DERIVED REPRESENTATION SCHEMES AND NON-COMMUTATIVE GEOMETRY

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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January 2012

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Cornell University 2012

After surveying relevant literature (on representation schemes, homotopical algebra, and non-commutative algebraic geometry), we provide a simple algebraic construction of relative derived representation schemes and prove that it constitutes a derived functor in the sense of Quillen. Using this construction, we introduce a derived Kontsevich-Rosenberg principle. In particular, we construct a (non-abelian) derived functor of a functor introduced by Van den Bergh that offers one (particularly significant) realization of the principle. We also prove a theorem allowing one to finitely present derived representation schemes of an associative algebra whenever one has an explicit finite presentation for an almost free resolution of that algebra; using this theorem, we calculate several examples (including some computer calculations of homology).

BIOGRAPHICAL SKETCH

George Khachatryan was born in Moscow, Russia on December 5, 1984, and moved to the United States with his family in 1990. After graduating from the Kinkaid School in 2003, he attended the University of Chicago. From 2007 to 2008, George attended the University of Cambridge, receiving a Certificate of Advanced Study in Mathematics (CASM) with Distinction. From 2008 to 2011, he was a graduate student at Cornell University under the supervision of Yuri Berest.

To my grandparents.

ACKNOWLEDGEMENTS

For introducing me to mathematics and showing me the beauty of the subject, I would like to thank Mira Bernstein and the staff at Canada/USA Mathcamp, as well as Mark Sapir and Paul Sally. Lenya Ryzhik supported me in my first attempts at research. Victor Ginzburg gave me excellent problem sets (which first got me interested in representation theory) and introduced me to my advisor, Yuri Berest. Chris Brookes at Cambridge was kind enough to set a special essay topic on A_∞-algebras to prepare me for my research at Cornell.

For the present work, I would like to thank above all my advisor, Yuri Berest, who conceived the project and guided me in my efforts; Yuri spent countless hours helping me understand the landscape of the field and the context of the problem, and I couldn't have wished for a more supportive and generous advisor. This work would also have been impossible without Ajay Ramadoss, who suggested approaches to numerous problems and closely collaborated with me and Yuri on many of the results presented here. A graduate student is lucky if he or she has even one guide, while I had the incredible fortune to have the support of both Yuri and Ajay, whom I regard not only as my mentors, but also as my friends.

I would like to thank Martin Kassabov for valuable conversations (and in particular for his suggestion of using semidirect products to define a linearization of the derived representation scheme, discussed in Section 6.3) and R. Frank Moore for help with computer calculations of homology (presented in Section 7.3), which were done using a DG algebras package recently developed by him for Macaulay2; these calculations not only serve to illustrate the strengths of this approach, but have also led to further conjectures (whose proofs are outside of the scope of this thesis, but will appear in later works). I would also like to thank the members of my committee, Allen Knutson and Reyer Sjamaar, for their time and support.

I would like to acknowledge the financial support of several institutions. My undergraduate studies were funded by a College Honors Scholarship from the University of Chicago, supplemented by a National Merit Scholarship and a grant from the Office of Naval Research. My graduate work at Cambridge and Cornell was funded by an NSF Graduate Research Fellowship and by a Cornell Sage Fellowship.

Finally, I would like to thank my parents and my fiance for their love and encouragement, and my friends for their mathematical camaraderie.

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CHAPTER 1

INTRODUCTION

1.0.1 Background and motivation

The set of representations of a finitely generated unital k-algebra A in a vector space V can naturally be given the structure of an affine scheme, called the *representation scheme* $\mathrm{Rep}_V A$. There is a strong connection (known since the 70s) between the representation theory of A and the geometry of $\mathrm{Rep}_V A$. More recently, representation schemes have come to play an important role in non-commutative algebraic geometry: in accordance with a principle proposed by M. Kontsevich and A. Rosenberg in [KR], the schemes $\mathrm{Rep}_V A$ should "approximate" the geometry of the non-commutative space "$\mathrm{Spec}\, A$" as $\dim(V) \to \infty$. In particular, every non-commutative geometric structure on "$\mathrm{Spec}\, A$" should naturally induce a corresponding commutative structure on $\mathrm{Rep}_V(A)$. This viewpoint provides a litmus test for proposed definitions of non-commutative analogs of classical geometric notions.

Examples of the Kontsevich-Rosenberg principle are usually *ad hoc*, differing in each case to suit the specifics of the construction (see, for example, the constructions for functions, vector fields, and differential forms described in Section 3.3 on p. 36). In [VdB1], M. Van den Bergh introduced an additive functor

$$(-)_V : \mathtt{Mod}_{A^e} \to \mathtt{Mod}_{A_V},$$

where $A^e := A \otimes A^{\mathrm{op}}$, the algebra A_V is the coordinate algebra of $\mathrm{Rep}_V A$, and the two categories $\mathtt{Mod}_{A^e}$ and $\mathtt{Mod}_{A_V}$ are the categories of left modules over the corresponding algebras. Many non-commutative geometric structures on "$\mathrm{Spec}\, A$"

(e.g., functions, differential forms, vector fields) are left A^e-modules, while the corresponding (commutative) geometric structures on $\mathrm{Rep}_V(A)$ are left A_V-modules; Van den Bergh's functor $(-)_V$ provides a (unified) realization of the Kontsevich-Rosenberg principle for such cases, taking the non-commutative structures given by generally accepted definitions to the corresponding classical structures on the representation scheme.

In a different direction, there has been work on deriving (in a non-abelian sense) representation schemes (most notably, by I. Ciocan-Fontanine and M. Kapranov in [CK2] and by B. Toën in [TV]). One motivation for this study comes from derived deformation theory (DDT); a derived representation scheme can be regarded as a "resolution of singularities" of a classical one. The existing approaches have a serious limitation in that they are quite abstract, and thus do not lend themselves easily to applications or to the computation of examples.

The primary purpose of the present thesis is to provide a simple algebraic construction of derived representation schemes, and then to use it to construct a (non-abelian) derived functor of Van den Bergh's functor $(-)_V$. We demonstrate the practicality of the approach by calculating some examples. More substantive applications – which include a generalization of a classical theorem of Procesi to the derived setting, where it becomes a statement relating the derived representation scheme to cyclic homology – will appear elsewhere.

One motivation for this approach is that the Kontsevich-Rosenberg principle only works well in practice when the algebra A is formally smooth (for a definition of formal smoothness, see p. 38). Thus, it is expected that the derived representation scheme will play a role in the geometry of arbitrary non-commutative algebras analogous to that played by $\mathrm{Rep}_V A$ in the geometry of (formally) smooth algebras.

1.0.2 DG representation schemes

The scheme $\mathrm{Rep}_V(A)$ represents the functor Rep_V^A taking a commutative algebra B to the set of algebra homomorphisms from A to the algebra of $n \times n$ matrices with values in B. In terms of the coordinate algebra A_V of $\mathrm{Rep}_V(A)$, this means we have an adjunction

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}(A_V, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes_k B)\,, \tag{1.1}$$

where $\mathtt{CommAlg}_k$ is the full subcategory of $\mathtt{Alg}_k$ consisting of commutative algebras.

By classical results of Bergman and Cohn, the extension of Rep_V^A to the category of associative algebras $\mathtt{Alg}_k$ also has a representing object, called the n-*matrix reduction* of A and denoted $\sqrt[V]{A}$. This algebra, which more recently has been called the (coordinate ring of the) *non-commutative representation scheme*, admits a simple algebraic description:

$$\sqrt[V]{A} = (\mathrm{End}(V) *_k A)^{\mathrm{End}(V)}\,,$$

by which we mean the subalgebra of invariants,

$$\{w \in \mathrm{End}(V) *_k A \;:\; [w, m] = 0 \;\forall m \in \mathrm{End}(V)\}.$$

By composition of adjunctions (as abelianization is adjoint to the forgetful functor from commutative to associative algebras) and the Yoneda lemma, the coordinate algebra A_V of the scheme $\mathrm{Rep}_V(A)$ is isomorphic to $\left(\sqrt[V]{A}\right)_{\mathrm{ab}}$.

As it turns out, this construction can be simultaneously generalized along three different directions: first, we can consider an associative DG algebra R instead of the associative algebra A, second, we can relativize the construction,[1] and third, we can generalize V to a chain complex of vector spaces. More concretely, let $\mathtt{DGA}_S$ be the category of associative unital DG chain algebras over S (itself an associative unital DG chain algebra), and let V be a chain complex of k-vector spaces of finite total dimension. Fix a DG algebra morphism $S \to \underline{\mathrm{End}}\, V$, where $\underline{\mathrm{End}}\, V$ is the graded endomorphism ring (which is naturally a DG algebra – see p. 87 for a full definition), and define

$$\sqrt[V]{S\backslash -} : \mathtt{DGA}_S \to \mathtt{DGA}_k, \qquad R \mapsto (\underline{\mathrm{End}}(V) *_S R)^{\underline{\mathrm{End}}(V)}.$$

We will write R as $S\backslash R$ when we wish to emphasize that R is considered as an algebra over S. The algebra $\underline{\mathrm{End}}(V) *_S R$ is the free product (coproduct) in $\mathtt{DGA}_S$ and

$$\sqrt[V]{S\backslash R} = \{w \in \underline{\mathrm{End}}\, V *_S R \;:\; [w, m] = 0 \;\forall m \in \underline{\mathrm{End}}\, V\},$$

where the commutator is taken in the graded sense.

We have defined $\sqrt[V]{S\backslash R}$ to be a generalization of the "non-commutative representation scheme," and indeed it satisfies the DG analog of the main adjunction (thus generalizing a theorem of Cohn and Bergman – see Theorem 2.18 of [S]). This is our Theorem 69 on p. 92:

Theorem. *Let $R \in \mathtt{DGA}_S$, $B \in \mathtt{DGA}_k$, and (V, d) a chain complex of vector spaces of*

[1]This terminology has the potential to be confusing, since a *relative representation scheme* is not a relative scheme in the sense of algebraic geometry. Rather, its an affine k-scheme of relative representations. We regard R as an algebra over another DG algebra S, fix an S-module structure on V, and then consider the space of those representations that respect these S-structures. In geometric terms, if $h : \mathrm{Rep}_V(R) \to \mathrm{Rep}_V(S)$ is the morphism of schemes induced by the structure map $S \to R$, then the relative representation scheme can be identified with the fiber of h at the base point corresponding to the chosen (fixed) S-module structure on V.

finite total dimension. Then, we have an adjunction

$$\mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R}, B) = \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes_k B).$$

Analogously to the classical case, we define the relative (coordinate algebra of the) DG representation scheme of R as

$$(S\backslash R)_V = \left(\sqrt[V]{S\backslash R}\right)_{\mathrm{ab}}.$$

This is an object in $\mathtt{CDGA}_k$, the full subcategory of $\mathtt{DGA}_k$ consisting of commutative algebras. As one would hope, $(S\backslash -)_V$ satisfies a DG version of the adjunction that defines the representation scheme (Corollary 78 on p. 102):

Corollary. *Let $B \in \mathtt{CDGA}_k, R \in \mathtt{DGA}_S$, and (V, d) a complex of vector spaces of finite total dimension. Then, we have an adjunction*

$$\mathrm{Hom}_{\mathtt{CDGA}_k}\left((S\backslash R)_V, B\right) = \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes_k B).$$

1.0.3 Derived representation schemes

There are model category structures on $\mathtt{DGA}_S$ and $\mathtt{CDGA}_S$, with fibrations being surjections and weak equivalences being quasi-isomorphisms. The adjunction of the preceding corollary is in fact a Quillen adjunction, and this yields the following theorem (Theorem 82 on p. 106):

Theorem. *The total derived functors of $(-)_V$ and $\underline{\mathrm{End}}\, V \otimes_S$, which we will call $\mathsf{D}\,(-)_V$ and $\mathcal{E}$ (respectively), exist and form an adjoint pair*

$$\mathsf{D}\,(-)_V : \mathcal{H}o(\mathtt{DGA}_S) \rightleftarrows \mathcal{H}o(\mathtt{CDGA}_k) : \mathcal{E}.$$

By abuse of notation, we will apply the functor $\mathsf{D}\,(-)_V$ to elements of $\mathtt{DGA}_S$, by which we actually mean that we apply first the natural functor

$$\gamma_{\mathtt{DGA}_S} : \mathtt{DGA}_S \to \mathcal{H}o(\mathtt{DGA}_S)$$

defining the homotopy category (see the overview of model categories of Section 4.1), and only then the total derived functor $\mathsf{D}\,(-)_V$. If we start with an associative algebra A, we can regard it as a DG algebra concentrated in degree zero. We call the object $\mathsf{D}\,(A)_V$ the (coordinate algebra of the) derived representation scheme of A.

The preceding theorem appears quite theoretical, but in fact, since the model structures in question can be described very concretely, gives an explicit description of the derived representation scheme of an associative algebra A. Namely, one begins by resolving A with an almost free DG algebra F (which in the case when $S = k$ means that the underlying graded algebra of F is free – see p. 76 for a definition in the general case), so that one gets a quasi-isomorphism $F \to A$. Then, the algebra $(S\backslash F)_V \in \mathtt{CDGA}_k$ is a representative of the equivalence class of $\mathsf{D}\,(A)_V$ in $\mathcal{H}o(\mathtt{CDGA}_k)$, i.e., it is both fibrant and cofibrant, and

$$\gamma_{\mathtt{CDGA}_k} : \mathtt{CDGA}_k \to \mathcal{H}o(\mathtt{CDGA}_k), \qquad (S\backslash F)_V \mapsto \mathsf{D}\,(A)_V \,.$$

This whole procedure is completely analogous to the standard procedure for deriving functors in classical (abelian) homological algebra: one extends a functor on modules to one on chain complexes, then resolves a chosen module with a complex of free modules, and finally applies the (extended) functor to it. In the classical case, one recovers the original functor as the zeroth homology of the derived one; this is the situation in the non-abelian case, too (Theorem 84 on p. 107):

Theorem. *Let A be an associative algebra over S and V be concentrated in degree 0. Then,*

$$\mathrm{H}_0\mathsf{D}\,(S\backslash A)_V = (S\backslash A)_V.$$

Because homology is an invariant of quasi-isomorphism class, the graded algebra $\mathrm{H}_\bullet\mathsf{D}\,(A)_V$ is an invariant, depending only on V and A. Thus, the construction here can be regarded as a homology theory, which one can call *representation homology*.

