

# SHELLING THE COSET POSET

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It is shown that the coset lattice of a finite group has shellable order complex if and only if the group is complemented. It is also shown that the order complex is Cohen-Macaulay under the same circumstances. The group theoretical tools used are relatively elementary, and avoid the classification of finite simple groups and of minimal finite simple groups.

## BIOGRAPHICAL SKETCH

Russ Woodrooffe was born July 3rd, 1975. He used to make fun of his math teachers for majoring in math, but came around during his time as an undergraduate at the University of Michigan, between 1993 and 1997, where he dual-majored in mathematics and computer science. Also during this time, he worked for Eidelman Associates, where he programmed and helped design the WinDraft document assembly engine. The summer of 1996 he did an REU project on a variant of the Travelling Salesperson Problem with Alexander Barvinok. He studied abroad at the Budapest Semesters program in the fall of 1997, before coming to Cornell in the fall of 1998. He has been a graduate student at Cornell from 1998 until 2005.

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I would not be in math if it were not for Tom Storer, who first showed me that math could be beautiful and satisfying, and Jack Meiland, who made me think about what I wanted out of my time at the university. Alexander Barvinok supervised my first dip into research, for which I am grateful. I've been interested in combinatorics for as long as I can remember, but thanks to Tom Hales for awakening my interest in group theory in his algebra course at Michigan.

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# Chapter 1

## Introduction

We start by recalling the definition of a shelling. All posets, lattices, simplicial complexes, and groups in this paper are finite.

**Definition 1.** If  $\Delta$  is a simplicial complex, then a *shelling* of  $\Delta$  is an ordering  $F_1, \dots, F_n$  of the facets (maximal faces) of  $\Delta$  such that  $F_k \cap (\cup_{i=0}^{k-1} F_i)$  is a nonempty union of facets of  $F_k$ . If  $\Delta$  has a shelling, we say it is *shellable*.

We will use this definition in the context of a poset  $P$  by recalling the *order complex*  $|P|$  to be the simplicial complex with vertex set  $P$  and faces chains in  $P$ . We say that a poset  $P$  is shellable if  $|P|$  is. Recall also that  $P$  is *graded* if all its maximal chains have the same length.

The idea of a shelling (and the property of shellability) were first formally introduced by Bruggesser and Mani in [8], though similar ideas had been assumed implicitly since the beginning of the 20th century. See Chapter 8 of [22] for a development of some of the history and basic results on shellability. Since its introduction, it has been studied extensively by combinatorialists. Particularly, in the 1980's and 90's Björner and Wachs wrote several papers ([3, 4, 5, 6]) developing the theory of shellability for posets. Of particular importance to those interested in group theory are [5] and [6], as they extend the older definition of shellability (which only applied to graded posets) to apply to any poset. This extension makes Theorem 3 much more interesting!

We henceforth assume that a reader has seen the basic definitions and results of, say, [5], although we try to state clearly what we are using.

Recall that the subgroup lattice (denoted  $L(G)$ ) is the lattice of all subgroups of a group  $G$ . Shellings of subgroup lattices have been of interest for quite some time. In fact, one of the main results of Björner's first paper on shelling posets ([3]) was to show that supersolvable groups have shellable subgroup lattices. (Recall a supersolvable group is a group having chief series with every factor of prime order). As mentioned before, at that time, shellability was a property that applied only to graded posets. Under this definition, Björner had the shellable subgroup lattices completely characterized, if we recall the following theorem of Iwasawa:

**Theorem 2.** (Iwasawa [12]) *Let  $G$  be a finite group. Then  $L(G)$  is graded if and only if  $G$  is supersolvable.*

Of course, when Björner and Wachs updated the definition of a shelling to allow non-graded posets in [5, 6], shellable subgroup lattices were no longer characterized. This gap was soon filled by John Shareshian:

**Theorem 3.** (Shareshian [17]) *Let  $G$  be a finite group. Then the subgroup lattice  $L(G)$  is shellable if and only if  $G$  is solvable.*

A nice summary article on shellability and group theory was written by Volkmar Welker in [21]. This article is now somewhat out of date, and it has some errors, but it is very useful as an overview of the topic. The reader should be warned, however, that at the time it was written shellability was still considered to apply only to graded posets.

Shareshian's result is surprising and pretty, and it would be nice to find something similar for other lattices on groups. In this paper, we consider cosets. The *coset poset*  $\mathfrak{C}(G)$  (poetically named by K. Brown in [7]) is the poset of all cosets of proper subgroups of  $G$ , ordered by inclusion. The *coset lattice*  $\hat{\mathfrak{C}}(G)$  is

$\mathfrak{C}(G) \cup \{G, \emptyset\}$ , likewise ordered by inclusion. (The meet operation is intersection, while  $H_1x_1 \vee H_2x_2 = \langle H_1, H_2, x_1x_2^{-1} \rangle x_2$ ). It is easy to prove that  $\mathfrak{C}(G)$  is shellable if and only if  $\hat{\mathfrak{C}}(G)$  is, so we study the two interchangeably. If  $\mathfrak{C}(G)$  is shellable, we will call  $G$  *coset-shellable*.

The history of the coset poset is discussed in the last chapter of [16]. Most results proved have been either negative results, or else so similar to the situation in the subgroup lattice as to be uninteresting. More recently, K. Brown rediscovered the coset poset, and studied its homotopy type while proving some divisibility results on the so called probabilistic zeta function ([7]). In particular, he shows that if  $G$  is a solvable group, then  $|\mathfrak{C}(G)|$  has the homotopy type of a bouquet of spheres.

