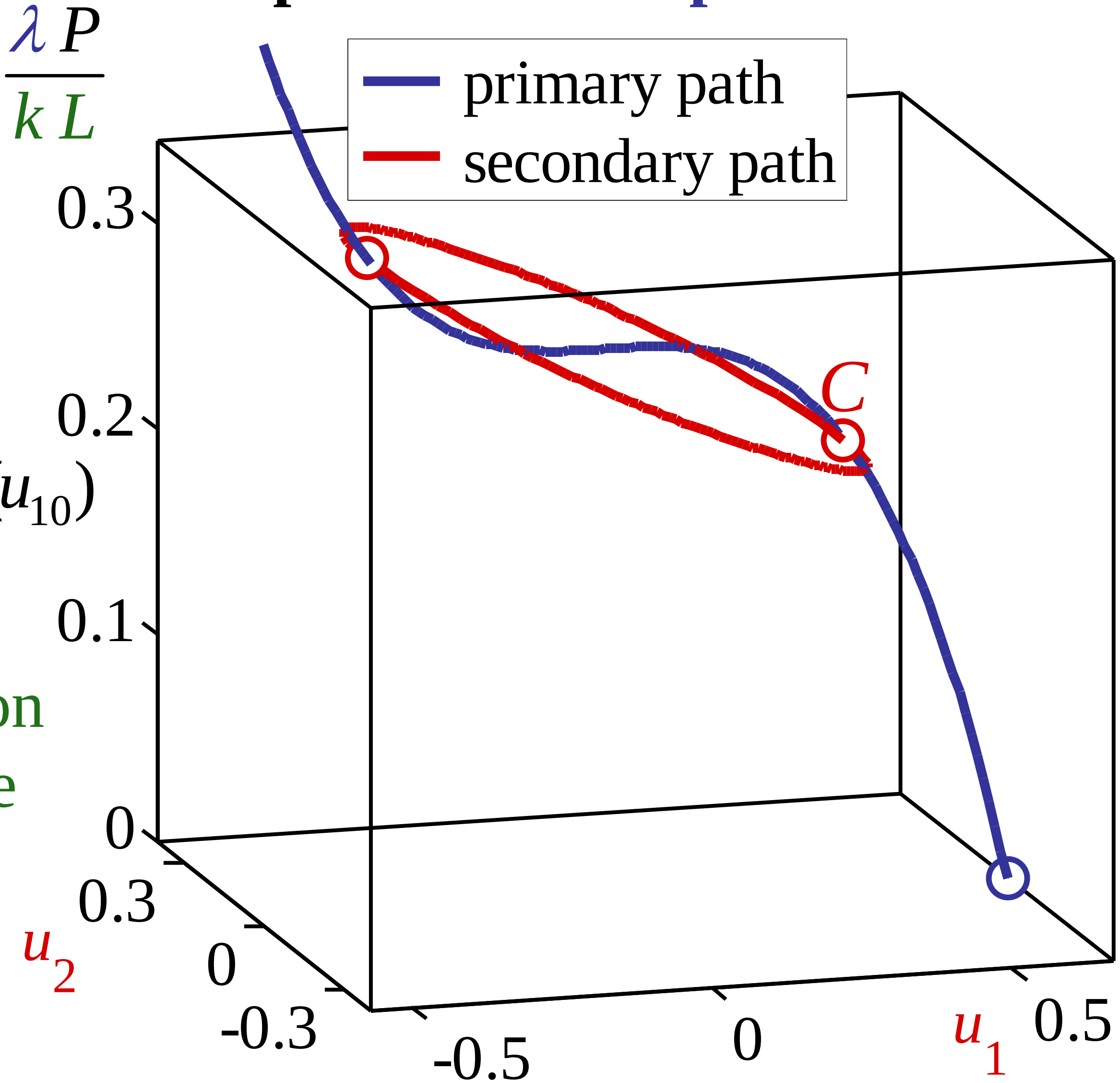
Is hilltop buckling necessarily imperfection sensitive?