Using a completely different approach, I. Ciocan-Fontanine and M. Kapranov defined in [CK2] a derived action space, $\mathrm{RAct}(A, V)$, whose coordinate algebra satisfies the same adjunction as $\mathsf{D}\,(A)_V$. Thus, by the Yoneda lemma,

$$\mathsf{D}\,(A)_V \cong k\,[\mathrm{RAct}(A, V)]$$

whenever $\mathrm{RAct}(A, V)$ is defined (which is when $S = k$ and V is concentrated in degree 0). Thus, $\mathsf{D}\,(A)_V$ can be regarded as a generalization of (the coordinate algebra of) $\mathrm{RAct}(A, V)$. Just as importantly, the explicitness of its definition will enable us to make concrete calculations and further constructions, most notably of a derived Van den Bergh functor.

Another advantage of our approach is that we actually prove that $\mathsf{D}\,(-)_V$ is a derived functor in the sense of Quillen, and – an even stronger condition – is part of a Quillen pair. This means that it has an adjoint functor, which is a strong condition with both theoretical and practical significance for applications.

1.0.4 The derived Van den Bergh functor

For an algebra $A \in \texttt{Alg}_k$ and a finite-dimensional vector space V, we call the map $\pi : A \to \mathrm{End}(V) \otimes_k A_V$ corresponding to the identity $A_V \to A_V$ under the adjunction 1.1 (on p. 3) the *universal representation* of A on V. Using π, we can regard $\mathrm{End}(V) \otimes_k A_V$ as a bimodule over A, or, equivalently, as a left module over A^e. Since A_V is commutative, the image of A_V under the natural inclusion $A_V \hookrightarrow \mathrm{End}(V) \otimes_k A_V$ lies in the center of this bimodule. Hence, we can regard $\mathrm{End}(V) \otimes_k A_V$ as an A^e-A_V-bimodule. M. Van den Bergh's functor (which, as mentioned previously, can be regarded as a concrete realization of the Kontsevich-Rosenberg principle) is defined in [VdB1] (see Section 3.3) by

$$(-)_V : \texttt{Bimod}_A \to \texttt{Mod}_{A_V}, \qquad M \mapsto M \otimes_{A^e} (\mathrm{End}(V) \otimes_k A_V) \, .$$

Our aim is to construct an appropriate derived functor of this functor, which would replace it in the case when A is not formally smooth. Proceeding in a similar vein to Van den Bergh's construction, we define the functors

$$\begin{aligned} \sqrt[V]{-} \quad &: \quad \texttt{DGBimod}_R \to \texttt{DGBimod}_{\sqrt[V]{R}}, \quad M \mapsto \left(\sqrt[V]{R} \otimes_k V^*\right) \otimes_R M \otimes_R \left(V \otimes_k \sqrt[V]{R}\right), \\ (-)_V \quad &: \quad \texttt{DGBimod}_R \to \texttt{DGMod}_{R_V}, \quad M \mapsto M_V := \sqrt[V]{M} \otimes_{\left(\sqrt[V]{R}\right)^e} R_V . \end{aligned}$$

The second of these functors is a DG generalization of the Van den Bergh functor. The first, meanwhile, is a non-commutative DG version. (As one would expect, there is a non-commutative non-DG version, although this functor does not – to the author's knowledge – appear in the literature.)

Recall that if R is a DG algebra and M, N are DG modules over R, then $\underline{\mathrm{Hom}}_R(M, N)$ carries a natural chain complex structure (see p. 131 for details). The functor $\sqrt[V]{-}$ satisfies an adjunction analogous to the one satisfied by the non-commutative representation scheme functor, which we also denoted by $\sqrt[V]{-}$ (see Theorem 107 on p. 132):

Theorem. *There is a canonical isomorphism of complexes*

$$\underline{\mathrm{Hom}}_{\left(\sqrt[V]{R}\right)^e}\left(\sqrt[V]{M}, N\right) \cong \underline{\mathrm{Hom}}_{R^e}\left(M, \underline{\mathrm{End}}\, V \otimes_k N\right).$$

From this we get the following generalization to the DG setting of a result of Van den Bergh (see [VdB1]):

Corollary. *There is a canonical isomorphism of complexes*

$$\underline{\mathrm{Hom}}_{R_V}\left(M_V, N\right) \cong \underline{\mathrm{Hom}}_{R^e}\left(M, (\underline{\mathrm{End}}\, V) \otimes_k N\right).$$

Now, to construct the (non-abelian) derived functors of Van den Bergh's functor, we follow the standard procedure in differential homological algebra. Recall that a DG module M over a DG algebra R has a semi-free resolution $L \to M$, which is a generalization of a free resolution for ordinary modules over ordinary algebras (see [FHT2]). Given an algebra $A \in \mathtt{Alg}_k$ and a complex M of bimodules over A, we first choose an almost free resolution $f : F \to A$ in $\mathtt{DGA}_k$ and consider M as a DG bimodule over F via f. Then, we choose a semi-free resolution $L(F, M) \to M$ in the category $\mathtt{DGBimod}_F$ and apply to $L(F, M)$ the functor $(-)_V$. The result of this construction is described by the following theorem.

Theorem. *Let A be an associative k-algebra, and let M be a complex of bimodules over A. The assignment $M \mapsto L(R, M)_V$ induces a well-defined functor between the derived categories*

$$\mathsf{D}(-)_V : \mathsf{D}(\mathtt{Bimod}_A) \to \mathsf{D}(\mathtt{DGMod}_{F_V}),$$

and this functor is independent of the choice of resolutions $F \to A$ and $L \to M$, up to auto-equivalence of $\mathsf{D}(\mathtt{DGMod}_{F_V})$ inducing the identity on homology.

As one would hope, in the case when M is concentrated in degree zero (i.e., is simply a bimodule over A), we have $\mathrm{H}_0\mathsf{D}(M)_V \cong M_V$.

As an alternative to taking the more concrete approach, it is possible to prove the existence of the derived Van den Bergh functor using Quillen's theorem on adjunctions, just as we did with the derived representation scheme. This is done by Theorem 111 (on p. 136).

1.0.5 Calculations

Drawing inspiration from the geometric representation theory of quivers (for details, see Subsection 7.1.1), it is actually possible to obtain an explicit finite presentation[2] $\tilde{F}$ for $\mathsf{D}\left(A\right)_V$ whenever one has an explicit finite presentation for an almost free resolution $F \to A$. This construction is a generalization of a classical construction of a presentation for A_V, which is described in Subsection 3.1.2. An analogous theorem exists for the derived Van den Bergh functor.

This allows one to calculate homology with the aid of a computer. In many cases, this is too computationally intensive to be practical, but it is not difficult to find (nontrivial) examples where it is indeed feasible.

These results are significant in part because explicit computations of examples for derived functors in homotopical algebra are rare (owing to the complexity of most constructions involved).

[2]More precisely, one obtains an explicitly presented algebra $\tilde{F} \in \mathtt{CDGA}_k$ such that $\gamma_{\mathtt{CDGA}_k}(\tilde{F}) \cong \mathsf{D}\left(A\right)_V$.

1.0.6 Alternative proofs

There are several places in this thesis where alternative proofs are given. The main reason for this is that the present work aims to establish tools which can be used for various applications, and some approaches lend themselves better to given applications than others. Moreover, there are several potential avenues for extending the results presented here, and having different proofs to choose from may be helpful to this end.

For example, we give two different proofs for the existence of the derived functor $\mathsf{D}\,(-)_V$, one via Quillen's adjunction theorem, and another via the machinery of M-homotopies. The first proof is significant because it confirms that the construction indeed gives the conceptually "correct" definition of a derived functor, and – even more importantly – provides an adjoint to the derived functor on the level of homotopy categories. The second approach, meanwhile, allows for the results here to be applied in settings where suitable model structures don't exist, such as for establishing a relationship with cyclic homology and studying derived $GL(V)$-invariants.

1.0.7 Concerning sign rules

The calculations presented in the thesis are given in detail; in particular, sign rules have been written out explicitly (contrary to the recent custom in the field of simply writing "$\pm$" and verifying results only up to signs). While selecting the appropriate sign rule in each case proved difficult, the benefits of knowing the precise signs are not inconsiderable, since this is what has allowed specific examples to be computed.

1.0.8 Overview of contents

In Chapter 3, we mostly survey background results, including the definition of the representation scheme, Cohn and Bergman's results on n-matrix reduction, the Kontsevich-Rosenberg principle, and Van den Bergh's functor. For many results, we provide substantially more detailed proofs than those currently available in the literature.

In Chapter 4, we briefly recall Quillen's formalism of model categories, and then focus on carefully describing the model structures we will need in the subsequent chapters, most notably on categories of DG algebras; we also discuss almost free resolutions (giving explicit ways in which suitable resolutions can be obtained) and M-homotopies, a useful tool for proving that various constructions with DG algebras are independent of the choice of resolution. This material is mostly present (in one form or another) in the literature, but here we gather it together systematically and present it with a level of detail that will enable concrete calculations to be made later.

Chapter 5 contains the main results on derived representation schemes. Here, we construct DG representation schemes (generalizing results of Bergman and Cohn) and then provide two different proofs of the existence of derived representation schemes, one using Quillen's theorem on adjunctions in model categories, and another using M-homotopies.

Chapter 6 opens with an example of how the *ad hoc* approach to the Kontsevich-Rosenberg principle transfers to the derived setting in the case of vector fields; this example is somewhat tedious, but is important as it validates the approach. Next, we introduce the DG Van den Bergh functor and prove the main adjunction, following which we construct the derived Van den Bergh functor. Finally, we discuss in general terms an alternative approach that could be taken via semidirect products.

In the concluding chapter, Chapter 7, we present the theorem allowing one to produce finite presentations of the derived representation scheme when given a finite presentation of an almost free resolution of an associative algebra. To illustrate the simplicity of the approach, we give several examples and present some computer calculations of homology.

1.0.9 A significant application (to appear elsewhere)

One significant application of the results presented in this thesis, due to Yuri Berest and Ajay Ramadoss, is a generalization of a well-known theorem of Procesi, which states that the image of the natural trace map[3]

$$\mathrm{Tr} : S^{\bullet}(A/[A,A]) \to A_V$$

generates the subalgebra of invariants $(A_V)^{GL(V)}$, where the action of $GL(V)$ corresponds to conjugation of representations (for details on the classical Procesi theorem, see Subsection 3.3.2). In the context of the Kontsevich-Rosenberg principle, the Procesi theorem supports the view that $S^{\bullet}(A/[A,A])$ should be regarded as the algebra of "functions" on the non-commutative space $\mathrm{Spec}\, A$.

It has been proposed that cyclic homology $\mathrm{HC}_{\bullet}$ should play the role of a derived "global sections" functor in the non-commutative setting. Indeed, for

[3]Note that $A/[A,A]$ is not by itself an algebra, since $[A, A]$ is the vector subspace of commutators, not the commutator ideal $A[A, A]A$.

any $S\backslash R \in \mathtt{DGA}_S$, it is possible to construct a natural map

$$\mathrm{Tr}\left((S\backslash R)_V\right) : \mathrm{HC}_{n-1}(S\backslash R) \to \mathrm{H}_n\mathsf{D}\left(S\backslash R\right)_V .$$

Moreover, these maps (taken over varying V) assemble to provide a derived "stabilization" (i.e., limit as $\dim V \to \infty$) version of Procesi's theorem. In particular, this shows that there is a very close relationship between cyclic homology and representation homology. Details can be found in [BKR1].

1.0.10 Some future directions

There are several directions in which this work could be extended. On the theoretical side, it is likely that an appealing generalization to DG categories can be developed; besides restating the present results in that (quite popular) language, this may yield a computational tool for studying algebras with no classical representation theory (such as the Weyl algebras A_n). This direction is especially promising in light of the results concerning explicit presentations (in particular, Corollary 122 of Chapter 7), which drew inspiration from results on quivers but only used the case of a single-vertex quiver, a multiple-vertex version likely corresponding to precisely such a generalization to DG categories.

Another direction could involve using operads to study related functors. For example, I. Ciocan-Fontanine and M. Kapranov define a "derived Hilbert scheme" in [CK1] using similar arguments to those of [CK2], but with some added operadic machinery. It is possible that a similar operadic modification of the present approach would yield simple constructions of derived Hilbert schemes and other such objects.

More concretely, it would be interesting to calculate a larger number of examples of representation homology, as this could yield further conjectures. In particular, no examples have yet been calculated in cases when V is not concentrated in degree 0, since this turns the problem from a finitely generated linear problem into a finitely generated non-linear one (which translates into an infinitely generated linear one). Finding a workaround would open the possibility of further exploring the (still mysterious) case of V not concentrated in degree 0.

CHAPTER 2

NOTATION CONVENTIONS

Throughout, k is a field of characteristic 0. The notation A is usually reserved for a unital associative algebra over k; often, we will further require that A be finitely generated. We use R as the standard notation for a (not necessarily finitely generated) unital associative differential graded (DG) algebra over k. All chain complexes and DG algebras, unless specified otherwise, are homologically graded (i.e., have differential d of degree -1).

