Web Appendix for "Latent Factor Regression Models for Grouped Outcomes" by D. B. Woodard, T. Love, S. W. Thurston, D. Ruppert, S. Sathyanarayana and S. H. Swan

A DERIVATION OF THE MATRIX B

We wish to specify $\tau^2_{\zeta,k}$ and $\mathbf{B}$ in such a way that the factor vector $\boldsymbol{\xi}_i$, defined by

$$\boldsymbol{\xi}_i = \boldsymbol{\beta}_{\mathrm{D},\eta}\eta_i + \boldsymbol{\beta}_{\mathrm{D},z}\mathbf{Z}_i + \mathbf{B}\boldsymbol{\xi}_i + \boldsymbol{\zeta}_i \tag{A.1}$$

where $\zeta_{i,k} \overset{\text{ind}}{\sim} N(0, \tau^2_{\zeta,k})$, can be rewritten as follows.

$$\xi_{i,k} = \beta_{\mathrm{D},\eta,k}\eta_i + \boldsymbol{\beta}_{\mathrm{D},z,k}\mathbf{Z}_i + \phi_i + \psi_{i,k} \qquad\qquad k = 1,\ldots,d$$

where $\phi_i \overset{\text{ind}}{\sim} N(0, \tau^2_\phi)$, $\psi_{i,k} \overset{\text{ind}}{\sim} N(0, \tau^2_{\psi,k})$, and $\boldsymbol{\beta}_{\mathrm{D},z,k}$ is the kth row of the matrix $\boldsymbol{\beta}_{\mathrm{D},z}$. First, let's rewrite (A.1) as

$$\boldsymbol{\xi}_i = (\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\beta}_{\mathrm{D},\eta}\eta_i + (\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\beta}_{\mathrm{D},z}\mathbf{Z}_i + (\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\zeta}_i$$

Since $(\mathbf{I}-\mathbf{B})^{-1}$ is a one-to-one transformation, we can reparametrize using $\boldsymbol{\beta}^*_{\mathrm{D},\eta} = (\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\beta}_{\mathrm{D},\eta}$ and $\boldsymbol{\beta}^*_{\mathrm{D},z} = (\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\beta}_{\mathrm{D},z}$. Suppressing the *s, we now have

$$\boldsymbol{\xi}_i = \boldsymbol{\beta}_{\mathrm{D},\eta}\eta_i + \boldsymbol{\beta}_{\mathrm{D},z}\mathbf{Z}_i + (\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\zeta}_i.$$

Our problem now has been simplified to that of specifying $\tau^2_{\zeta,k}$ and $\mathbf{B}$ in such a way that $(\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\zeta}_i \overset{\text{d}}{=} \boldsymbol{\psi}^*_i$, where $\boldsymbol{\psi}^*_i$ is the vector with elements $\phi_i + \psi_{i,k}$ for $k = 1,\ldots,d$. Let's equivalently find an invertible matrix $\mathbf{A} = (\mathbf{I}-\mathbf{B})^{-1}$ such that $\mathbf{A}\boldsymbol{\zeta}_i \overset{\text{d}}{=} \boldsymbol{\psi}^*_i$ and such that the diagonal entries of $\mathbf{A}^{-1}$ are equal to one. Since the vector $\boldsymbol{\psi}_i$ has a multivariate normal distribution, and since ϕ_i is normally distributed independent of $\boldsymbol{\psi}_i$, the vector $\boldsymbol{\psi}^*_i$ has a multivariate normal distribution. So we can write $\boldsymbol{\psi}^*_i \overset{\text{d}}{=} \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\mathbf{X}_i$ where $\mathbf{X}_i$ is a vector of i.i.d. standard normal random variables, $\boldsymbol{\mu} = E(\boldsymbol{\psi}^*_i) = 0$, and $\boldsymbol{\Sigma} = \mathrm{Var}(\boldsymbol{\psi}^*_i)$ is the invertible variance-covariance matrix of $\boldsymbol{\psi}^*_i$. So for any values of the $\tau^2_{\zeta,k} > 0$, taking $\mathbf{A} = \boldsymbol{\Sigma}^{1/2}\mathbf{C}$ where $\mathbf{C}$ is a diagonal matrix with entries $1/\tau_{\zeta,k}$, we have that $\mathbf{A}\boldsymbol{\zeta}_i \overset{\text{d}}{=} \boldsymbol{\psi}^*_i$. The inverse $\mathbf{A}^{-1} = \mathbf{C}^{-1}\boldsymbol{\Sigma}^{-1/2}$ exists; furthermore, by taking $1/\tau_{\zeta,k}$ equal to the kth diagonal entry of $\boldsymbol{\Sigma}^{-1/2}$ ensures that $\mathbf{A}^{-1}$ has diagonal entries equal to one.