Load-displacement path - saddle point

$\kappa = 1 - \cos(u_{10})$

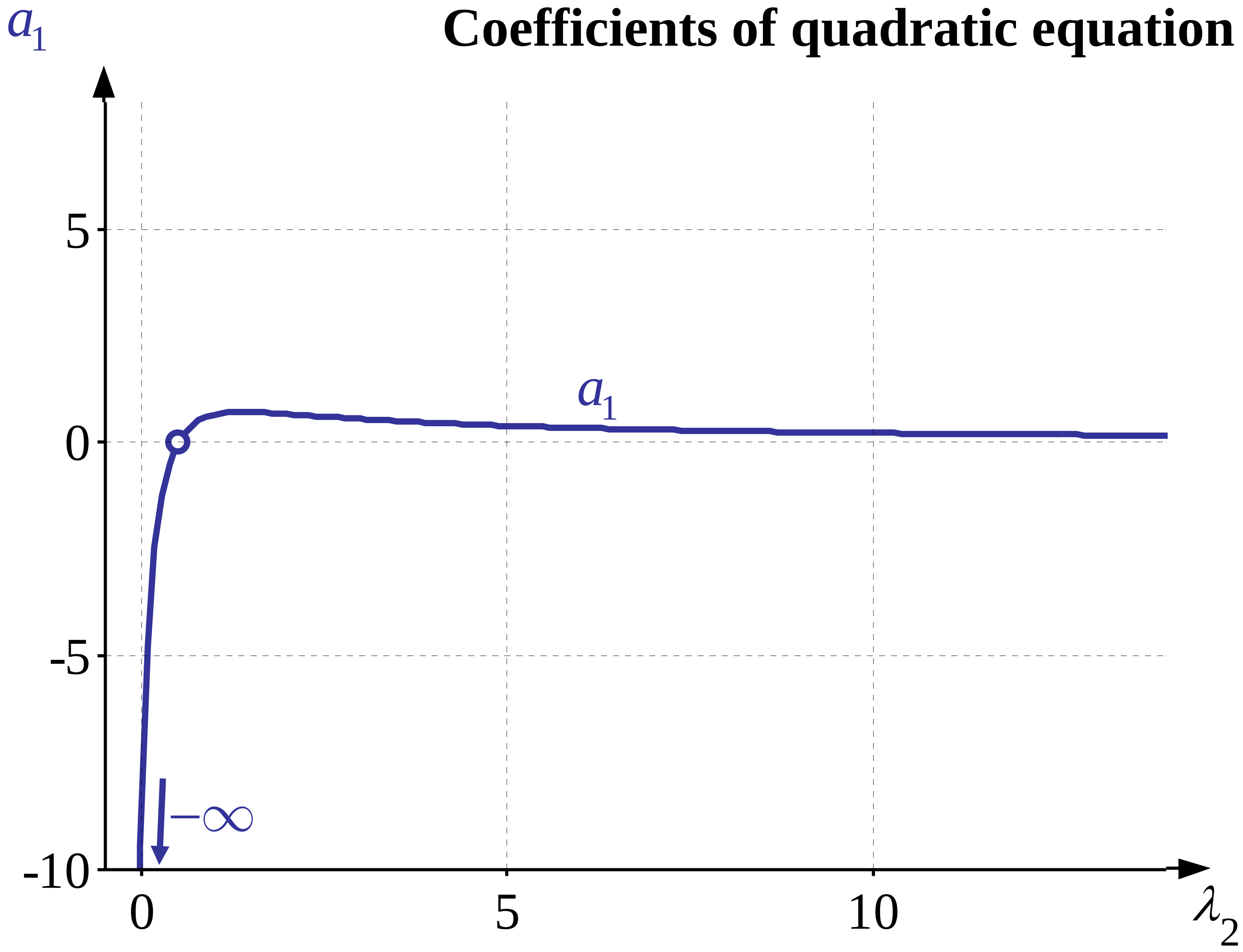
$\lambda_2 > 0$

imperfection insensitive