We denote by $\mathtt{Alg}_k$ the category of unital associative algebras over k, by $\mathtt{CommAlg}_k$ its full subcategory of commutative algebras, and by $\mathtt{GrAlg}_k$ the category of graded unital associative algebras over k. When we mean to restrict a graded category to the full subcategory of objects concentrated in non-negative degree, we add a $+$ superscript; thus, we write the category of DG algebras over k concentrated in non-negative degrees as $\mathtt{DGA}_k^+$ to distinguish it from $\mathtt{DGA}_k$, which contains all (unital associative) DG algebras.

Whenever the sign $\otimes$ appears with no subscript, it is implied that the tensor product is over the base field k. The same holds for the free product sign $*$.

Whenever working in graded categories, we use the Koszul sign rule, which stipulates that whenever two symbols are transposed, a sign emerges corresponding to the product of the symbols' degrees. Thus, the graded commutator is $[x, y] = xy - (-1)^{|x||y|}yx$. More subtly, if $f : A \to B$ and $g : C \to D$ are graded linear maps (of graded vector spaces), then we have

$$f \otimes g : A \otimes C \to B \otimes D, \qquad a \otimes c \mapsto f(a) \otimes (-1)^{|g|\cdot|a|} g(c).$$

Therefore, for another graded linear map $f' \otimes g' : A \otimes C \to B \otimes D$, we have

$$(f \otimes g) \circ (f' \otimes g')(a \otimes c) = (-1)^{|g'||a|+|g|(|f'|+|a|)} f(f'(a)) \otimes g(g'(c)),$$

or, equivalently,

$$(f \otimes g) \circ (f' \otimes g') = (-1)^{|f'||g|} (f \circ f') \otimes (g \circ g') \, .$$

The above identities make sense in the case when all elements (including the graded linear maps) involved are homogeneous. We will often write formulas that concern homogeneous elements without explicitly specifying that these formulas refer only to that case; the reader will be able to determine this from the context, most significantly from the taking of degrees of elements.

We write the endomorphism algebra of a vector space V as $\mathrm{End}(V)$. When V is a chain complex, we write the DG algebra of endomorphisms (for the definition, see p. 87) as $\underline{\mathrm{End}}\, V$. We will omit the parentheses around V in this case because the underline resolves the risk of confusion in expressions of the form $\underline{\mathrm{End}}\, V \otimes R$.

Throughout, we use the notation $(-)_V$ and $\mathsf{D}\,(-)_V$ in two different contexts: for representation schemes, and for the Van den Bergh functor. Which is meant is clear from the context, since the representation scheme functor is applied to algebras, while the Van den Bergh functor to bimodules. In a similar vein, we use the notation $(-)_V$ and $\sqrt[V]{-}$ for both the classical (ungraded) constructions and for their DG generalizations.

CHAPTER 3

REPRESENTATION SCHEMES AND THE KONTSEVICH-ROSENBERG PRINCIPLE

3.1 Representation Schemes

Let A be a finitely generated unital associative algebra over a field k of characteristic 0 and V a finite-dimensional k-vector space. The *representation scheme* of A in V, denoted $\mathrm{Rep}_V A$, is the scheme universally parametrizing all algebra morphisms $A \to \mathrm{End}(V)$.

An early motivation for studying representation schemes was to use geometric methods in the study of the set of representations of an associative algebra (see, for example, H. Kraft's work [K2] or the more recent survey [CB2]). Because $\mathrm{Rep}_V A$ is the natural way to endow this set with a geometric structure, there is a close connection between the representation theory of A and the geometric properties of $\mathrm{Rep}_V A$.

3.1.1 Definition of representation schemes

For B a commutative algebra over k, a family of representations of A in V parametrized by $X = \mathrm{Spec}(B)$ is defined as a morphism of k-algebras

$$\phi : A \to \mathrm{End}(V) \otimes B.$$

To interpret this definition, note that $\mathrm{End}(V) \otimes B$ can be regarded as the algebra of B-valued matrices; for every closed point of X, we get a representation of A.

Moreover, this family of representations varies algebraically over X, since the representations' coordinates are determined by algebraic functions on X (i.e., elements of B).

A universal such family would be a scheme $Y = \mathrm{Rep}_V A$ together with a k-algebra morphism $\pi : A \to \mathrm{End}(V) \otimes k[Y]$ such that for any family ϕ as above, there would exist a unique morphism $f : X \to Y$ such that $(\mathrm{id}_{\mathrm{End}(V)} \otimes f^*) \circ \pi = \phi$, where $f^* : k[Y] \to k[X]$ is the morphism induced by f.

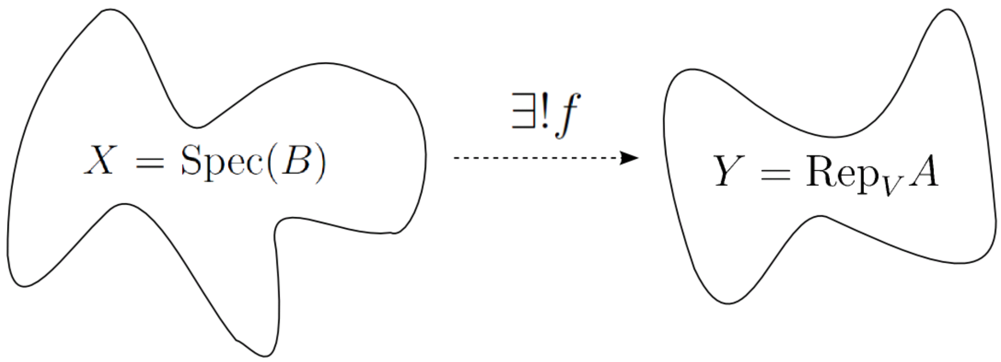

Figure 3.1: Definition of $\mathrm{Rep}_V A$

In other words, the **representation scheme** $\mathrm{Rep}_V A$ is defined to be the spectrum of the k-algebra (if it exists) representing the functor

$$\mathrm{Rep}_V^A : \mathtt{CommAlg}_k \to \mathtt{Sets}, \qquad B \mapsto \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes B). \tag{3.1}$$

We will see (in the following subsection) that the functor Rep_V^A is representable; we will denote the representing algebra by A_V. By the Yoneda lemma, A_V (and thus also its spectrum $\mathrm{Rep}_V A$) is unique up to isomorphism.

The existence and uniqueness of A_V have been known for some time; see, for example, [A].

The morphism π, which is called the **universal representation** of A in V, is the element of

$$\mathrm{Rep}_V^A(A_V) = \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes A_V)$$

corresponding to the identity $\mathrm{id} : A_V \to A_V$.

Remark 1. *The universal representation establishes a natural bijection between representations of A in V and closed points of $\mathrm{Rep}_V A$. Specifically, a closed point $x \in \mathrm{Rep}_V A$ defines a morphism $\lambda_x : A_V \to k$, and thus induces a representation of A defined by*

$$(\mathrm{id}_{\mathrm{End}(V)} \otimes \lambda_x) \circ \pi : A \to \mathrm{End}(V).$$

Conversely, a representation of A can be regarded as a morphism $\rho : A \to \mathrm{End}(V) \otimes k$, and this yields by the universal mapping property a morphism $\lambda_x : A_V \to k$ defining a closed point $x \in \mathrm{Rep}_V A$. These constructions are mutually inverting.

Remark 2. *In much of the recent literature, the notation $\mathrm{Rep}_n A$ (where n is the dimension of V) is used for what we call $\mathrm{Rep}_V A$. Of course, the representation schemes of an algebra in two isomorphic vector spaces are isomorphic; however, we have chosen the notation with V for two reasons: first, it is more consistent with other notation that we have need of (such as writing A_V for the coordinate ring of $\mathrm{Rep}_V A$), and second, it will allow us to generalize from V a vector space to V a complex in subsequent chapters without awkward shifts in notation.*

3.1.2 Explicitly presenting A_V

When given an explicit presentation for the finitely generated algebra A, it is possible to produce an explicit presentation for A_V. Besides proving representability of the functor Rep_V^A and allowing one to compute examples, this process can be seen as providing an alternative, more explicit (and, in some ways, more geometrically illuminating) definition for $\mathrm{Rep}_V A$.

Following [G1], Let $F = \langle x_1, ..., x_m \rangle$ be the free k-algebra on m generators and V an n-dimensional vector space. Fixing a basis for V, we have a natural identification $\mathrm{End}(V) = \mathbb{M}_n(k)$, where $\mathbb{M}_n(k)$ is the algebra of $n \times n$ matrices with coefficients in k.

Now, an action of F on V is determined by assigning to each x_i an element of $\mathbb{M}_n(k)$. Thus, the n-dimensional representations of F are naturally parametrized by

$$\underbrace{\mathbb{M}_n(k) \oplus ... \oplus \mathbb{M}_n(k)}_{m \text{ times}},$$

which we identify with affine n^2m-space, k^{n^2m}.

Now, for a unital associative algebra A with generators $\{y_1, ..., y_m\}$, we have a surjective morphism of k-algebras

$$s : F \twoheadrightarrow A, \qquad y_i \mapsto x_i, \ 1 \leq i \leq m.$$

A representation ρ of A determines a representation ρs of F :

$$F \overset{s}{\twoheadrightarrow} A \xrightarrow{\rho} \mathbb{M}_n(k).$$

Meanwhile, a representation κ of F has the form ρs precisely when $\kappa|_{\ker(s)} = 0$. Therefore, we have a natural bijection:

$$\{\text{representations } \kappa \text{of } F \text{with } \kappa|_{\ker(s)}\text{=0}\} \longleftrightarrow \{\text{representations of } A\}.$$

The requirement that $\kappa|_{\ker(s)} = 0$ gives, for each $x \in \ker(s)$, a family of n^2 polynomial equations in the coordinates of $\mathbb{M}_n(k)$. Taking the union of these equations over all $x \in \ker(s)$, we obtain a (not necessarily reduced) subscheme of $k^{n^2 m}$ parametrizing representations of A. We call the coordinate algebra of this scheme $A_{V,s}$.

Theorem 3. *The algebra $A_{V,s}$ represents the functor* Rep_V^A.

Proof. Begin with the case of a free algebra F with generators $\{x_1, ..., x_m\}$ and trivial presentation $s : F \to F$, where $s = \mathrm{id}_F$. Then $F_{V,s}$ is the free commutative k-algebra on $n^2 m$ generators

$$\{x_i^{jl} \quad : \quad 1 \leq i \leq n, \quad 1 \leq j, l \leq m\}.$$

Fix a basis for V. We have a standard basis E_{jl} for $\mathbb{M}_n(k)$ and a dual basis E^{jl} for $\mathbb{M}_n(k)^*$. Now, define

$$\pi_F : F \longrightarrow \mathbb{M}_n(k) \otimes F_{V,s}, \qquad x_i \mapsto \sum_{j,l} E_{jl} \otimes x_i^{jl}.$$

Then, for every family $\phi : F \to \mathrm{End}(V) \otimes B$, there exists (by the universal property of the free commutative algebra $F_{V,s}$) a unique morphism

$$f : F_{V,s} \to B, \qquad x_i^{jl} \mapsto (E^{jl} \otimes \mathrm{id}_B) \cdot \phi(x_i)$$

satisfying $\phi = (\mathrm{id}_{\mathrm{End}(V)} \otimes f) \circ \pi_F$. Thus, $F_{V,s}$ is indeed the coordinate algebra of the representation scheme of F, and we are justified in calling it simply F_V.

To generalize to an algebra A with a resolution $s : F \to A$, consider $p : F_V \to A_{V,s}$, the surjection corresponding to the embedding $\mathrm{Spec}(A_{V,s}) \hookrightarrow \mathrm{Spec}(F_V)$. Form the composition

$$F \xrightarrow{\pi_F} \mathbb{M}_n(k) \otimes F_V \xrightarrow{\mathrm{id}_{\mathrm{End}(V)} \otimes p} \mathbb{M}_n(k) \otimes A_{V,s},$$

which (by construction of p) is 0 on $\ker(s) \subseteq F$ and thus lifts to a morphism $\pi_{A,s} : A \to \mathbb{M}_n(k) \otimes A_{V,s}$.

To verify that $\pi_{A,s}$ has the desired universal property, take any morphism $\phi : A \to \mathbb{M}_n(k) \otimes B$, and extend it to

$$\phi \circ s : F \to \mathbb{M}_n(k) \otimes B.$$

Then, we have a unique map $f_F : F_V \to B$ such that $\phi \circ s = (\mathrm{id}_{\mathrm{End}(V)} \otimes f_F) \circ \pi_F$. Since f_F is (by construction) 0 on $\ker(p)$, it lifts to a unique morphism $f_A : A_{V,s} \to B$ satisfying $\phi = (\mathrm{id}_{\mathrm{End}(V)} \otimes f_A) \circ \pi_{A,s}$.

$\square$

By the Yoneda lemma, any two such algebras $A_{V,s}$ are isomorphic, and thus we are justified in simply writing A_V. Note that in particular, this means that the construction above does not depend (up to isomorphism) on the choice of the rank of the free algebra F and morphism $s : F \twoheadrightarrow A$.

Remark 4. *We will see another (more elegant) proof of the representability of the functor Rep_V^A in Section 3.2. We will also prove (using completely different means) a deeper fact (Corollary 122 on p. 148) which will have the present proposition as an easy consequence.*

Example 5. *Let V be a k-vector space of dimension 2. Taking the presentation*

$$\mathbb{C}\langle X, Y\rangle \twoheadrightarrow \mathbb{C}[x,y],$$

$$X \mapsto x, \quad Y \mapsto y$$

and applying the procedure outlined in the beginning of this subsection, we obtain

$$(\mathbb{C}[x,y])_V \cong \frac{\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]}{(R_{11}, R_{12}, R_{21}, R_{22})},$$

where

$$R_{11} = x_{11}y_{11} + x_{12}y_{21} - y_{11}x_{11} - y_{12}x_{21} \qquad R_{12} = x_{11}y_{12} + x_{12}y_{22} - y_{11}x_{12} - y_{12}x_{22}$$

$$R_{21} = x_{21}y_{11} + x_{22}y_{21} - y_{21}x_{11} - y_{22}x_{21} \qquad R_{22} = x_{21}y_{12} + x_{22}y_{22} - y_{21}x_{12} - y_{22}x_{22}$$

This ring is Cohen-Macaulay, but not Gorenstein. It has dimension 6.