In chapter 2 we show that there are finite groups  $G$  which have a shellable subgroup lattice, but a non-shellable coset lattice. In particular, we show that for  $\mathfrak{C}(G)$  to be shellable,  $G$  must be supersolvable, and every Sylow subgroup of  $G$  must be elementary abelian. In chapter 3 we use linear algebra to construct an invariant on subgroups of such groups. Finally, in chapter 4 we use this invariant to construct a so-called EL-shelling, and to finish the proof of our main theorem:

**Theorem 4.** (Main Theorem) *If  $G$  is a finite group, then  $\mathfrak{C}(G)$  is shellable if and only if  $G$  is supersolvable with all Sylow subgroups elementary abelian.*

Our theorem is even more interesting when we connect it with a paper of P. Hall ([11]). We recall that a group  $G$  is *complemented* if for every subgroup  $H \subseteq G$ , there is a *complement*  $K$  which satisfies i)  $K \cap H = 1$  and ii)  $HK = KH = G$ . Hall proved the equivalence of the first three properties in the following restatement of our theorem:

**Theorem 5.** (Restatement of Main Theorem) *If  $G$  is a finite group, then the following are equivalent:*

- 1)  $G$  is supersolvable with all Sylow subgroups elementary abelian,
- 2)  $G$  is complemented,
- 3)  $G$  is a subgroup of the direct product of groups of square-free order,
- 4)  $G$  is coset-shellable,
- 5)  $\mathfrak{C}(G)$  is homotopy Cohen-Macaulay,
- 6)  $\mathfrak{C}(G)$  is sequentially Cohen-Macaulay over some field, and
- 7)  $\mathfrak{C}(G)$  is Cohen-Macaulay over some field.

Parts (5), (6) and (7) are discussed in chapter 2.3, where we define the three used versions of the Cohen-Macaulay property.

Complemented groups have also been called *completely factorizable* groups, and have been studied by other people, see for example [1], or [13].

# Chapter 2

## Coset posets that are not shellable

### 2.1 $p$ -groups

It is often easier to show that something is not shellable, than to show that it is. So we start our search for shellings of the coset lattice by finding groups for which  $\mathfrak{C}(G)$  is certainly *not* shellable. The following lemma will be very useful in this endeavor:

**Lemma 6.** *If  $P$  is a shellable poset, then every interval in  $P$  is also shellable. (Thus, if  $G$  is coset-shellable, then so is every subgroup  $H \subseteq G$ ).*

*Proof.* Since every interval in a poset  $P$  is a “link” (for more information, see the beginning of chapter 2.3), the first part follows immediately from Proposition 10.14 in [6].

For the second part, we note that the interval  $[\emptyset, H]$  in  $\hat{\mathfrak{C}}(G)$  is isomorphic to  $\hat{\mathfrak{C}}(H)$ . □

**Corollary.** *If  $G$  is a finite coset-shellable group, then  $G$  is solvable.*

*Proof.* Note that the interval  $[1, G]$  in  $\hat{\mathfrak{C}}(G)$  is isomorphic to the subgroup lattice of  $G$ . Apply Lemma 6 and Shareshian’s theorem.

A proof of this fact that does not rely on Shareshian’s theorem will also be given, in chapter 2.2. □

At first glance, one might hope that perhaps all solvable groups have a shellable coset poset. Soon enough, however, one considers the coset-poset of  $\mathbb{Z}_4$ , pictured in figure 2.1. We see that  $\mathfrak{C}(\mathbb{Z}_4)$  is not even connected, and connectivity is an easy

consequence of the definition of shellability as long as all facets have dimension at least 1.

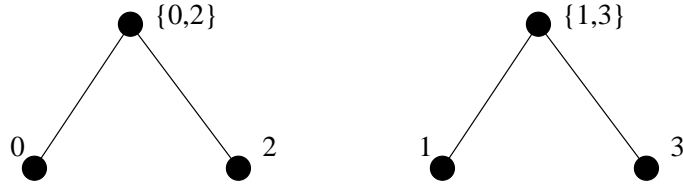


Figure 2.1: The coset poset of  $\mathbb{Z}_4$ . Not even connected!

A similar situation holds for arbitrary  $p$ :  $\mathbb{Z}_{p^2}$  has only one nontrivial proper subgroup, so  $\mathfrak{C}(\mathbb{Z}_{p^2})$  falls into  $p$  connected components, and in particular is not shellable. Hence, no group  $G$  with a subgroup isomorphic with  $\mathbb{Z}_{p^2}$  can be coset-shellable. Can we eliminate any other  $p$ -groups from the possibility of coset-shellability? In fact we can. We recall the following theorem of K. Brown:

**Theorem 7.** (Proposition 11 from [7]) *Let  $G$  be a finite solvable group with a chief series  $1 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_k = G$ . Then  $\mathfrak{C}(G)$  has the homotopy type of a bouquet of  $(d - 1)$ -spheres, where  $d$  is the number of indices  $i = 1, \dots, k$  such that  $N_i/N_{i-1}$  has a complement in  $G/N_{i-1}$ .*

It follows from the proof in [7] that for  $G \neq 1$  the number of spheres is, in fact, nonzero.

**Proposition 8.** 1) *If  $H$  is a finite  $p$ -group which is coset-shellable, then  $H$  is elementary abelian.*

2) *If  $G$  is a finite group which is coset-shellable, then all Sylow subgroups of  $G$  are elementary abelian.*

*Proof.* 1) If  $H$  is a finite  $p$ -group, then  $L(H)$  is graded (by, for example, Iwasawa's Theorem, Theorem 2), hence  $\mathfrak{C}(H)$  is also graded. But it is well known (see for

example [5]) that a graded, shellable poset  $P$  has homotopy type of a bouquet of  $r$ -spheres, where  $r$  is the length of a maximal chain in  $P$ . By the above theorem, we see that every chief factor of  $H$  must be complemented, and hence  $H$  has trivial Frattini subgroup  $\Phi(H)$  (otherwise any minimal normal subgroup contained in  $\Phi(H)$  is an uncomplemented chief factor).