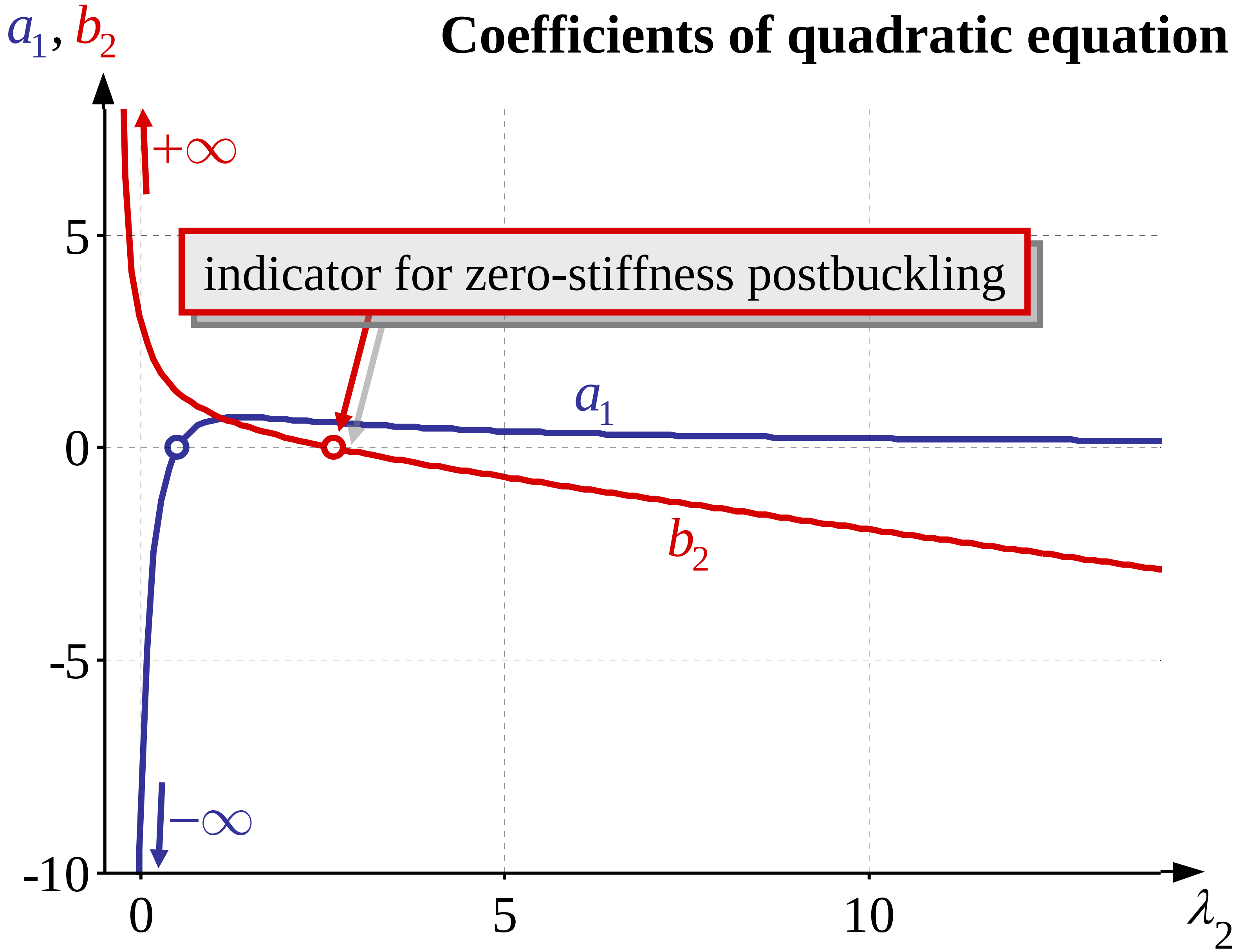
Is hilltop buckling necessarily imperfection sensitive?

Coefficients of quadratic equation



Is hilltop buckling necessarily imperfection sensitive?