Corollary 6. *The assignment $A \mapsto A_V$ defines a functor $(-)_V : \mathtt{Alg}_k \to \mathtt{CommAlg}_k$, and there is an adjunction*

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}(A_V, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes B).$$

Proof. It is a general categorical fact that a universal construction that always has a solution – and Theorem 22 means this is our case – gives rise to a pair of adjoint functors (see Theorem 2 of Chapter IV in [McL]). Nevertheless, let's define the functor $(-)_V$ explicitly. Given a morphism of unital associative algebras $f : A_1 \to A_2$, we get a diagram

$$\begin{tikzcd} A_1 \arrow[r, "\pi_1"] \arrow[d, "f"] & \mathrm{End}(V) \otimes (A_1)_V \\ A_2 \arrow[r, "\pi_2"] & \mathrm{End}(V) \otimes (A_2)_V \end{tikzcd}$$

Now, applying the universal property of $\pi_1 : A_1 \to \mathrm{End}(V) \otimes (A_1)_V$ to the morphism $\pi_2 \circ f$, we obtain a unique morphism $g : (A_1)_V \to (A_2)_V$ such that $\mathrm{id}_{\mathrm{End}(V)} \otimes g$ makes the diagram commute. Define $(f)_V = g$. $\square$

Corollary 7. *The algebra A_V is isomorphic to the commutative algebra on generators $\{a^{jl} : a \in A, 1 \leq j, l \leq n\}$ with relations*

$$\alpha a^{jl} = (\alpha a)^{jl}\ \forall \alpha \in k, \quad a^{jl} + b^{jl} = (a+b)^{jl}, \quad \sum_t a^{jt} b^{tl} = (ab)^{jl}, \quad 1^{jl} a^{j'l'} = \delta^{lj'} a^{j'l'}.$$

(Cf. pp. 2-3 of [VdB1].)

Proof. Pick a resolution $s : F \to A$. Following Theorem 3, let's present $A_V := A_{V,s}$ through $p : F_1 \to A_V$, where F_1 is a free commutative algebra on generators

$\{x_i^{jl}\}$. Denote $R_1 := \ker(p)$. The corollary states that this algebra is isomorphic to one presented as F_2/R_2, where F_2 is the free commutative algebra on generators $\{a^{jl}\}$ and R_2 is the ideal generated by the listed relations. To prove that $F_1/R_1 \cong F_2/R_2$, it is sufficient to construct morphisms $f : F_1 \to F_2$ and $g : F_2 \to F_1$ such that

$$f(R_1) \subseteq R_2, \qquad g(R_2) \subseteq R_1,$$
$$g \circ f \equiv \mathrm{id}_{F_1} (\mathrm{mod}\, R_1), \qquad f \circ g \equiv \mathrm{id}_{F_2} (\mathrm{mod}\, R_2).$$

Define f by sending $x_i^{jl} \mapsto \iota\left(p(x_i^{jl})\right)$, where $\iota : A \to F_2$ is the natural inclusion $a \mapsto a$. To determine g on an element a^{jl}, first select an element $b \in s^{-1}(a)$, where $s : F \to A$ is as above, and then set $g(a) = m^{jl}(b)$, where $m_{ij} : F \to F_1$ is the linear map sending

$$x_{i_1} x_{i_2} \ldots x_{i_t} \mapsto \sum_{j_1, \ldots, j_{t-1}} x_{i_1}^{jj_1} x_{i_2}^{j_1 j_2} \ldots x_{i_t}^{j_{t-1} l}.$$

It is a straightforward verification to see that these f, g satisfy the required conditions. $\square$

Remark 8. *This corollary may be interpreted as saying that A_V can be regarded as the commutative algebra of elements which behave like entries of matrices whose (matrix) multiplication reflects the structure of A.*

3.1.3 The action of $GL(V)$ on $\mathrm{Rep}_V A$

Let $GL(V) \subset \mathrm{End}(V)$ be, as usual, the group of invertible endomorphisms of V. The natural left action by conjugation of $GL(V)$ on $\mathrm{End}(V)$ induces a left action on $\mathrm{End}(V) \otimes A_V$ given by

$$g \cdot (\phi \otimes x) = g \phi g^{-1} \otimes x, \qquad g \in GL(V),\ \phi \in \mathrm{End}(V),\ x \in A_V$$

and extended by linearity. Now for $g \in GL(V)$, define

$$\pi_g : A \to \mathrm{End}(V) \otimes A_V \qquad a \mapsto g \cdot \pi(a),$$

where π is the universal representation introduced in Subsection 3.1.1. By the universal property of A_V, there exists a unique morphism of schemes

$$f_g : \mathrm{Rep}_V A \to \mathrm{Rep}_V A$$

such that

$$\pi_g = (\mathrm{id}_{\mathrm{End}(V)} \otimes f_g^*) \circ \pi,$$

where $f_g^* : A_V \to A_V$ is the induced morphism of k-algebras. The map f_g (and thus f_g^*, too) is an isomorphism, since it has inverse $f_{g^{-1}}$.

The assignment $g \mapsto f_g \in \mathrm{Aut}(\mathrm{Rep}_V A)$ defines a group homomorphism $GL(V) \to \mathrm{Aut}(\mathrm{Rep}_V A)$, i.e. a natural left action of $GL(V)$ on $\mathrm{Rep}_V A$. Dually, the assignment $g \mapsto f_g^* \in \mathrm{Aut}(A_V)$ defines a group homomorphism $GL(V)^{\mathrm{op}} \to \mathrm{Aut}(A_V)$, i.e. a natural right action of $GL(V)$ on A_V.

Remark. *To see the geometric meaning of this action, recall from Subsection 3.1.1 the one-to-one correspondence via π between closed points of $\mathrm{Rep}_V A$ and n-dimensional representations of A. Let $\rho : A \to \mathrm{End}(V)$ be a representation and $x_\rho \in \mathrm{Rep}_V A$ the corresponding closed point. Then,*

$$g \cdot x_\rho = x_{g \cdot \rho},$$

where

$$g \cdot \rho : A \to \mathrm{End}(V), \qquad a \mapsto g\rho(a)g^{-1}$$

is the representation isomorphic to ρ given by conjugation by g.

Proposition 9. *Two representations ρ, ρ' of A are equivalent if and only if their corresponding closed points $x_\rho, x_{\rho'} \in \mathrm{Rep}_V A$ are in the same orbit of the action by $GL(V)$.*

The following two theorems (whose proofs can be found on pp. 91-94 of [K1]) demonstrate the close relationship between the representation theory of A and the geometry of $\mathrm{Rep}_V A$.

Theorem 10. *There is a one-to-one correspondence between closed orbits of the $GL(V)$ action on $\mathrm{Rep}_V A$ and semisimple representations of A in V.*

Theorem 11. *Let ρ be a representation, x_ρ the corresponding closed point of the representation scheme, and M_ρ the A-module determined by ρ on V. Then, the orbit of x_ρ under the $GL(V)$ action on $\mathrm{Rep}_V A$ is open if and only if M_ρ has no nontrivial self-extensions, i.e.* $\mathrm{Ext}^1_A(M_\rho, M_\rho) = 0$.

Because choice of a basis gives a natural identification of $GL(V)$ with $GL_n(k)$, the action described in this section can be regarded as a GL_n action (and, in the literature, usually is).

3.2 n-Matrix Reduction and Non-commutative Representation Schemes

In the early 1970s, G. M. Bergman and P.M. Cohn ([B1, C2]) studied a functor (which Cohn called *n-matrix reduction*) which can be regarded as the natural non-commutative analog of the functor Rep_V^A. Bergman's main motivation for studying the functor was to construct counterexamples in homological algebra, while Cohn used it to define spectra of non-commutative rings. We will use the notation $\sqrt[V]{A}$ for this functor.

The functor $\sqrt[V]{-}$ not only allows for a more elegant proof of the representability of Rep_V^A, but also will play a critical role in our construction of derived representation schemes in Chapter 5.

3.2.1 Universal adjunctions in $\texttt{PMod}_R$ and the functor $\sqrt[V]{A}$

Let R be an associative k-algebra. In [B1], Bergman develops a theory for how one can adjoin to the category of finitely generated projective right modules over R, which we denote $\texttt{PMod}_R$, certain additional morphisms and relations. Thus, one aims to form an R-algebra S which has a similar module theory to R (excepting the adjoined morphisms and relations), and is universal with respect to other such R-algebras.

Let us recall some definitions which will allow us to formulate the key result more concretely. A k-linear category is one whose Hom-sets are k-vector spaces such that the composition maps are bilinear; a k-linear functor is one that maps Hom-sets by k-linear maps.[1] For S an algebra over a k-algebra R, there is a natural functor

$$\otimes S : \texttt{PMod}_R \to \texttt{PMod}_S, \quad M \mapsto M \otimes_R S.$$

In [B1], Bergman proves the following (*op. cit.*, Theorem 3.3):

Theorem 12. *Let us be given a k-algebra R, small k-linear categories $\mathcal{A}$ and $\mathcal{B}$, and k-linear functors $\mathcal{F} : \mathcal{A} \to \texttt{PMod}_R$ and $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{G}$ is a bijection on objects. Then, there exists an R-algebra S and a k-linear functor $\mathcal{H} : \mathcal{B} \to \texttt{PMod}_S$ that make the*

[1] In other words, k-linear categories are simply categories enriched over the category of k-vector spaces, and k-linear functors are the corresponding enriched functors.

following diagram commute:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{G}"'] & \mathtt{PMod}_R \arrow[d, "\otimes S"] \\ \mathcal{B} \arrow[r, dashed, "\mathcal{H}"] & \mathtt{PMod}_S \end{tikzcd}$$

and which are universal for this property: i.e., for any R-algebra S' and functor $\mathcal{H}'$: $\mathcal{B} \to \mathtt{PMod}_{S'}$ making the analogous diagram commute, there is a unique morphism of R-algebras $f : S \to S'$ such that $\mathcal{H}' = \mathcal{T} \circ \mathcal{H}$, where

$$\mathcal{T} : \mathtt{PMod}_S \to \mathtt{PMod}_{S'}, \quad M \mapsto M \otimes_S S'$$

is the tensor functor induced by the S-algebra structure $f : S \to S'$.

The categories $\mathcal{A}$ and $\mathcal{B}$ serve as a kind of labeling mechanism. To better understand the statement of the theorem, let's consider the two key examples:

1. Let $\mathcal{A}$ and $\mathcal{B}$ have two objects each, $X_\mathcal{A}, Y_\mathcal{A} \in \mathrm{Ob}(\mathcal{A})$ and $X_\mathcal{B}, Y_\mathcal{B} \in \mathrm{Ob}(\mathcal{B})$ with $\mathcal{G}(X_\mathcal{A}) = X_\mathcal{B}$ and $\mathcal{G}(Y_\mathcal{A}) = Y_\mathcal{B}$. As for the Hom-sets, set

$$\mathrm{Aut}(X_\mathcal{A}) = \mathrm{Aut}(X_\mathcal{B}) = \mathrm{Aut}(Y_\mathcal{A}) = \mathrm{Aut}(Y_\mathcal{B}) = k$$

$$\mathrm{Hom}(X_\mathcal{A}, Y_\mathcal{A}) = 0, \qquad \mathrm{Hom}(X_\mathcal{B}, Y_\mathcal{B}) = k$$

$$\mathrm{Hom}(Y_\mathcal{A}, X_\mathcal{A}) = \mathrm{Hom}(Y_\mathcal{B}, X_\mathcal{B}) = 0$$

   In this case, we are adjoining a homomorphism $\mathcal{F}(X_\mathcal{A}) \to \mathcal{F}(Y_\mathcal{A})$.

2. Take the same setup as in the first example, with just two exceptions:

$$\mathrm{Hom}(X_\mathcal{A}, Y_\mathcal{A}) = k, \qquad \mathrm{Hom}(X_\mathcal{B}, Y_\mathcal{B}) = 0.$$

   Consider the morphism $\mathbf{1} \in \mathrm{Hom}(X_\mathcal{A}, Y_\mathcal{A}) \cong k$, whose image under $\mathcal{F}$ we will call f. With this construction, we are adjoining the relation that sets $f = 0$.

As one can see from these two examples, the objects of $\mathcal{A}$ (which are in bijection with those of $\mathcal{B}$) serve to label some collection of objects in $\mathtt{PMod}_R$. Then, any morphisms in $\mathcal{B}$ which do not come via $\mathcal{G}$ from $\mathcal{A}$ serve to adjoin universal morphisms between the corresponding objects of $\mathtt{PMod}_R$, while any morphisms in $\mathcal{A}$ which are sent to 0 by $\mathcal{G}$ serve to eliminate the corresponding morphisms of $\mathtt{PMod}_R$.

By appropriately configuring the setup $\mathcal{A} \xrightarrow{\mathcal{G}} \mathcal{B}$, it is possible to adjoin universal isomorphisms, universal one-sided-invertible maps, universal idempotents, and so on.