But since  $H$  is a finite  $p$ -group, it is true that  $\Phi(H) = H'H^p$  (see for example [14, 5.3.2]). Hence  $H$  must be abelian of exponent  $p$ , i.e., isomorphic to  $\mathbb{Z}_p^n$ , i.e., elementary abelian.

2) Apply Lemma 6 to the interval  $[\emptyset, H]$  in  $\hat{\mathfrak{C}}(G)$ , where  $H$  is a Sylow subgroup of  $G$ . □

## 2.2 Non-supersolvable groups

We now have that for a finite group  $G$  to be coset-shellable,  $G$  must be solvable with elementary abelian Sylow subgroups. A little more holds:  $G$  must in fact be supersolvable. To prove this, it suffices by 6 and the discussion of the previous section to restrict ourselves to groups  $G$  such that:

- 1)  $G$  is not supersolvable,
- 2) All proper subgroups of  $G$  are supersolvable, and
- 3) All Sylow subgroups of  $G$  are elementary abelian.

A closely related idea is that of minimal non-complemented groups, if we drop condition (3). A complete characterization of such groups is given in [13], although we do not use their characterization.

In light of Shareshian's Theorem (Theorem 3), it might seem at first glance that a stronger version of condition 1 would be to require  $G$  to be solvable but not

supersolvable. The following result of Doerk, however, shows that this would be redundant.

**Lemma 9.** (*Doerk, [9, Hilfssatz C]*) *If every maximal subgroup of  $G$  is supersolvable, then  $G$  is solvable.*

We notice also that this frees our characterization of groups that are not coset-shellable from Shareshian's Theorem (Theorem 3), which relies on Thompson's classification of minimal finite simple groups. We will also need this for Section 2.3.

For any normal subgroup  $N$  in  $G$ , let  $q : G \rightarrow G/N$  be the quotient map. Then we take  $\mathfrak{C}_0(G)$  to be the subposet of  $\mathfrak{C}(G)$  of all  $Hx$  such that  $q(Hx) \neq G/N$ . Thus,  $\mathfrak{C}_0(G)$  is obtained from  $\mathfrak{C}(G)$  by removing cosets  $Kx$  when  $KN = G$ . We will use the following proposition to show that  $\mathfrak{C}(G)$  has the wrong homotopy type to be shellable.

**Theorem 10.** (K. Brown [7], Proposition 8 and following discussion) *The quotient map  $q : G \rightarrow G/N$  induces a homotopy equivalence  $\mathfrak{C}_0(G) \rightarrow \mathfrak{C}(G/N)$ .*

The following lemma from group theory will be useful.

**Lemma 11.** *Let  $G$  be a solvable group, with  $H$  a proper subgroup that is not normal. Then*

- 1) *If  $N$  is an abelian normal subgroup of  $G$  with  $NH = G$ , then  $H$  is maximal in  $G$  if and only if  $N/N \cap H$  is a chief factor for  $G$ .*
- 2)  *$H$  is maximal if and only if it is a complement to a chief factor  $N_{i+1}/N_i$ , ie, if and only if  $HN_{i+1} = G$  and  $H \cap N_{i+1} = N_i$ .*

Part 1 may be found in [14, Theorem 5.4.2]. Part 2 follows from part 1 by taking  $N_{i+1}$  to be minimal such that  $HN_{i+1} = G$ .

We use Lemma 11 in proving the following:

**Lemma 12.** *Let  $G$  be a group satisfying conditions 1-3 above. Let  $n$  be the length of a longest chain in  $\mathfrak{C}(G)$ . Then  $G$  has a unique minimal normal subgroup  $N$  of non-prime order, such that  $|\mathfrak{C}_0(G)|$  (over this  $N$ ) is the subcomplex of  $|\mathfrak{C}(G)|$  generated by all chains of dimension  $n$ .*

*Proof.* Our proof goes in four steps:

1) Every chief factor of  $G$  is complemented. This follows by a theorem of Gaschütz (found in [15, p191], but not in earlier editions of the same book) which says that a normal abelian  $p$ -subgroup  $N$  is complemented in  $G$  if and only if  $N$  is complemented in a Sylow  $p$ -subgroup containing it. Since  $G$  is solvable,  $N_{i+1}/N_i$  is an abelian  $p$ -group, and the 2nd isomorphism theorem gives that  $G/N_i$  has all Sylow subgroups elementary abelian (hence is a complemented group). It follows that  $N_{i+1}/N_i$  is complemented.

2) A chief factor  $N_{i+1}/N_i$  is of non-prime order only if  $N_i = 1$ . Suppose otherwise, that  $N_{i+1}/N_i$  is of non-prime order with  $N_i \neq 1$ . Then  $N_i/N_{i-1}$  is complemented, so there is a group  $K$  with  $G/N_i \cong K/N_{i-1}$ , so  $G/N_i$  is a complemented group. Since  $N_{i+1}/N_i$  is not of prime order, there is an  $H_1$  with  $N_i \subsetneq H_1 \subsetneq N_{i+1}$ . Let  $H_2/N_i$  be a complement to  $H_1/N_i$ , and let  $N_* = H_2 \cap N_{i+1}$ . Then  $H_2$  normalizes  $N_*$  by containment,  $N_{i+1}$  normalizes  $N_*$  since  $N_{i+1}/N_i$  is elementary abelian, and so  $H_2 N_{i+1} \supseteq H_2 H_1 = G$  normalizes  $N_*$ . But  $N_{i+1}/N_i$  was a chief factor, and we have our contradiction.