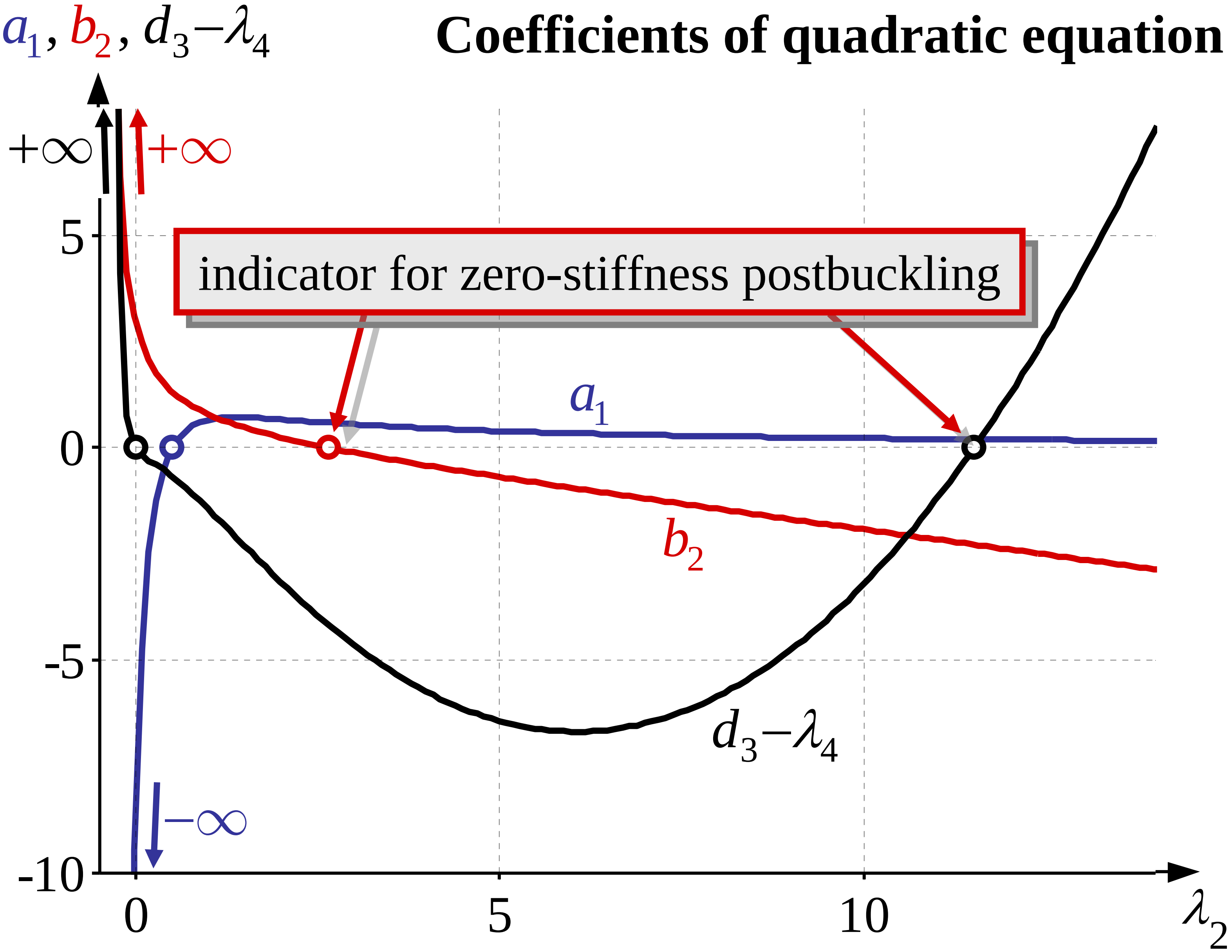
Coefficients of quadratic equation



indicator for zero-stiffness postbuckling

Is hilltop buckling necessarily imperfection sensitive?