Definition. *Let A be a unital associative algebra over k. In the statement of Theorem 12, let $\mathcal{A}$ and $\mathcal{B}$ each have a single object, $X_{\mathcal{A}}$ and $X_{\mathcal{B}}$, with $\mathrm{Aut}(X_{\mathcal{A}}) \cong k$ while $\mathrm{Aut}(X_{\mathcal{B}}) \cong A$. Furthermore, set $R = k$ and let $\mathcal{F}(X_{\mathcal{A}})$ be the free k-module (i.e., k-vector space) V of rank (i.e., dimension) n. Then, we call the resulting A-algebra S the n-**matrix reduction** of A, and we denote it by $\sqrt[V]{A}$.*

Put another way, $\sqrt[V]{A}$ is obtained by adjoining a universal morphism $A \to \mathrm{End}(V)$. Thus, $\sqrt[V]{A}$ comes with a distinguished morphism

$$\tilde{\pi} : A \to \mathrm{End}_{\sqrt[V]{A}}(V \otimes \sqrt[V]{A}) \cong \mathrm{End}(V) \otimes \sqrt[V]{A}.$$

We have chosen the notation $\tilde{\pi}$ to suggest analogy with the universal representation; as we will see shortly, this analogy can be made quite precise (Corollary 17 on p. 33).

Remark. *In [B1], Bergman uses the notation $\mathfrak{w}_n(A)$ for $\sqrt[V]{A}$. His primary motivation in studying this functor comes from the fact that $\mathfrak{w}_n(A)$ inherits certain properties of A but not others, and thus is useful in constructing counterexamples. More recently, in [LBW], the notation $\sqrt[n]{A}$ was introduced for the n-matrix reduction. We adopt the* *notation* $\sqrt[V]{A}$ *for the same reasons for which we chose to use* $\mathrm{Rep}_V A$ *(rather than the more conventional* $\mathrm{Rep}_n A$*) for the representation scheme.*

3.2.2 The relationship between the functors $\sqrt[V]{-}$ and $(-)_V$

In this subsection, we will study the special relationship between the functors $\sqrt[V]{-}$ and $(-)_V$. Most importantly, this will lay the groundwork for the next subsection, which will give a construction for A_V which is at the same time algebraically explicit and independent of the choice of presentation of the algebra A.

Theorem 13. *There is an adjunction*[2]

$$\mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A}, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes B).$$

Proof. This is simply a restatement of the definition of $\sqrt[V]{A}$ (in terms of Theorem 12). $\square$

Remark 14. *Let's unpack this adjunction. A morphism* $f : A_V \to B$ *corresponds under the adjunction to the morphism* $(\mathrm{id}_{\mathrm{End}(V)} \otimes f) \circ \pi$*. Conversely, given a morphism* $g : A \to \mathrm{End}(V) \otimes B$*, first form* $\iota *_k g$*, where* $\iota : \mathrm{End}(V) \to \mathrm{End}(V) \otimes B$ *is defined by* $\iota(m) = m \otimes 1$*. Then, restrict to invariants and identify* $(\mathrm{End}(V) \otimes B)^{\mathrm{End}(V)} = B$*, yielding the morphism* $(\iota *_k g)^{\mathrm{End}(V)} : \sqrt[V]{A} \to B$*.*

Next, recall the abelianization functor $(-)_{\mathrm{ab}} : \mathtt{Alg}_k \to \mathtt{CommAlg}_k$, sending $A \mapsto {}^{A}/_{A[A,A]A}$.

Theorem 15. *There is a natural isomorphism* $A_V \cong (\sqrt[V]{A})_{\mathrm{ab}}$*.*

[2]This adjunction "explains" the root sign in the notation $\sqrt[V]{-}$.

Proof. Theorem 13 states that $\sqrt[V]{-}$ is left adjoint to the functor

$$\mathfrak{m}_V : \mathtt{Alg}_k \to \mathtt{Alg}_k, \qquad B \mapsto \mathrm{End}(V) \otimes B.$$

Drawing this adjunction on a single diagram along with the forgetful functor $\mathcal{U} : \mathtt{CommAlg}_k \to \mathtt{Alg}_k$ and its left adjoint, the abelianization functor $(-)_{\mathrm{ab}}$, we get

$$\begin{tikzcd} \mathtt{Alg}_k \arrow[r, bend left, "\sqrt[V]{-}"] & \mathtt{Alg}_k \arrow[l, bend left, "\mathfrak{m}_V"] \arrow[r, bend left, "(-)_{\mathrm{ab}}"] & \mathtt{CommAlg}_k \arrow[l, bend left, "\mathcal{U}"] \end{tikzcd}, \tag{3.2}$$

where the left adjoints are shown on top. A composition of adjoint functors is an adjoint functor (see, for example, [McL]), and thus we get an adjunction

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}\left(\left(\sqrt[V]{A}\right)_{\mathrm{ab}}, B\right) = \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes B).$$

Therefore, because they have the same right adjoint (see Corollary 6), the functors $(-)_V$ and $\left(\sqrt[V]{-}\right)_{\mathrm{ab}}$ are isomorphic. $\square$

Note that the preceding theorem provides an alternative proof of the representability of the functor Rep_V^A. Namely, the proof of this theorem shows that $\left(\sqrt[V]{A}\right)_{\mathrm{ab}}$, which is defined for every unital associative algebra A, represents Rep_V^A. Unlike the first proof we gave, this proof is independent of the choice of presentation for the algebra A.

Remark 16. *We already knew that a morphism $f : A_V \to B$ corresponds under the adjunction to the morphism $(\mathrm{id}_{\mathrm{End}(V)} \otimes f) \circ \pi$. Now, having obtained the adjunction for $(-)_V$ from the one for $\sqrt[V]{-}$, we can give a description for the inverse mapping. Given a morphism $g : A \to \mathrm{End}(V) \otimes B$:*

(i) Form $\iota *_k g$, where $\iota : \mathrm{End}(V) \to \mathrm{End}(V) \otimes B$ is defined by $\iota(m) = m \otimes 1$.

(ii) Restrict to invariants and identify $(\mathrm{End}(V) \otimes B)^{\mathrm{End}(V)} = B$, *yielding the morphism* $(\iota *_k g)^{\mathrm{End}(V)} : \sqrt[V]{A} \to B$.

(iii) Abelianize, getting a map $A_V \to B$ *(since* B *is already commutative).*

To summarize, the morphism g *corresponds under the adjunction of Corollary 6 to*

$$\left((\iota *_k g)^{\mathrm{End}(V)}\right)_{\mathrm{ab}}.$$

Remark. *Looking at the results of the previous subsection from a different point of view, we see that the algebra* $\sqrt[V]{A}$ *represents the functor*

$$\widetilde{\mathrm{Rep}_V^A} : \mathtt{Alg}_k \to \mathtt{Sets}, \qquad B \mapsto \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes B). \tag{3.3}$$

Comparing this with formula 3.1, we see that this is just the functor Rep_V^A *extended from the category of commutative algebras to that of all associative algebras; this is why we have denoted it* $\widetilde{\mathrm{Rep}_V^A}$:

$$\begin{tikzcd}
\mathtt{CommAlg}_k \arrow[r, "\mathrm{Rep}_V^A"] \arrow[d, hook] & \mathtt{Sets} \\
\mathtt{Alg}_k \arrow[ur, "\widetilde{\mathrm{Rep}_V^A}"'] &
\end{tikzcd}$$

For this and other reasons, the algebra $\sqrt[V]{A}$ *can be regarded (see, for example, [G1]) as the coordinate algebra of the **"non-commutative representation scheme**."*

Corollary 17. *Let* $p : \sqrt[V]{A} \twoheadrightarrow A_V$ *be the abelianization morphism. Then,*

$$\pi = (\mathrm{id}_{\mathrm{End}(V)} \otimes p) \circ \tilde{\pi},$$

where $\pi : A \to \mathrm{End}(V) \otimes A_V$ *is the universal representation and* $\tilde{\pi}$ *is the universal morphism* $A \to \sqrt[V]{A}$ *adjoined in the construction (via universal localization) of* $\sqrt[V]{A}$.

Proof. The adjunction of $(-)_V$ and $\mathfrak{m}_V$ is a composition of two adjunctions (per diagram 3.2). In particular, these two adjunctions give us

$$\mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes A_V) = \mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A}, A_V) = \mathrm{Hom}_{\mathtt{CommAlg}_k}(A_V, A_V).$$

Beginning in the category $\mathtt{CommAlg}_k$ (on the far right), and then passing the morphism id_{A_V} through each of the two adjunctions, we first get $p : \sqrt[V]{A} \to A_V$ and then $(\mathrm{id}_{\mathrm{End}(V)} \otimes p) \circ \tilde{\pi}$.

However, we know that this is the same as just passing the morphism id_{A_V} through the composed adjunction, which yields π. □

3.2.3 Describing $\sqrt[V]{A}$ explicitly

Consider the functor $\mathfrak{m}_V : B \mapsto \mathrm{End}(V) \otimes B$, which is the "coordinate-free" version of the functor $\mathbb{M}_n$ sending an algebra B to the k-algebra of $n \times n$ matrices over B. The functor $\mathfrak{m}_V$ factors as the composition of two functors

$$\mathtt{Alg}_k \xrightarrow{\mathfrak{M}_V} \mathtt{Alg}_{\mathrm{End}(V)} \xrightarrow{\mathfrak{I}_V} \mathtt{Alg}_k,$$

where $\mathfrak{I}_V$ is the forgetful functor and $\mathfrak{M}_V$ is the functor sending an algebra B to $\mathrm{End}(V) \otimes B$, regarded as an $\mathrm{End}(V)$-algebra via the natural morphism $\mathfrak{m}_V(k \hookrightarrow B)$.

Lemma 18. *The functor $\mathfrak{M}_V : \mathtt{Alg}_k \to \mathtt{Alg}_{\mathrm{End}(V)}$ is an equivalence of categories, with the inverse (up to isomorphism) given by*

$$(-)^{\mathrm{End}(V)} : \mathtt{Alg}_{\mathrm{End}(V)} \to \mathtt{Alg}_k,$$

the functor sending an $\mathrm{End}(V)$-algebra B to its subalgebra of $\mathrm{End}(V)$-invariants,

$$B^{\mathrm{End}(V)} = \{x \in B \ : \ \phi x = x\phi \quad \forall \phi \in \mathrm{End}(V)\}.$$

Proof. See 10 of [B1]. Alternatively, this result is a consequence of Lemma 68, which we prove on p. 89. $\square$

Now, the forgetful functor $\mathfrak{I}_V$ has the well-known adjoint (coproduct) functor, $\amalg_{\mathrm{End}(V)} : B \mapsto \mathrm{End}(V) *_k B$, while $\mathfrak{M}_V$, being an equivalence, has its inverse as an adjoint (both right and left). So, we have two pairs of adjoints (with the left adjoints drawn on top):

$$\begin{tikzcd} \mathtt{Alg}_k \arrow[r, bend right, "\mathfrak{M}_V"'] & \mathtt{Alg}_{\mathrm{End}(V)} \arrow[l, bend right, "(-)^{\mathrm{End}(V)}"'] \arrow[r, bend right, "\mathfrak{I}_V"'] & \mathtt{Alg}_k \arrow[l, bend right, "\amalg_{\mathrm{End}(V)}"'] \end{tikzcd} \tag{3.4}$$

Theorem 19. *For any associative k-algebra A, there is a natural isomorphism*

$$\sqrt[V]{A} \cong (\mathrm{End}(V) *_k A)^{\mathrm{End}(V)}.$$

Proof. As before, recall that the composition of adjoints is itself an adjoint; thus, the composition $(-)^{\mathrm{End}(V)} \circ \amalg_{\mathrm{End}(V)}$ is left adjoint to the functor $\mathfrak{m}_V = \mathfrak{I}_V \circ \mathfrak{M}_V$. But the functor $\sqrt[V]{-}$ is also left adjoint to $\mathfrak{m}_V$, and any two functors which have the same right adjoint are naturally isomorphic. $\square$

The significance of this result is that it provides an explicit algebraic construction for $\sqrt[V]{A}$.

Remark. *The approach taken in this subsection is essentially that of [B1]. However, the same argument can be found in Cohn's work ([C2], pp. 207-208). Cohn takes a less coordinate-free approach, defining a category of "n-matrix rings," which are rings with n^2 distinguished elements $\{e_{ij}\}_{1 \le i,j \le n}$ satisfying the "matrix rules"*

$$e_{ij}e_{i'j'} = \delta_{ji'}e_{ij'}, \qquad \sum e_{ii} = 1.$$

Of course, this is equivalent to considering algebras over $\mathbb{M}_n(k)$. Cohn suggests that the n-matrix reduction $\sqrt[V]{A}$ "may be thought of as the ring A with the elements of A interpreted as $n \times n$ matrices" (loc. cit.).

3.2.4 Another explicit description for $\sqrt[V]{A}$

The results of Subsection 3.2.2 suggest that $\sqrt[V]{A}$ should be regarded as a non-commutative analog of A_V. In light of this, it is natural to hope that a result along the lines of Corollary 7 will hold for $\sqrt[V]{A}$ as well. Indeed, this turns out to be the case.

Theorem 20. *The algebra $\sqrt[V]{A}$ is isomorphic to the associative algebra on generators $\{a^{jl} : a \in A, 1 \leq j, l \leq n\}$ with relations*

$$\alpha a^{jl} = (\alpha a)^{jl} \; \forall \alpha \in k, \quad a^{jl} + b^{jl} = (a+b)^{jl}, \quad \sum_t a^{jt} b^{tl} = (ab)^{jl}, \quad 1^{jl} a^{j'l'} = \delta^{lj'} a^{j'l'}.$$

Unlike most of the other results of this chapter, the author is not aware of this theorem's appearance anywhere in the literature. We will withhold the proof, since the result is an easy consequence of a deeper result (on derived non-commutative representation schemes) that we will prove below, Corollary 122 (p. 148). Nevertheless, it is likely that the present theorem can also be proven using more elementary (albeit cumbersome) means.

Analogously to the remark following Corollary 7, this result may be interpreted as saying that $\sqrt[V]{A}$ can be regarded as the associative algebra of elements which behave like entries of matrices whose (matrix) multiplication reflects the structure of A. (Cf. the quotation from Cohn in the preceding subsection.)