3) There is a minimal normal subgroup  $N$  of  $G$  of non-prime order, which is a complement to all maximal subgroups of nonprime index. Since  $G$  is not supersolvable, it is immediate that there exists some minimal normal subgroup of non-prime order  $N$ . Since  $N$  has a complement (by part 1), we have that  $G/N$

is supersolvable, hence any maximal  $K$  which contains  $N$  is of prime index by Iwasawa's theorem (Theorem 2). It follows that any maximal  $K$  of non-prime index does not contain  $N$ , hence,  $KN = G$ , and by Lemma 11  $K$  and  $N$  complement one another.

4) If  $C$  is a maximal chain of length less than  $n$ , then the top element of  $C$  is  $Kx$  for some coset of some complement  $K$  of  $N$ , and  $C \setminus \{Kx\}$  can be extended to a chain of length  $n$ . Let  $K_1x$  be the coset immediately under  $Kx$  in  $C$ . Then since  $K$  is supersolvable,  $[K : K_1]$  is of prime order. Then  $[G : NK_1] = [K : K_1]$  is also of prime order. Moreover,  $NK_1$  is supersolvable, so if  $|N| = p^a$ , then there is a chain  $K_1 = H_0 < H_1 < \dots < H_a = NK_1$  between  $K_1$  and  $NK_1$ . The desired chain then follows  $C$  up to  $K_1x$ , and ends at the top with  $K_1x < H_1x < \dots < H_ax = NK_1x$ .

We have shown that  $|\mathfrak{C}_0(G)|$  is obtained from  $|\mathfrak{C}(G)|$  by removing the facets of dimension less than  $n$ , thus that  $|\mathfrak{C}_0(G)|$  is the subcomplex of  $|\mathfrak{C}(G)|$  generated by all  $n$ -faces.  $\square$

We relate this to the following result from Björner and Wachs:

**Lemma 13.** (*Björner/Wachs [5]*) *If  $\Delta$  is shellable, then the subcomplex generated by all faces of dimensions between  $r$  and  $s$  is also shellable.*

We are now ready to prove our goal for this section.

**Theorem 14.** *If  $G$  is not supersolvable, then  $G$  is not coset-shellable.*

*Proof.* By the preceding discussion, it suffices to consider  $G$  solvable with every subgroup complemented. Let  $N$  be the minimal normal subgroup constructed in Lemma 12. Then the resulting  $|\mathfrak{C}_0(G)|$  is the subcomplex of  $|\mathfrak{C}(G)|$  generated by the faces of dimension  $n$ . From Theorem 7 we have that  $|\mathfrak{C}_0(G)|$  is homotopic to a bouquet of spheres of dimension at most the length of the chief series for  $G$ .

But since  $G$  is not supersolvable, the chief series has length less than  $n$ , so the dimension of the spheres in  $|\mathfrak{C}_0(G)|$  is less than  $n$ . It follows that  $\mathfrak{C}_0(G)$  is not shellable, hence by Lemma 13 that  $\mathfrak{C}(G)$  is not shellable.  $\square$

### 2.3 Cohen-Macaulay coset lattices

In fact, we have proven slightly more in section 2. A property closely related to shellability is that of being Cohen-Macaulay or sequentially Cohen-Macaulay. Recall that the *link* of a face  $F_0$  in a simplicial complex  $\Delta$  is  $\text{lk}_\Delta F_0 = \{F \in \Delta : F \cup F_0 \in \Delta, F \cap F_0 = \emptyset\}$ . Links in the order complexes of posets are closely related to intervals. More specifically, if  $C$  is a maximal chain containing  $x$  and  $y$ , and  $C'$  is  $C$  with all  $z$  such that  $x < z < y$  removed, then it is easy to see that  $\text{lk}_P C'$  is the order complex of the interval  $(x, y)$ . In general, the link of a chain in a bounded poset is the so-called “join” of intervals.

Let  $k$  be a field. A simplicial complex  $\Delta$  is *Cohen-Macaulay over  $k$*  if for every face  $F \in \Delta$ ,  $\tilde{H}_i(\text{lk}_\Delta F, k) = 0$  for  $i < \dim \text{lk}_\Delta F$ , ie, if every link has the homology of a wedge of top dimensional spheres. It will come as little surprise after the preceding discussion of links in posets that one can prove the following fact: a poset  $P$  is Cohen-Macaulay if and only if every interval  $(x, y)$  in  $P$  has the homological wedge of spheres property (see [2] for a proof of this and further discussion of links and joins). The complex  $\Delta$  is *homotopy Cohen-Macaulay* if every such link is homotopic to a wedge of top dimensional spheres. Since a graded shellable poset has the homotopy type of a wedge of top dimensional spheres, and since every interval in a shellable poset is shellable, we see that (the order complex of) a graded shellable poset is homotopy Cohen-Macaulay. It is clear that a homotopy Cohen-Macaulay complex is Cohen-Macaulay over any field.

There is an extension of the Cohen-Macaulay property to non-pure complexes. The *pure  $i$ -skeleton* of a simplicial complex  $\Delta$  is the subcomplex generated by all faces of dimension  $i$ . We say that  $\Delta$  is *sequentially Cohen-Macaulay* if its pure  $i$ -skeleton is Cohen-Macaulay for all  $i$ . It is clear that a pure, sequentially Cohen-Macaulay complex is Cohen-Macaulay.

A useful reference on Cohen-Macaulay complexes is [18]. Useful properties of sequentially Cohen-Macaulay complexes are given in [20]. We recall some properties from the latter.

**Lemma 15.** *Let  $\Delta$  be a simplicial complex,  $P$  a poset:*

- 1) *If  $\Delta$  is shellable, then  $\Delta$  is sequentially Cohen-Macaulay.*
- 2) *If  $P$  is sequentially Cohen-Macaulay, then all intervals in  $P$  are also sequentially Cohen-Macaulay.*

Then in the previous two sections we have actually shown

**Proposition 16.** *If  $\mathfrak{C}(G)$  is sequentially Cohen-Macaulay, then  $G$  is a complemented group.*

*Proof.* The proof of Proposition 8 shows that if  $P$  is a  $p$ -group, but not elementary abelian, then  $\mathfrak{C}(P)$  has the homotopy type of a wedge of spheres of the wrong dimension. Hence the homology does not vanish below the top dimension, and  $\mathfrak{C}(P)$  is not (sequentially) Cohen-Macaulay. Lemma 15 part 2 then gives that all Sylow subgroups of a group  $G$  with  $\mathfrak{C}(G)$  sequentially Cohen-Macaulay must be elementary abelian.