Coefficients of quadratic equation

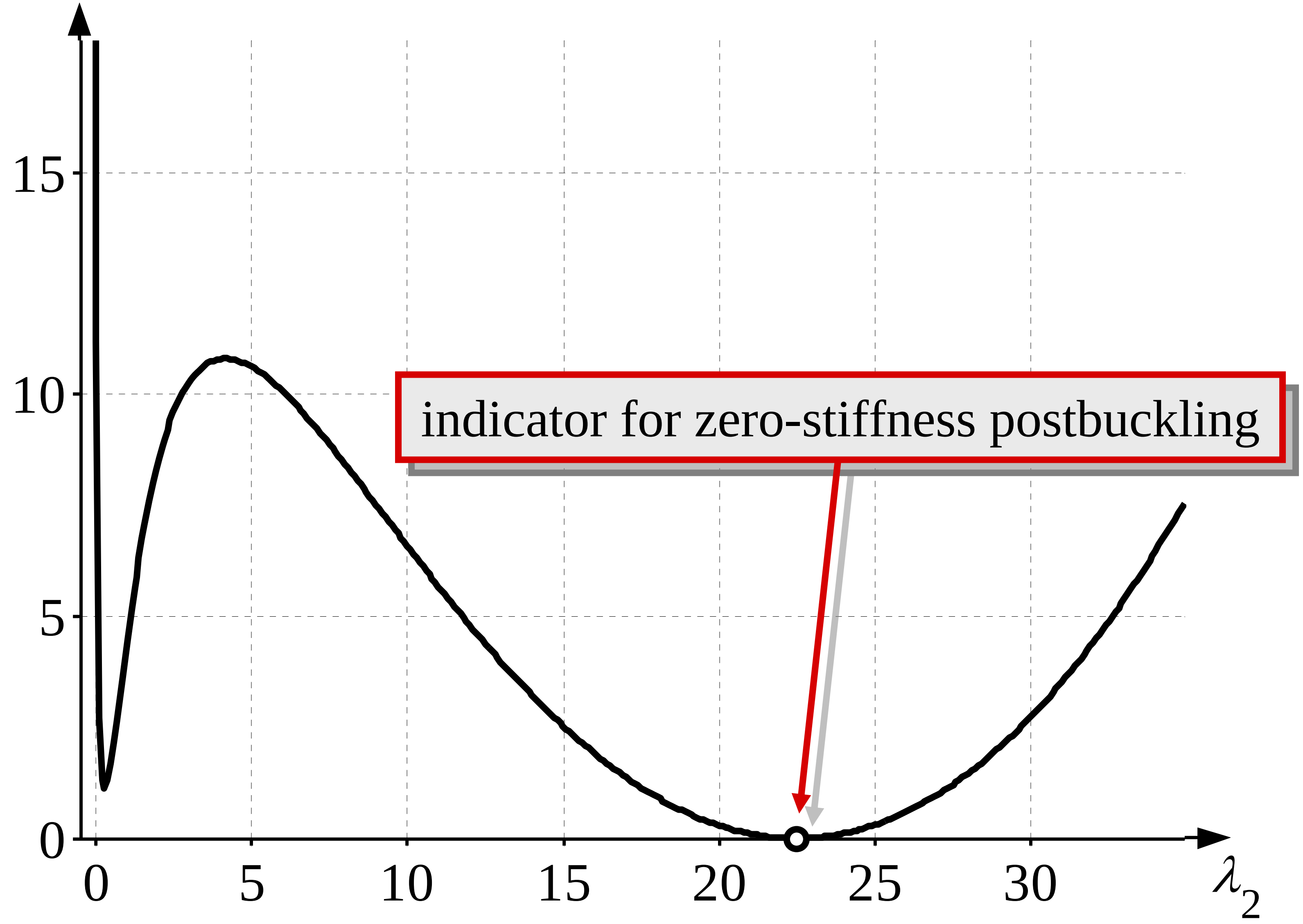


indicator for zero-stiffness postbuckling

Is hilltop buckling necessarily imperfection sensitive?

$b_2^2(\kappa) - 4\,a_1(\kappa)(d_3(\kappa) - \lambda_4(\kappa))$

Discriminant



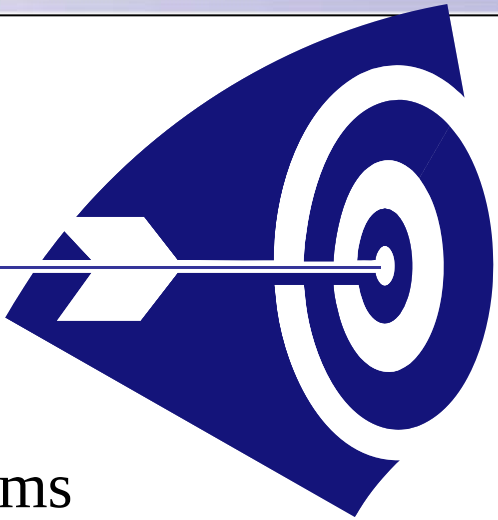
indicator for zero-stiffness postbuckling

Is hilltop buckling necessarily imperfection sensitive?

Conclusions

- two characteristic classes of hilltop buckling
- different behavior for $\kappa \rightarrow +\infty$
- Hilltop buckling is necessarily

  imperfection sensitive.



Further research

- FEM analysis of multiple-DOF systems
- identification of qualitative properties pivotal for the conversion from imperfection sensitivity into insensitivity
- investigation of the reasons for the initial postbuckling behavior according to class ① and ②, respectively
- proof of the *a priori* predictability of zero-stiffness postbuckling behavior