3.3 The Kontsevich-Rosenberg Principle

One studies the geometry of a commutative ring B by considering the collection of homomorphisms $B \to k$ into a field k. Since such homomorphisms are one-dimensional representations of the algebra B, it is natural to consider the representation schemes $\mathrm{Rep}_V(A)$ as a generalization of the prime spectrum of B. This is especially appealing when one wishes to expand the class of algebras studied from commutative algebras to all associative algebras,[3] since the set of one-dimensional representations of an associative algebra fails to capture much of the structure of the algebra (simply because the commutativity of k forces any homomorphism $A \to k$ to be zero on the commutator ideal $A[A, A]A$).

In [KR], M. Kontsevich and A. Rosenberg proposed that every non-commutative geometric structure on "$\mathrm{Spec}\, A$" should naturally induce a corresponding commutative structure on $\mathrm{Rep}_V(A)$. This viewpoint – which in particular provides a litmus test for proposed definitions of non-commutative analogs of classical geometric notions – has been very influential in the development of non-commutative algebraic geometry.

3.3.1 Smoothness

We begin by recalling Grothendieck's **Infinitesimal Lifting Property**, which plays an important role in deformation theory:

Proposition 21. *If k is algebraically closed and the affine scheme $\mathrm{Spec}\, A$ corresponding to a finitely generated commutative unital k-algebra A is smooth, then for any commutative unital k-algebra B and nilpotent ideal $I \subset B$, every morphism $\phi : A \to B/I$ has*

[3] For example, as early as the 1970s, Cohn used $\sqrt[V]{A}$ (exploiting the adjunction of Theorem 13 described on p. 31 of the present work) to define spectra of non-commutative algebras (see [C1, C2]).

a lifting $\tilde{\phi}$ making the following diagram commute:

$$\begin{tikzcd} B \arrow[r, two heads] & \frac{B}{I} \\ & A \arrow[u, "\phi"'] \arrow[ul, dotted, "\exists\tilde{\phi}"] \end{tikzcd} \tag{3.5}$$

Proof. The proof is sketched in Exercise 8.6 of Chapter II in [H]. $\square$

Grothendieck takes this property as essentially a definition of smoothness, defining a commutative unital k-algebra A to be **formally smooth**[4] whenever it satisfies the property. Quillen extends this definition to the category of associative unital algebras:

Definition 22. *The associative unital algebra A is **quasi-free** if for any associative unital algebra B and nilpotent ideal $I \subset B$, every morphism $\phi : A \to B/I$ has a lifting $\tilde{\phi}$ making the diagram 3.5 commute.*

Lemma 23. *Let A be an associative unital k-algebra. If A is quasi-free, then so is $\sqrt[V]{A}$.*

Proof. Say we are given a diagram as follows (where B is a unital associative algebra B with nilpotent ideal $I \subset B$):

$$\begin{tikzcd} B \arrow[r, two heads, "p"] & \frac{B}{I} \\ & \sqrt[V]{A} \arrow[u, "\phi"'] \end{tikzcd}$$

There is a natural isomorphism $\mathrm{End}(V) \otimes \left(\frac{B}{I}\right) = \frac{\mathrm{End}(V)\otimes B}{\mathrm{End}(V)\otimes I}$. Now, moving ϕ

[4]Note that for this definition, we do not require k to be algebraically closed, nor do we require A to be finitely generated.

through the adjunction of Theorem 13, we obtain ϕ' in the diagram

$$\begin{tikzcd}
\mathrm{End}(V) \otimes B \arrow[rr, two heads, "p'"] & & \frac{\mathrm{End}(V)\otimes B}{\mathrm{End}(V)\otimes I} \\
& & A \arrow[u, "\phi'"'] \arrow[ull, dotted, "\exists\tilde{\phi}'"]
\end{tikzcd}$$

which lifts to $\tilde{\phi}'$ because A is quasi-free.

Now, consider the maps

$$\begin{aligned}
\iota_1 : \quad \mathrm{End}(V) &\to \mathrm{End}(V) \otimes \left(\tfrac{B}{I}\right), & m &\mapsto m \otimes 1, \\
\iota_2 : \quad \mathrm{End}(V) &\to \mathrm{End}(V) \otimes B, & m &\mapsto m \otimes 1.
\end{aligned}$$

Because $\iota_1 = p \circ \iota_2$, we apply the universal property of $*_k$ to obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{End}(V) \otimes B \arrow[rr, two heads, "p'"] & & \frac{\mathrm{End}(V)\otimes B}{\mathrm{End}(V)\otimes I} \\
& & \mathrm{End}(V) *_k A \arrow[u, "\iota_1 *_k \phi'"'] \arrow[ull, "\iota_2 *_k \tilde{\phi}'"]
\end{tikzcd}$$

Next, apply to this diagram the functor $(-)^{\mathrm{End}(V)}$. By Remark 14 (p. 31), this gives us the (commutative) diagram

$$\begin{tikzcd}
B \arrow[r, two heads, "p"] & \frac{B}{I} \\
& \sqrt[V]{A} \arrow[u, "\phi"'] \arrow[ul, "\tilde{\phi}"]
\end{tikzcd}$$

where $\tilde{\phi} := \left(\iota_2 *_k \tilde{\phi}'\right)^{\mathrm{End}(V)}$. Thus, we have constructed a lifting to our initial diagram, as desired. $\square$

The following theorem is our first illustration of the Kontsevich-Rosenberg principle.

Theorem 24. *Let A be an associative unital k-algebra. If A is quasi-free, then A_V is formally smooth (in the sense of Grothendieck).*

Proof. Say we are given a diagram as follows (where B is a unital commutative algebra with nilpotent ideal $I \subset B$):

$$\begin{tikzcd} B \arrow[r, two heads, "p"] & \frac{B}{I} \\ & A_V \arrow[u, "\psi"'] \end{tikzcd}$$

Extend ψ to a morphism $\phi : \sqrt[V]{A} \to {}^{B}/_{I}$ defined by $\phi = \psi \circ c$, where $c : \sqrt[V]{A} \to A_V$ is the abelianization map. Then, the preceding lemma gives us a lifting $\tilde{\phi} : \sqrt[V]{A} \to B$ satisfying

$$\phi = p \circ \tilde{\phi}.$$

Applying the functor $(-)_{\mathrm{ab}}$ to this equality, we get

$$\begin{tikzcd} B \arrow[r, two heads, "p"] & \frac{B}{I} \\ & A_V \arrow[u, "\psi"'] \arrow[ul, "(\tilde{\phi})_{\mathrm{ab}}"] \end{tikzcd}$$

because $(-)_{\mathrm{ab}}$ acts trivially on $B \xrightarrow{p} {}^{B}/_{I}$ and by definition $\phi_{\mathrm{ab}} = \psi$. Thus, we have constructed a lifting $\tilde{\psi} = \left(\tilde{\phi}\right)_{\mathrm{ab}}$, as desired. $\square$

3.3.2 Functions

In [KR], the symmetric algebra $S^{\bullet}({}^{A}/_{[A,A]})$ is proposed as the "**algebra of non-commutative functions**" on the non-commutative space $\operatorname{Spec} A$.

Let $\mathrm{tr} : \mathrm{End}(V) \to k$ be the linear trace map. Because $\mathrm{tr}(XY - YX) = 0$, the map

$$A \xrightarrow{\pi} \mathrm{End}(V) \otimes A_V \xrightarrow{\mathrm{tr}\otimes\mathrm{id}} A_V$$

descends to a map of vector spaces ${}^{A}/_{[A,A]} \to A_V$, where $[A, A]$ is the vector subspace (which is not in general an ideal!) of commutators. This map gives (via the universal property of the symmetric algebra) a morphism of algebras

$$\mathrm{Tr} : S^{\bullet}({}^{A}/_{[A,A]}) \to A_V.$$

The existence of such a natural map justifies (in accordance with the Kontsevich-Rosenberg principle) the preceding definition. Non-commutative analogs of certain classical modules over A (including differential forms and vector fields) come with natural $S^{\bullet}({}^{A}/_{[A,A]})$-module structures.

Let $(A_V)^{GL(V)}$ be the subalgebra of **invariants of the action of** $GL(V)$ **on** Rep_V^A (described in Subsection 3.1.3 on p. 25). In other words, $(A_V)^{GL(V)} \subset A_V$ consists of those functions on Rep_V^A that are constant on every $GL(V)$ orbit.

Proposition 25. *The image of* Tr *is contained in* $(A_V)^{GL(V)}$.

Proof. The image of Tr is just the subalgebra generated by the image I of the morphism

$$A \xrightarrow{\pi} \mathrm{End}(V) \otimes A_V \xrightarrow{\mathrm{tr}\otimes\mathrm{id}} A_V.$$

Therefore, since each $g \in GL(V)$ acts as a homomorphism $f_g : A_V \to A_V$, it is enough to prove that $f_g(x) = x$ for every $g \in GL(V)$ and every $x \in I$.

Let $\phi_g : \mathrm{End}(V) \to \mathrm{End}(V)$ be the automorphism given by conjugation by $g \in GL(V)$. Then, the following diagram commutes (since the left square commutes by the definition of f_g and the right square by construction):

$$\begin{tikzcd}
A \arrow[rr, "\pi"] \arrow[d, "\pi"] & & \mathrm{End}(V) \otimes A_V \arrow[r, "\mathrm{tr}\otimes\mathrm{id}_{A_V}"] \arrow[d, "\mathrm{id}_{\mathrm{End}(V)}\otimes f_g"] & A_V \arrow[d, "f_g"] \\
\mathrm{End}(V) \otimes A_V \arrow[rr, "\phi_g\otimes\mathrm{id}_{A_V}"] & & \mathrm{End}(V) \otimes A_V \arrow[r, "\mathrm{tr}\otimes\mathrm{id}_{A_V}"] & A_V
\end{tikzcd}$$

But the bottom edge satisfies

$$(\mathrm{tr} \otimes \mathrm{id}_{A_V}) \circ (\phi_g \otimes \mathrm{id}_{A_V}) = (\mathrm{tr} \otimes \mathrm{id}_{A_V}),$$

since conjugation preserves trace. Therefore, comparing the two outer paths from the top left-hand corner to the bottom right-hand one, we get

$$f_g \circ (\mathrm{tr} \otimes \mathrm{id}_{A_V}) \circ \pi = (\mathrm{tr} \otimes \mathrm{id}_{A_V}) \circ \pi,$$

which means precisely that f_g is the identity on I. $\square$

By the preceding proposition, the map $\mathrm{Tr} : S^\bullet({}^A/_{[A,A]}) \to A_V$ can be regarded as a map into $(A_V)^{GL(V)}$. The following important result of Procesi (see [Pr]) predates the formulation of the Kontsevich-Rosenberg principle by two decades.

Theorem 26. *The map* $\mathrm{Tr} : S^\bullet({}^A/_{[A,A]}) \to (A_V)^{GL(V)}$ *is surjective.*

3.3.3 Vector fields

Definition 27. *Let M be a bimodule over an associative k-algebra A. Recall that a **derivation** $\phi : A \to M$ is a linear map*

$$\phi : A \to M, \qquad \phi(xy) = \phi(x)y + x\phi(y).$$

We call the set of all such derivations $\mathrm{Der}(A, M)$. In the special case when M is simply A (with the canonical bimodule structure), we shorten this to just $\mathrm{Der}(A)$.

Now, recall that a (global) **vector field on an affine scheme** $\operatorname{Spec} B$ is given by a derivation $\phi \in \operatorname{Der}(B)$. It is natural to take derivations of associative algebras as the non-commutative analogs (see, for example, [G1]), i.e. as "**non-commutative vector fields**." To formulate an instance of the Kontsevich-Rosenberg principle for vector fields, we first need the following alternative characterization of derivations.

Proposition 28. *Let A be an associative k-algebra. Form $A(\epsilon)$ by adjoining a central element ϵ satisfying $\epsilon^2 = 0$. Let p be the natural k-algebra morphism*

$$p : A(\epsilon) \to A, \qquad (x + \epsilon \cdot y) \mapsto x,$$

and let $\mathcal{H}$ be the set of morphisms $f : A \to A(\epsilon)$ satisfying $p \circ f = \mathrm{id}_A$. Then, there is a one-to-one correspondence between $\operatorname{Der}(A)$ and $\mathcal{H}$ given by sending the derivation ϕ to the morphism

$$f : x \mapsto x + \epsilon \cdot \phi(x).$$

Proof. This is a routine verification; it is also a special case of Proposition 95 (see p. 119). $\square$

So, let $\phi : A \to A$ be a derivation of an associative k-algebra A. Form the corresponding $f : A \to A(\epsilon)$, and then compose it with the map π_ϵ given by extending the universal representation π by scalars,

$$\pi_\epsilon : A(\epsilon) \to \operatorname{End}(V) \otimes A_V(\epsilon), \qquad (x + \epsilon \cdot y) \mapsto \pi(x) + \epsilon \cdot \pi(y).$$

Pass the resulting morphism $\pi_\epsilon \circ f : A \to \operatorname{End}(V) \otimes A_V(\epsilon)$ through the adjunction of Corollary 6 (p. 24) to obtain a morphism $g : A_V \to A_V(\epsilon)$. This morphism satisfies $p' \circ g = \mathrm{id}_{A_V}$, where

$$p' : A_V(\epsilon) \to A_V, \qquad (x + \epsilon \cdot y) \mapsto x$$

is the natural projection,[5] and thus gives rise to a derivation $\psi \in \mathrm{Der}(A_V)$. Thus, we have a natural construction associating to any non-commutative vector field $\phi \in \mathrm{Der}(A)$ a vector field $\psi \in \mathrm{Der}(A_V)$ on the affine scheme $\mathrm{Rep}_V(A)$.