Similarly, in the proof of Theorem 14 we show that,  $\mathfrak{C}_0(G)$  is not Cohen-Macaulay. By Lemma 12 we have that  $\mathfrak{C}_0(G)$  is the pure  $n$ -skeleton of  $\mathfrak{C}(G)$ , and then the definition gives that  $\mathfrak{C}(G)$  is not sequentially Cohen-Macaulay.  $\square$

It is clear from definition that a pure sequentially Cohen-Macaulay complex is Cohen-Macaulay. Proposition 16 and the fact that complemented groups are supersolvable then gives that (6) and (7) are equivalent, and that both imply (1-3) in our Restatement of the Main Theorem, Theorem 5. Then  $(4) \implies (5) \implies (6)$  is clear from the definition of homotopy Cohen-Macaulay, and it remains only to prove  $(4) \implies (1-3)$ . This will be the subject of chapter 4.

# Chapter 3

## Some linear algebra

We now take a brief break from shellings and homotopy type to do some linear algebra. First, we introduce some notation. Fix a vector space  $V$  with (ordered) basis  $\mathfrak{B} = \{e_1, \dots, e_n\}$ , and consider a subspace  $U \subseteq V$ . Let  $\{g_1, \dots, g_k\}$  be a set of generators for  $U$ . Then we can write the coordinates of the  $g_i$ 's as row vectors

$[g_i]_{\mathfrak{B}}$ , put these in a matrix  $\begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}_{\mathfrak{B}}$ , and reduce to reduced row echelon form  $M$ .

Denote the set of pivot columns for  $M$  (i.e., the columns with a leading 1 in some row of  $M$ ) as  $I_{V, \mathfrak{B}}(U)$ , or just  $I(U)$  if the choice of  $V$  and  $\mathfrak{B}$  are clear.

**Lemma 17.**  *$I(U)$  is an invariant for the subspace  $U$  of  $V$  with respect to  $\mathfrak{B}$ .*

We need only show that  $I(U)$  does not depend on the choice of generators for  $U$ . Suppose generators  $\{h_i\}$  give row reduced matrix  $M_h$  and generators  $\{g_i\}$  give row reduced matrix matrix  $M_g$ . But then the row reduced matrix of  $\{h_i\} \cup \{g_i\}$  must be both  $M_h$  and  $M_g$  by uniqueness of reduced row echelon form, hence  $M_g = M_h$ . In particular, the pivot columns are the same.

We mention some elementary properties of our invariant

**Proposition 18.** *Fix  $V$  and  $\mathfrak{B}$  as above, and let  $U_1, U_2$  be subspaces of  $V$ . Then*

- 1)  $|I(U_1)| = \dim U_1$
- 2) *If  $U_1 \subseteq U_2$ , then  $I(U_1) \subseteq I(U_2)$ .*

We will need the following lemma in our application of  $I(U)$  to the next chapter. Briefly, part (2) will correspond with having a unique lexicographically first path in intervals of  $\mathfrak{C}(G)$ .

**Lemma 19.** Fix  $V$  and  $B$  as above, and let  $U_1 \subseteq U_2$  be subspaces of  $V$ . Then

1) If  $k$  is the largest number in  $I(U_2) \setminus I(U_1)$ , then there is a unique subspace  $W_\uparrow$  such that  $U_1 \subseteq W_\uparrow \subseteq U_2$  and  $I(W_\uparrow) = I(U_1) \cup \{k\}$ .

2) If  $j$  is the smallest number in  $I(U_2) \setminus I(U_1)$ , then there is a unique subspace  $W_\downarrow$  such that  $U_1 \subseteq W_\downarrow \subseteq U_2$  and  $I(W_\downarrow) = I(U_2) \setminus \{j\}$ .

*Proof.* 1) It is immediate from the definition of  $I_{U_2}$  that there is some  $g \in U_2$  with a 1 in the  $k$ th coordinate, and 0's in all preceding coordinates when written as a vector with respect to  $B$ . Suppose  $g_1$  and  $g_2$  both have this property. Then  $g_1 - g_2$  is 0 in all coordinates up to and including  $k$ , hence  $g_1 - g_2 \in U_1$ . It follows that the desired  $W_\uparrow = \langle U_1, g \rangle$  is unique.

2) First, such a subspace exists. Suppose  $U_2 = \langle g_1, \dots, g_n \rangle$ , where the  $g_i$ 's are row reduced as in the definition of  $I(U)$ . Reorder so that  $g_1, \dots, g_l$  are the generators (rows) with pivots in  $I(U_2) \setminus I(U_1)$ , ordered from least to greatest. Then  $U_2 = \langle U_1, g_1, \dots, g_l \rangle$  (where  $l = |I(U_2) \setminus I(U_1)|$ ), and  $W_\downarrow = \langle U_1, g_2, \dots, g_l \rangle$  is a space with the desired properties.