B IDENTIFIABILITY

We show the identifiability of our most general model (A), for the case where η_i is included in the model but $\mathbf{Z}_i$ is not; $\mathbf{Z}_i$ can be handled analogously to η_i. We assume that there is more than one domain, and that the values η_i are i.i.d. from some unknown distribution with mean zero and variance one (the case of any finite mean and variance proceeds in the same fashion but is notationally cumbersome). We also assume that each domain either has more than one outcome, or if it has only one outcome j then we assume that $\beta_{o,\eta,j} = \tau^2_{\psi,d(j)} = 0$.

Proof of identifiability can proceed by setting the empirical moments of $\mathbf{Y}_i$ and η_i equal to the moments implied by the model (2.3), and showing that the solution for the parameter values is unique (Bollen 1989). For model (2.3) such uniqueness can be shown by looking at the first and second moments; in particular we will use $E(\mathbf{Y}_i)$, the first-moment vector for the outcomes; $\text{Var}(\mathbf{Y}_i)$, the variance-covariance matrix of the outcomes; and $\text{Cov}(\mathbf{Y}_i, \eta_i)$, the vector of covariances between η_i and the elements of the vector $\mathbf{Y}_i$. First, $E(Y_{ij}) = \alpha_j$, so α_j is identified for each outcome j. Next,

$$\begin{aligned}\text{Cov}(Y_{ij}, \eta_i) =& \text{Cov}(Y_{ij} - \alpha_j, \eta_i) \ = E\left[(Y_{ij} - \alpha_j)\eta_i\right] \\ &= \left[\beta_{o,\eta,j} + \lambda_j \beta_{\text{D},\eta,d(j)}\right] E(\eta_i^2) \ = \beta_{o,\eta,j} + \lambda_j \beta_{\text{D},\eta,d(j)}\end{aligned}$$

for each outcome j, so $\left[\beta_{o,\eta,j} + \lambda_j \beta_{\text{D},\eta,d(j)}\right]$ is identified. The elements of the variance-covariance matrix $\text{Var}(\mathbf{Y}_i)$ are

$$\begin{aligned}&\text{Cov}(Y_{ij}, Y_{i\ell}) = E\left[(Y_{ij} - \alpha_j)(Y_{i\ell} - \alpha_\ell)\right] \\ &= (\beta_{o,\eta,j} + \lambda_j \beta_{\text{D},\eta,d(j)})(\beta_{o,\eta,\ell} + \lambda_\ell \beta_{\text{D},\eta,d(\ell)}) + \lambda_j \lambda_\ell \tau^2_\phi + \mathbf{1}_{\{d(j)=d(\ell)\}} \lambda_j \lambda_\ell \tau^2_{\psi,d(j)} + \mathbf{1}_{\{j=\ell\}} \sigma^2_j.\end{aligned}$$

Using this expression, since $\left[\beta_{o,\eta,j} + \lambda_j \beta_{\text{D},\eta,d(j)}\right]$ is identified $\forall j$, since $\lambda_j = 1$ for the first outcome in each domain, and since there is more than one domain, τ^2_ϕ is identified. Since τ^2_ϕ is identified, λ_j is identified for each j (by taking ℓ to be the first outcome in another domain). For any domain $k \in \{1, \ldots, d\}$ that has more than one outcome, $\tau^2_{\psi,k}$ is then identified. For a domain with a single outcome, $\tau^2_{\psi,k} = 0$ so we don't have to be concerned about its identifiability. Since everything else is identified in the expression for $\text{Cov}(Y_{ij}, Y_{i\ell})$, σ^2_j is also identified for each j.

Next we address the question of whether $\beta_{o,\eta,j}$ and $\beta_{\text{D},\eta,d(j)}$ are separately identified for each j. Since we have assigned $\beta_{o,\eta,j}$ a random effect prior distribution centered at zero, and since $\left[\beta_{o,\eta,j} + \lambda_j \beta_{\mathrm{D},\eta,d(j)}\right]$ and λ_j are identified for each j, $\beta_{o,\eta,j}$ and $\beta_{\mathrm{D},\eta,d(j)}$ are separately identified. So Model A is fully identified.