3.3.4 Differential forms

Let B be a commutative k-algebra. Recall (for details, see [Ma]) that the module of **differential 1-forms** is defined as the B-module $\Omega^1(B)$ with a derivation[6] $d : B \to \Omega^1(B)$ that satisfies the following universal property: for any B-module M with derivation $\delta : B \to M$, there exists a unique module morphism $f : \Omega^1(B) \to M$ making the diagram commute:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega^1(B) \arrow[d, dotted, "f"] \\ & M \end{tikzcd} \tag{3.6}$$

Explicitly, the module $\Omega^1(B)$ is generated by the set of symbols $\{db \ : \ b \in B\}$ under the relations

$$d(\alpha b_1 + b_2) = \alpha \cdot db_1 + db_2, \qquad d(b_1 b_2) = db_1 \cdot b_2 + b_1 \cdot db_2 \qquad \forall\, b_1, b_2 \in B,\ \alpha \in k, \tag{3.7}$$

with the map d given by $d(b) = db$. In these terms, the map f corresponding to a derivation δ is given by

$$f : \ b_1 \cdot db_2 \mapsto b_1 \cdot \delta(b_2).$$

[5] To see this, note first that $p'' \circ \pi_\epsilon \circ f = \pi$, where $p'' : \mathrm{End}(V) \otimes A_V(\epsilon) \to \mathrm{End}(V) \otimes A_V$ is the natural projection. Then, use the description of the adjunction furnished by Remark 14 (p. 32) to show that this condition translates to the condition $p' \circ g = \mathrm{id}_{A_V}$.

[6] Since B is commutative, any module is automatically a bimodule.

The **de Rham complex** $\Omega^\bullet(B)$ is then the skew-commutative DG algebra defined as the alternating algebra (over B)

$$\Omega^\bullet(B) = \bigwedge_B^{\bullet} \Omega^1(B)$$

equipped with a differential $d_{\Omega^\bullet(B)}$ of degree $+1$ defined to correspond with the universal map $d : B \to \Omega^1(B)$ on $\Omega^0(B) = B$ and then extended to all of $\Omega^\bullet(B)$ by the Leibniz rule.

Any element $b_1' db_1 \wedge b_2' db_2 \wedge ... \wedge b_m' db_m$ of $\Omega^\bullet(B)$ can be rewritten (gathering the coefficients at the front) as $(b_1' b_2' ... b_m') db_1 \wedge db_2 \wedge ... \wedge db_m$. It is customary to drop the wedge symbol and simply write elements of $\Omega^\bullet(B)$ as (sums of elements of the form) $b_0\, db_1\, db_2 \,...\, db_m$.

The functor $\bigwedge_B^\bullet$ is the free functor in the category of graded skew-commutative B-algebras (adjoint to the forgetful functor to B-modules). We can combine the universal properties of Ω^1 and of $\bigwedge_B^\bullet$ to obtain the following universal mapping property for $\Omega^\bullet$.

Proposition 29. *Let B be a commutative k-algebra and $\iota : B \hookrightarrow \Omega^\bullet(B)$ the B-algebra structure map of $\Omega^\bullet(B)$. Then, for any skew-commutative DG algebra C and morphism of algebras $g : B \to C^0$ into the zero-degree component of C, there exists a unique morphism of DG algebras $h : \Omega^\bullet(B) \to C$ making the following diagram commute:*

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[dr, "g"'] & \Omega^\bullet(B) \arrow[d, dotted, "h"] \\ & C \end{tikzcd}$$

Proof. On elements of $\Omega^0(B)$, the morphism h is determined by g (with which it must correspond). Then, on elements of $\Omega^1(B)$ the morphism h is given (uniquely) by the universal property of the module of differential 1-forms applied to the B-module C^1 (take $d = d_{\Omega^\bullet(B)} \circ \iota$ and $\delta = d_C \circ g$). Following this, h extends to all of $\Omega^\bullet(B)$ by the universal property of $\bigwedge_B^\bullet$. $\square$

This classical theory has a (remarkably consistent) non-commutative analog; we need only to everywhere replace "module" with "bimodule" and change exterior products to tensor products (which are the corresponding free functors for the category of associative graded algebras). See, for example, [CQ1].

More concretely, every associative k-algebra A has a bimodule $\Omega^1_{\mathrm{nc}}(A)$ with derivation $d : A \to \Omega^1_{\mathrm{nc}}(A)$ which satisfies a universal property completely analogous to that of the diagram 3.6. The bimodule $\Omega^1_{\mathrm{nc}}(A)$ can be presented as the bimodule on the symbols $\{da \;:\; a \in A\}$ satisfying relations exactly like Relations 3.7. The tensor algebra $\mathrm{T}^\bullet_A \Omega^1_{\mathrm{nc}}(A)$ of this bimodule, denoted $\Omega^\bullet_{\mathrm{nc}}(A)$ and called the algebra of **non-commutative differential forms** of A, is naturally a DG algebra over A, with differential $d_{\Omega^\bullet_{\mathrm{nc}}(A)}$ given by extending d using the Leibniz rule. As in the commutative case, there is a natural inclusion map $\iota_{\mathrm{nc}} : A \hookrightarrow \Omega^\bullet_{\mathrm{nc}}(A)$. The DG A-algebra $\Omega^\bullet_{\mathrm{nc}}(A)$ satisfies a universal property analogous to that of Proposition 29: for every associative DG algebra C and algebra morphism $A \to C^0$, there is a unique morphism of DG algebras $\Omega^\bullet_{\mathrm{nc}}(A) \to C$ making the corresponding diagram commute. On account of this universal property, Quillen (see [CQ1]) calls the DG algebra $\Omega^\bullet_{\mathrm{nc}}(A)$ the **differential envelope** of A.

Remark 30. *Just as in the commutative case, any element*

$$a_1'(da_1)a_1'' \otimes a_2'(da_2)a_2'' \otimes \ldots \otimes a_m'(da_m)a_m''$$

of $\Omega^\bullet(B)$ can be rewritten to gather the coefficients at the front. To do this, begin with *the rightmost term $a'_m(da_m)a''_m$ and rewrite it (using the Leibniz rule) as*

$$a'_m(d(a_m a''_m)) - a'_m a_m (da''_m),$$

subsequently transferring the left coefficients through the tensor product to the $m-1$ term. Proceeding inductively (moving to the left), we will ultimately obtain a sum of 2^m terms each with coefficients only at the leftmost end. Thus, we can adopt the same convention as in the commutative case, and write non-commutative differential forms as sums of elements of the form $a_0\, da_1\, da_2 \dots da_m$.

Despite these many parallels between $\Omega^\bullet_{\mathrm{nc}}(A)$ and the classical de Rham complex, there is one very significant difference: the DG algebra $\Omega^\bullet_{\mathrm{nc}}(A)$ does not have an interesting cohomology theory (see 11.4 of [G1]):

$$\mathrm{H}^i(\Omega^\bullet_{\mathrm{nc}}(A)) = \begin{cases} k & \text{for } i = 0 \\ 0 & \text{for } i > 0 \end{cases}$$

The following definition remedies this shortcoming.

Definition 31. *The* ***non-commutative de Rham complex*** *of A (also called the* ***Karoubi-de Rham complex****) is defined to be the cochain complex (i.e., differential graded vector space)*

$$\mathrm{DR}^\bullet_{\mathrm{nc}}(A) = \Omega^\bullet_{\mathrm{nc}}(A) / \left[\Omega^\bullet_{\mathrm{nc}}(A), \Omega^\bullet_{\mathrm{nc}}(A)\right],$$

where the commutator is taken in the graded sense, $[x, y] = xy - (-1)^{|x||y|}yx$.

Note that this puts us in the unusual position of having to make a distinction between the non-commutative differential forms (which form a DG algebra) and the non-commutative de Rham complex (which is a cochain complex without an algebra structure).

To see the Kontsevich-Rosenberg principle at work in this case, define the morphism $g : A \to \Omega^\bullet(A_V)$ to be the composition

$$A \xrightarrow{\pi} \mathrm{End}(V) \otimes A_V \xrightarrow{\mathrm{tr}\otimes\mathrm{id}} A_V \overset{\iota}{\hookrightarrow} \Omega^\bullet(A_V)$$

and then apply the universal property of $\Omega^\bullet_{\mathrm{nc}}(A)$ to obtain a morphism h :

$$\begin{tikzcd} A \arrow[r, hook, "\iota_{\mathrm{nc}}"] \arrow[dr, "g"'] & \Omega^\bullet_{\mathrm{nc}}(A) \arrow[d, dotted, "h"] \\ & \Omega^\bullet(A_V) \end{tikzcd}$$

Define $\tilde{a} = (\mathrm{tr} \otimes \mathrm{id}) \circ (\pi)(a)$. Because the above diagram commutes, we can see that on $A \subset \Omega^\bullet_{\mathrm{nc}}(A)$, the map h is given by sending $a \mapsto \tilde{a}$. Because h is a morphism of DG algebras, this means that on arbitrary elements it sends

$$a_0\, da_1\, da_2 \ldots da_m \mapsto \tilde{a}_0\, d\tilde{a}_1\, d\tilde{a}_2 \ldots d\tilde{a}_m.$$

Because $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in \mathrm{End}(V)$, the map h vanishes on $[\Omega^\bullet_{\mathrm{nc}}(A), \Omega^\bullet_{\mathrm{nc}}(A)]$ and thus descends to a natural map

$$\mathrm{DR}^\bullet_{\mathrm{nc}}(A) \to \Omega^\bullet(A_V)$$

from the non-commutative de Rham complex to the classical one.

3.3.5 Van den Bergh's realization of the principle

While the constructions outlined in the preceding several sections were all quite natural, they were nevertheless *ad hoc*: every non-commutative structure in question required its own treatment. In [VdB1], M. Van den Bergh proposed a single functor providing a unified approach to several important constructions.

Definition 32. *Let M be an A-bimodule. Define M_V to be the left A_V-module generated by symbols $\{m^{jl} \ : \ m \in M, \ 1 \leq j,l \leq n\}$ satisfying, for all $m, m_1, m_2 \in M$, $a \in A$, and $\alpha \in k$, the relations*

$$(m_1 + m_2)^{jl} = m_1^{jl} + m_2^{jl}, \qquad (\alpha m)^{jl} = \alpha m^{jl},$$
$$(am)^{jl} = \sum_{r=1}^{n} a^{jr} m^{rl}, \qquad (ma)^{jl} = \sum_{r=1}^{n} a^{rl} m^{jr}.$$

Note that this is a very natural definition to make in light of Corollary 7; it suggests that we should regard M_V as the module whose elements are entries in vectors over the "ghost" matrix algebra whose entries the elements of A_V behave as. The assignment $M \mapsto M_V$ defines an additive functor

$$(-)_V : \mathtt{Mod}_{A^e} \to \mathtt{Mod}_{A_V},$$

where $A^e := A \otimes A^{\mathrm{op}}$. (Recall that the category of left A^e-modules is naturally isomorphic to the category of A-bimodules.)

While we use the same notation for this functor as for the functor assigning to an algebra A the coordinate ring of its representation scheme, there is no risk of confusion because the former is always applied to bimodules while the latter is always applied to algebras.

There is also an intrinsic definition of the functor $(-)_V$ (Lemma 3.3.1 in [VdB1]):

Lemma 33. *Regard* $\mathrm{End}(V) \otimes A_V$ *as an A-bimodule via the universal representation $\pi : A \to \mathrm{End}(V) \otimes A_V$. At the same time, regard it as a left A_V-module via the diagonal embedding $a \mapsto \mathbf{1} \otimes a$. Then there is a natural isomorphism of A_V-modules*

$$M_V \cong M \otimes_{A^e} (\mathrm{End}(V) \otimes A_V).$$

Whereas the *ad hoc* constructions of the previous section yielded natural maps from non-commutative structures on A to the corresponding commutative ones on A_V, the Van den Bergh functor yields isomorphisms of corresponding modules. For example,

$$(\mathrm{T}^{\bullet}_{A} M)_V \cong \mathrm{S}^{\bullet}_{A_V} M_V,$$

where on the left we have the tensor algebra and on the right the symmetric algebra. More interestingly,

$$(\Omega^{\bullet}_{\mathrm{nc}} A)_V \cong \Omega^{\bullet}(A_V).$$

As for derivations, it turns out that the appropriate non-commutative object to take in this case is not the module of derivations $\mathrm{Der}(A)$, but rather the A-bimodule of **double derivations**, denoted $\mathbb{D}\mathrm{er}(A)$. To define this, recall first that there are two A-bimodule structures on $A \otimes A$, the *outer* one, given by

$$a_1(b_1 \otimes b_2)a_2 = a_1 b_1 \otimes b_2 a_2,$$

and the *inner* one, given by

$$a_1(b_1 \otimes b_2)a_2 = b_1 a_2 \otimes a_1 b_2.$$

Now, $\mathbb{D}\mathrm{er}(A)$ is defined as the k-linear space of bimodule maps $A \to A \otimes A$ (with respect to the outer bimodule structure), with the bimodule structure on $\mathbb{D}\mathrm{er}(A)$ given via the inner bimodule structure.

Van den Bergh proves (Proposition 3.3.4 of [VdB1]) that when A is smooth (i.e., has bimodule category of cohomological dimension 1),

$$(\mathbb{D}\mathrm{er}\, A)_V \cong \mathrm{Der}\, A_V.$$

CHAPTER 4

HOMOTOPICAL ALGEBRA

The category of associative k-algebras is not an abelian one, and thus it is impossible to do classical homological algebra in this category. However, there exists a generalization of homological algebra – Quillen's "homotopical algebra" and the formalism of *model categories* (see, for example, [Q2]) – allowing one to construct homotopy categories and derived functors in a fairly wide variety of settings. This theory, while giving fewer concrete tools than one has available in an abelian category, nevertheless provides a unifying framework for studying "non-abelian" derived functors.

The goal of this chapter is to prepare the setting (including proving model structures on suitable categories) for deriving the representation scheme functor $A \mapsto A_V$, as well as for deriving (in a "non-abelian" setting) the Van den Bergh functor $M \mapsto M_V$. We will also develop methods for resolving associative algebras, allowing us to make concrete calculations.