Suppose  $W$  is another such space. Represent  $W = \langle U_1, h_2, \dots, h_l \rangle$  in the same way as we did for  $U_2$  in the preceding paragraph. Let  $W_0 = \langle g_2, \dots, g_l, h_2, \dots, h_l \rangle$ . Then since the  $h_i$ 's and  $g_i$ 's are all zero in coordinates up to and including  $j$ , so  $j \notin I(W_0)$ . Also,  $W_0 \subseteq U_2$  so  $I(W_0) \subseteq I(U_2)$ . But the  $g_i$ 's and  $h_i$ 's were row reduced with respect to  $U_1$ , so are zero in all pivots of  $U_1$ , so  $I(U_1) \cap I(W_0) = \emptyset$ . It follows that  $I(W_0) = I(U_2) \setminus \{j\} \cup I(U_1)$ , thus that  $W_0$  is  $l - 1$  dimensional, thus that  $W = W_\downarrow$ .  $\square$

This ends our excursion into linear algebra. We are now ready to apply the results of this chapter.

# Chapter 4

## Shelling the Coset Poset

To show that the coset poset of a finite complemented group  $G$  is shellable, we actually exhibit a *coEL*-labelling. First, let us recall the definition of an *EL*-labelling.

A *cover relation* is a pair  $x \leftarrow y$  in a poset  $P$  such that  $x \lesssim y$  and such that there is no  $z$  with  $x \lesssim z \lesssim y$ . In this situation, we say that  $y$  *covers*  $x$ . We recall that the usual picture one draws of a poset  $P$  is the *Hasse diagram*, where we arrange vertices corresponding with the elements of  $P$  such that  $x$  is below  $y$  if  $x < y$ , and draw an edge between  $x$  and  $y$  if  $x \leftarrow y$ .

Let  $\lambda$  be a labelling of the cover relations (equivalently, of the edges of the Hasse diagram) of  $P$  with elements of some poset  $L$  (for us,  $L$  will always be the integers). Then  $\lambda$  is an *EL-labelling* if for every interval of  $P$  we have i) there is a unique (strictly) increasing maximal chain on  $[x, y]$ , and ii) this maximal chain is first among maximal chains on  $[x, y]$  with respect to the lexicographic ordering. If  $\lambda$  is an *EL*-labelling of the dual of  $P$ , then we say  $\lambda$  is a *coEL-labelling*.

Björner first introduced *EL*-labellings in [3], and showed that if a poset  $P$  has an *EL*-labelling, then  $P$  is shellable. For this reason, posets with an *EL*-labelling (or *coEL*-labelling) are often called *EL-shellable* (or *coEL-shellable*). As mentioned before, we will construct a *coEL*-labelling of  $\hat{\mathcal{C}}(G)$ , which will be based on the invariants  $I(U)$  constructed in the previous chapter.

Let  $G = G_1 \times \cdots \times G_r$  be the direct product of square-free groups  $\{G_i\}$ . Fix  $p$ . Let  $H$  be a subgroup of  $G$ , with  $H^*$  a Sylow  $p$ -subgroup of  $H$ . Let  $G^*$  be a Sylow  $p$ -subgroup of  $G$ , with  $H^*$  contained in  $G^*$ . Notice that  $G^* \cap G_i$  is either

isomorphic to  $\mathbb{Z}_p$  or 1 (depending on whether  $p \mid |G_i|$ ). Let  $e_i$  be a generator of  $G^* \cap G_i$  when this intersection is nontrivial. Let  $\mathfrak{B}$  be an ordered basis of such generators  $e_i$ , taken from each nontrivial  $G^* \cap G_i$ . Think of the elementary abelian subgroup  $G^*$  as a vector space over  $\mathbb{Z}_p$ , and define  $I^p(H)$  to be  $I_{G^*, \mathfrak{B}}(H^*)$ .

**Lemma 20.**  *$I^p(H)$  is well defined.*

*Proof.* We need to check that  $I^p(H)$  is independent of the choice of Sylow  $p$ -subgroups  $H^*$  and  $G^*$ , and that  $I^p(H)$  is independent of the choice of the  $e_i$ 's in  $G^* \cap G_i$ .

Suppose that  $M_{\mathfrak{B}} = \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}$  is a set of generators for  $H^*$ , row reduced with

respect to  $\mathfrak{B}$ . Then conjugation by  $h$  will give us a set of generators for  $h^{-1}H^*h$ , and since conjugation fixes factors in the direct product, multiplication of rows by constants to restore 1's at the front will give us reduced row echelon form (with respect to  $h^{-1}\mathfrak{B}h = \{h^{-1}e_i h\}$ ). Hence, the pivots are not affected by conjugation, and since Sylow subgroups are conjugate, we have that  $I^p(H)$  is independent of the choice of Sylow subgroup.

Second, any different generator of  $G^* \cap G_i$  will differ from  $e_i$  by a (nonzero) constant. If we change basis from  $\mathfrak{B} = \{e_i\}$  to  $\mathfrak{B}' = \{c_i e_i\}$ , then we multiply the column corresponding with  $e_i$  in  $M_{\mathfrak{B}}$  by  $c_i^{-1}$  to find  $M_{\mathfrak{B}'}$  (the set of generators with respect to  $\mathfrak{B}'$ ). Again, multiplication of rows by constants will restore  $M_{\mathfrak{B}'}$  to reduced row echelon form, so the pivots are not affected by such a change of basis, so  $I^p(H)$  is independent of the choice of  $e_i \in G^* \cap G_i$ .  $\square$

We need a couple more lemmas.

**Lemma 21.** *Let  $M$  be a maximal subgroup of a supersolvable group  $G$ . If  $G = HM$ , then  $Hx \cap M$  is a maximal coset of  $Hx$ .*

*Proof.* Since  $G = HM$ , we can write  $Hx = Hm$  for some  $m \in M$ . So  $Hx \cap M \neq \emptyset$ . Also,  $|G| = |HM| = \frac{|H||M|}{|H \cap M|} = [H : H \cap M] \cdot \frac{|G|}{[G : M]}$ . Since  $[G : M]$  is prime, it follows that  $[H : H \cap M] = [G : M]$  is also prime, hence  $H \cap M$  maximal in  $H$  as desired.  $\square$

The following lemma is proved, for example, in [14, 5.4.8].

**Lemma 22.** *Let  $G$  be a finite supersolvable group. Then  $G$  has a chief series  $1 = N_0 \subseteq \cdots \subseteq N_k = G$  with  $[N_1 : N_0] \geq [N_2 : N_1] \geq \cdots \geq [N_{k-1} : N_k]$ .*

In particular, if  $p$  is the largest prime dividing  $G$  and  $q$  is the smallest; then  $G$  has a normal Sylow  $p$ -subgroup and a normal Hall  $q'$ -subgroup.

**Corollary 23.** *Let  $G$  be a finite supersolvable group. If  $p$  is the smallest prime dividing  $[H_0 : H_n]$  for some  $H_n \subseteq H_0$ , then there is a unique  $H_1$  with  $H_n \subseteq H_1 \subseteq H_0$  and such that  $p$  does not divide  $\frac{[H_0 : H_n]}{[H_0 : H_1]}$ .*

*Proof.* Let  $\pi = \{q : q \leq p, q \mid |H_0|\}$ , and  $K$  be a Hall  $\pi'$  group. Then  $K \triangleleft H_0$  by the lemma, hence  $KH_n$  is a subgroup of  $H_0$  with the desired properties.  $\square$

We are now ready to prove the main theorem. The high level idea is to use the changes in the invariants  $I^p(H)$  to label cover relations. Unfortunately, that gives us a lot of identically labelled chains. So we pick out some distinguished cover relations, and change their labels to have a unique increasing chain. The details follow:

**Theorem 24.** *If  $G$  is supersolvable with all subgroups elementary abelian, then  $\hat{\mathcal{C}}(G)$  is coEL-shellable, and so  $G$  is coset-shellable.*

*Proof.* We recall by the theorem of Hall restated in Theorem 5 that  $G \subseteq G_1 \times \cdots \times G_r$  where each  $G_i$  is of square-free order. If  $\hat{\mathfrak{C}}(G_1 \times \cdots \times G_r)$  is *coEL*-shellable, then it follows immediately from the definition that the interval  $[\emptyset, G] \cong \hat{\mathfrak{C}}(G)$  is as well. So we can assume without loss of generality that  $G = G_1 \times \cdots \times G_r$ , the direct product of groups of square-free order.

For each  $i$ , and each  $p$  dividing  $|G_i|$ , pick  $M_{p,i}^*$  to be a maximal subgroup of index  $p$  (a Hall  $p'$  subgroup) in  $G_i$ . Such  $M_{p,i}^*$ 's exist because  $G_i$  is solvable. Then set  $M_{p,i}$  to be  $M_{p,i}^* \times \prod_{j \neq i} G_j$ . (These will be used to pick out the distinguished edges mentioned above). Fix  $l(p, j)$  to be an order preserving map into the positive integers of the lexicographic ordering on the pairs  $(p, j)$  for all  $p$  dividing  $G$  and  $j = 1, \dots, r$ .

We now use  $l(p, j)$  to label the cover relations as follows. Suppose  $H_1x \subseteq H_0x$  is a cover relation in  $\hat{\mathfrak{C}}(G)$ . Since  $G$  is supersolvable,  $[H_0 : H_1] = p$  for some prime  $p$ , hence Sylow subgroups of  $H_1$  have dimension (as a vector space over  $\mathbb{Z}_p$ ) one lower than those of  $H_0$ . It follows that  $I^p(H_0) = I^p(H_1) \cup \{j\}$  for some  $j$ . Then label the edge  $H_0x \rightarrow H_1x$  as

$$\lambda(H_0x \rightarrow H_1x) = \begin{cases} -l(p, j) & \text{if } H_1x = H_0x \cap M_{p,j} \\ l(p, j) & \text{otherwise} \end{cases}$$

Finally, label  $\lambda(x \rightarrow \emptyset) = 0$ . We will show that  $\lambda$  is a *coEL*-labelling.

Intervals in  $\hat{\mathfrak{C}}(G)$  all have either the form  $[\emptyset, H_0x]$ , or  $[H_nx, H_0x]$ . We consider these types of intervals separately, and show there is a unique increasing chain which is lexicographically first.

On  $[\emptyset, H_0x]$ , we notice from Proposition 18 that every maximal chain from  $H_0x$  down to  $\emptyset$  has 0 on the last edge, and  $\pm l(p, j)$  (over all pairs  $(p, j)$  such that  $j \in I^p(H_0)$ ) on the preceding edges. In fact, for each such pair  $(p, j)$ , exactly one

of  $+l(p, j)$  or  $-l(p, j)$  occurs exactly once on any maximal chain. Since 0 is the last edge, the only possible increasing chain is the one with labels  $-l(p, j)$  in increasing order. By Lemma 21 there is such a chain, it is clearly unique and lexicographically first.

For  $[H_n x, H_0 x]$ , the situation is only slightly more complicated. Let a pair  $(p, j)$  be called *admissible* for the given interval if  $p$  divides  $[H_0 : H_n]$  and  $j \in I^p(H_0) \setminus I^p(H_n)$ . If  $l(p, j)$  is minimal among admissible  $(p, j)$ , then there is a unique  $H_1 x$  of index  $p$  in  $H_0 x$  with  $H_0 x \rightarrow H_1 x$  labeled  $\pm l(p, j)$  by Corollary 23 and Lemma 19. Moreover, any chain on  $[H_n x, H_0 x]$  has exactly one edge with label  $\pm l(p, j)$  for each admissible  $(p, j)$ .

Suppose  $C$  is an increasing chain on  $[H_n x, H_0 x]$ . Suppose  $H_i x \rightarrow H_{i+1} x$  in  $C$  is labelled  $+l(p, j)$ . Then  $l(p, j)$  is minimal among  $(p, j)$  admissible for  $[H_n x, H_i x]$  since the chain is increasing. Thus  $H_i x \rightarrow H_{i+1} x$  is the unique edge down from  $H_i x$  labeled with  $\pm l(p, j)$ , and since the label was positive we see that  $H_n x \not\subseteq M_{p,j}$ . It follows that the unique increasing chain on  $[H_n x, H_0 x]$  is the lexicographically first one labeled with  $-l(p, j)$  in increasing order for  $(p, j)$  such that  $H_n x \subseteq M_{p,j}$ , followed by  $l(p, j)$  for all other admissible  $(p, j)$ .  $\square$

# Chapter 5

## Examples

At first glance, the labelling constructed in Theorem 24 might seem to come “from left field.” It is helpful to work out what happens for the case where  $G$  is a group of square-free order. In this case, many of the complications we faced in the proof disappear. For example, we don’t have to worry about  $I^p$ , since if  $[H_0 : H_1] = p$ , then  $I^p(H_1) = \emptyset$  and  $I^p(H_0) = \{1\}$ . Similarly, we can just take  $l(p, j) = p$ , since the only possible value of  $j$  is 1. The only  $M_{p,j}$ ’s we have are  $M_{p,1}$ , which we can denote as  $M_p$ .

Thus we see that for any  $H_0, H_1$  with  $[H_0 : H_1] = p$  we get

$$\lambda(H_0x \rightarrow H_1x) = \begin{cases} -p & \text{if } H_1x = H_0x \cap M_p \\ p & \text{otherwise} \end{cases}$$

and  $\lambda(x \rightarrow \emptyset) = 0$ . An example for  $\mathbb{Z}_6$  is worked out in Figure 5.1. A helpful exercise might be to work out the labelling for  $S_3$ .

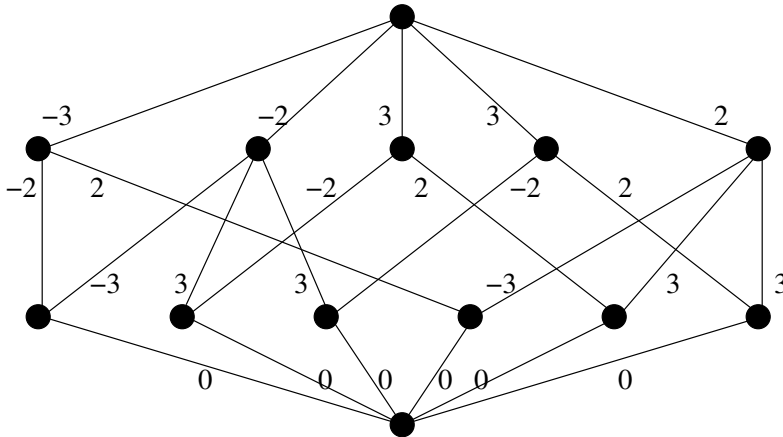


Figure 5.1: The  $coEL$ -labelling of  $\mathfrak{C}(\mathbb{Z}_6)$ . The leftmost two maximal cosets are  $M_3 = M_{3,1}$  and  $M_2 = M_{2,1}$ , respectively.

On the opposite extreme, it is not so hard to understand the *coEL*-shelling on  $\mathbb{Z}_p^n$  – it is just the change in  $I^p$ , with  $l(p, j)$  becoming  $j$ . We will not say anything more about this, but an example for  $\mathbb{Z}_2^2$  is worked out in Figure 5.2.

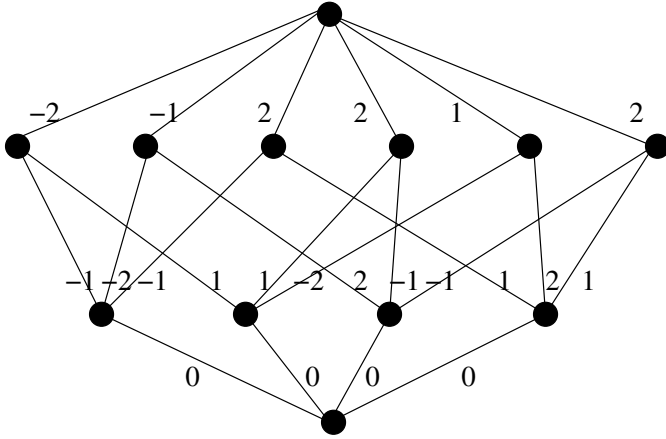


Figure 5.2: The *coEL*-labelling of  $\mathfrak{C}(\mathbb{Z}_2^2)$ . The leftmost two maximal cosets are  $M_{2,2}$  and  $M_{2,1}$ , respectively.

# Chapter 6

## Consequences and Conclusion

A (co)- $EL$ -labelling of a lattice  $L$  tells us a lot about the homotopy type of  $L \setminus \{0, 1\}$ . In particular, the *falling chains* (for our purposes, weakly decreasing maximal chains) in an  $EL$ -labelling give a basis for the nontrivial homology/cohomology group. See [5, Section 5] for a discussion of this in a more general setting. Our  $coEL$ -labelling for  $\mathfrak{C}(G)$  (where  $G$  is a complemented group) thus helps us understand the cohomology of the order complex in a very concrete way.

In showing the shellability of a solvable group's subgroup lattice, Shareshian ([17]) produces a so-called “coatom ordering.” Unfortunately, while the existence of a coatom ordering implies the existence of something with similar properties to a  $coEL$ -labelling (a “ $coCL$ -labelling”), Shareshian is not able to exhibit such a labelling. Such a labelling would be interesting, as it could presumably be used to give an alternative proof of a result of Thévenaz [19]. Perhaps techniques like we use here could be used on the chief series for a solvable group (where every factor is an elementary abelian  $p$ -group) to produce a (co)- $EL$ -labelling in the subgroup lattice.

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