4.1 Model Categories and Derived Functors

In classical homotopy theory (as described, for example, in Chapter IV of [Ha]), one localizes the category of CW complexes at the class of weak homotopy equivalences to form its homotopy category. Meanwhile, in homological algebra, one localizes the category of chain complexes at the class of quasi-isomorphisms to form the derived category. In both of these cases, certain functors descend to functors on the localized categories, providing a tool for studying the objects of the original categories.

Quillen's formalism of model categories can be regarded as a generalization of both of these theories; remarkably, any category satisfying a list of five axioms can be localized in a suitable way, and the notion of a derived functor can be defined. Moreover, certain general theorems follow from the axiomatics; most relevant to us will be Quillen's theorem on adjunctions (p. 57).

In this section, we will recall the definition of and state some of the basic results about model categories.

4.1.1 Model categories

Given a diagram of this form, a **lifting** is a map $h : B \to X$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

We say that i has the **left lifting property (LLP)** with respect to p – and, equivalently, that p has the **right lifting property (RLP)** with respect to i – if every diagram of this form has a lifting.

We say that a map f is a **retract** of a map g if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

such that the composition of the top and bottom rows is the identity (on X and Y, respectively).

Definition 34. *A **model category** is a category $\mathcal{C}$ with three distinguished classes of maps:*

*(i) **weak equivalences*** $(\xrightarrow{\sim})$, *which we also call **acyclic** maps,*

*(ii) **fibrations*** $(\twoheadrightarrow)$, *and*

*(iii) **cofibrations***$(\hookrightarrow)$,

each of which is closed under composition and contains all identity maps. The following five axioms must be satisfied:

MC1 *The category $\mathcal{C}$ has all finite limits and colimits.*

MC2 *In any diagram $X \xrightarrow{g} Y \xrightarrow{f} Z$, if any two of the three maps f, g, and fg are weak equivalences, then so is the third. (This is called the **"two of three" axiom**.)*

MC3 *Each of the three distinguished classes of maps is closed under taking retracts.*

MC4 *Given a diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

a lifting $h : B \to X$ exists whenever (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

MC5 *Any map f can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and (ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.*

Remark. *We will follow Dwyer and Spalinski [DS] in using the term "model category" to mean what Quillen calls a "**closed model category**."*

For any category $\mathcal{C}$ and object $A \in \mathbf{Ob}(\mathcal{C})$, the **under category** $A \downarrow \mathcal{C}$ is defined as the category whose objects are maps $A \to X$ in $\mathcal{C}$ and morphisms are maps $f : X \to Y$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

We can also write objects of $A \downarrow \mathcal{C}$ as $A \backslash C$ to emphasize that C is regarded not by itself but under A.

Proposition 35. *Let $\mathcal{C}$ be a model category, and $A \in \mathbf{Ob}(\mathcal{C})$. Then, the under category $A \downarrow \mathcal{C}$ is a model category, with a morphism $f : X \to Y$ being a member of a distinguished class precisely when the corresponding morphism $f \in \mathrm{Mor}(\mathcal{C})$ is.*

The proof of this proposition is a straightforward verification of each of the five axioms.

4.1.2 The homotopy category and derived functors

The value of a model structure on a category is that it allows one to construct the localization of that category at the class of weak equivalences and then give criteria for when a functor can be derived. This subsection will summarize the key results here; proofs can be found in any standard reference, such as for example [DS].

Because a model category $\mathcal{C}$ has all finite limits and colimits, it has in particular a terminal object $*$ and an initial object (since these are simply the limit and colimit, respectively, of the empty diagram). We say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is

fibrant when $A \to *$ is a fibration, and **cofibrant** when $\to A$ is a cofibration. In the present work, we will only need to work with **fibrant model categories**, i.e. model categories in which every object is fibrant; this assumption will substantially simplify the exposition of the present subsection.

A **cylinder object** for $A \in \mathrm{Ob}(\mathcal{C})$ is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a diagram

$$A \amalg A \to C \xrightarrow{\sim} A$$

factoring the natural map

$$(\mathrm{id}_A, \mathrm{id}_A) : A \amalg A \to A.$$

Label the two natural coproduct structure maps $A \to A \amalg A$ as i_1 and i_2. We say that two morphisms $f, g : X \to Y$ in $\mathcal{C}$ are **homotopic**[1] if there exists a cylinder object C for X along with a map $H : C \to Y$ such that the two compositions

$$X \overset{i_1}{\hookrightarrow} X \amalg X \to C \xrightarrow{H} Y, \qquad X \overset{i_2}{\hookrightarrow} X \amalg X \to C \xrightarrow{H} Y$$

are f and g, respectively.

In the case when X and Y are both fibrant and cofibrant, homotopy is an equivalence relation on the set of morphisms $X \to Y$. The set of equivalence classes of this relation is called $\pi(X, Y)$.

Applying axiom $\mathrm{MC5(i)}$ to the unique morphism $\to A$, we obtain a cofibrant object QA with an acyclic fibration $QA \overset{\sim}{\twoheadrightarrow} A$. This is called a **cofibrant replacement** of A. While a cofibrant replacement is not necessarily unique, it is unique up to homotopy equivalence: for any pair of cofibrant replacements QA, $Q'A$,

[1]Usually, the term "left homotopic" is used for this. However, since the assumption that all objects in $\mathcal{C}$ are fibrant will allow us to do away with the dual notion of "right homotopy," we can drop the word "left."

there exists a pair of morphisms

$$QA \underset{g}{\overset{f}{\rightleftarrows}} Q'A$$

such that fg and gf and both homotopic to the identity. For any morphism $f : X \to Y$ and cofibrant replacements QX, QY, there exists a lifting $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd} QX \arrow[r, "\tilde{f}"] \arrow[d, "\wr"] & QY \arrow[d, "\wr"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The lifting $\tilde{f}$ is not necessarily unique, but it is unique up to homotopy. These facts allow us to make the following definition.

Definition 36. *Let $\mathcal{C}$ be a fibrant model category. Then, the* ***homotopy category*** *$\mathcal{H}o(\mathcal{C})$ is defined as the category with the same objects as $\mathcal{C}$ and with morphism sets $\mathrm{Hom}(X, Y) = \pi(QX, QY)$. The functor $\gamma : \mathcal{C} \to \mathcal{H}o(\mathcal{C})$ is defined as the identity on objects, while sending each morphism f to the equivalence class of its lifting $\tilde{f}$.*

The following theorem states that $\mathcal{H}o(\mathcal{C})$ is the localization of $\mathcal{C}$ at the class of weak equivalences.

Theorem 37. *Let $\mathcal{C}$ be a fibrant model category and $\mathcal{D}$ any category. Given any functor $F : \mathcal{C} \to \mathcal{D}$ sending weak equivalences to isomorphisms, there is a unique functor $G : \mathcal{H}o(\mathcal{C}) \to \mathcal{D}$ such that $G \circ \gamma = F$.*

Definition 38. *Let $\mathcal{C}$ and $\mathcal{D}$ be fibrant model categories, and $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathcal{H}o(\mathcal{C})$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{H}o(\mathcal{D})$ the natural functors. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, a* ***total left derived functor*** *of F is a pair consisting of a functor and a natural transformation*

$$LF : \mathcal{H}o(\mathcal{C}) \to \mathcal{H}o(\mathcal{D}), \qquad t : LF \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$$

such that for any other such pair

$$G : \mathcal{H}o(\mathcal{C}) \to \mathcal{H}o(\mathcal{D}), \qquad s : G \circ \gamma_{\mathcal{C}} \to \gamma_{\mathcal{D}} \circ F$$

there exists a unique natural transformation $s' : G \to LF$ such that the diagram

$$\begin{tikzcd} & LF \circ \gamma_{\mathcal{C}} \arrow[d, "t"] \\ G \circ \gamma_{\mathcal{C}} \arrow[ur, dotted, "s'\gamma_{\mathcal{C}}"] \arrow[r, "s"] & \gamma_{\mathcal{D}} \circ F \end{tikzcd}$$

commutes, where $s'\gamma_{\mathcal{C}}$ is the natural transformation assigning to an object $A \in \mathcal{C}$ the morphism $s'(\gamma_{\mathcal{C}}(A))$.

More informally, this definition says that the functor LF approximates the functor F in a way measured by the natural transformation t, and any other such approximation factors through this one.

There is a dual notion of a **total right derived functor**, denoted RF and obtained by reversing the appropriate arrows.

Theorem 39. *Let $\mathcal{C}$ and $\mathcal{D}$ be fibrant model categories, and*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

a pair of adjoint functors.

(i) If F preserves cofibrations as well as acyclic cofibrations, then the total derived functors LF and RG exist and form an adjoint pair

$$LF : \mathcal{H}o(\mathcal{C}) \rightleftarrows \mathcal{H}o(\mathcal{D}) : RG.$$

(ii) The functor LF is given on objects $A \in \mathcal{H}o(\mathcal{C})$ by considering A as an object of $\mathcal{C}$, taking a cofibrant replacement QA, and concluding with $\gamma_{\mathcal{D}}(F(QA))$. The functor is given on morphisms $[f] \in \pi(QX, QY)$ by picking a representative f of $[f]$ and concluding with $\gamma_{\mathcal{D}}(F(f))$.

A situation as in Part (i) of the preceding theorem is called a **Quillen adjunction**, and the two adjoint functors are called a **Quillen pair**.

4.2 Model Structures on Categories of DG Algebras and Modules

A model structure on the category of commutative cochain DG algebras concentrated in non-negative degrees has been known since the 1970s (see, for example, [BG] or [GM]). More recently, in [J], J.F. Jardine used methods completely analogous to the commutative case to prove a model structure on the category of associative DG cochain algebras concentrated in non-negative degrees. In both of these model structures, weak equivalences are quasi-isomorphisms and fibrations are surjections; cofibrations are then defined via the left lifting property (LLP) with respect to acyclic fibrations.

For our purposes, we will require analogous model structures on the corresponding *chain* algebras. As it turns out, such structures exist, with fibrations defined as maps which are surjective on all (strictly) positive degrees. In the case of (not necessarily commutative) DG algebras, this structure has already been discussed in the literature (see, for example, [M] or [BP]). As for the commutative case, it can be seen as a special case of more general results of Hinich from [Hi].

In this section, we will present proofs of these model structures. Besides allowing for a self-contained exposition (without resorting to the operadic machinery of [Hi]), this will shed light on the specifics of the model structure (and in particular the nature of cofibrant objects in these model categories).

The approach taken here – and due to A. Ramadoss – combines elements of Jardine's work [J] with results on DG algebras from other sources (most notably [FHT1]) to give an elementary proof, i.e. one requiring neither operadic machinery nor the use of Quillen's small object argument.

4.2.1 $\mathtt{DGA}_S$, $\mathtt{CDGA}_S$, and their model structures

Recall that a **differential graded (DG) chain k-algebra** is a graded k-algebra R endowed with a linear map d (called the *differential*) of degree -1 satisfying the Leibniz rule

$$d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) \quad \forall x, y \in R,$$

where $|x|$ is the degree of x. If instead we require the degree of d to be $+1$, we obtain the notion of a DG cochain algebra. A morphism of DG chain algebras is a morphism of graded algebras commuting with the differentials on the domain and codomain.

Recall that a morphism $f : R_1 \to R_2$ of DG algebras induces a morphism of graded k-algebras on homology, $\mathrm{H}(f) : \mathrm{H}(R_1) \to \mathrm{H}(R_2)$. When $\mathrm{H}(f)$ is an isomorphism, we say that f is a **quasi-isomorphism**.

A DG algebra R is commutative if

$$x \cdot y = (-1)^{|x| \cdot |y|} y \cdot x \quad \forall x, y \in R.$$

Define $\mathtt{DGA}_k$ to be the category of unital DG chain algebras, and $\mathtt{DGA}_S$ the corresponding under category for a fixed algebra $S \in \mathtt{DGA}_k$. Analogously, let $\mathtt{CDGA}_k$

and $\mathtt{CDGA}_S$ be the full subcategories of commutative algebras of $\mathtt{DGA}_k$ and $\mathtt{DGA}_S$, respectively. For each of these categories, write a superscript $+$ sign to indicate the full subcategory of DG algebras concentrated in non-negative degrees.

Definition 40. *Define a model structure on $\mathtt{DGA}_k^+$ by defining a morphism $f : R_1 \to R_2$ to be*

(i) a weak equivalence *if f is a quasi-isomorphism,*

(ii) a fibration *if f is surjective on all strictly positive degrees, and*

(iii) a cofibration *if f has the left lifting property with respect to all acyclic fibrations (i.e., morphisms which are at the same time fibrations and weak equivalences).*

Define a completely analogous model structure on $\mathtt{CDGA}_k^+$, where the weak equivalences and fibrations are simply those morphisms which are in the corresponding classes as morphisms in $\mathtt{DGA}_k^+$, while cofibrations are morphisms with the left lifting property with respect to all acyclic fibrations in $\mathtt{CDGA}_k^+$.

Remark 41. *A natural question to ask is why fibrations are defined as surjections on all strictly positive degrees rather than on all degrees (as is the case in many similar situations). In fact, because our DG algebras are concentrated in non-negative degrees, any quasi-isomorphism which is a fibration in the above sense is necessarily a surjection in all degrees. Thus, no model structure with fibrations defined as simply surjections can exist, for if it did, its class of acyclic fibrations would coincide with that of the model structure defined in this subsection, and thus the classes of cofibrations would coincide, too. But this cannot be, since it is well-known that to specify a model structure, it is enough to specify only two of the three distinguished classes (see, for example, Proposition 3.13 and the subsequent remark in [DS]).*

The proof that this definition indeed yields a model structure is quite involved, and will take up the following two subsections, culminating in Corollary 53.

4.2.2 Limits and colimits in $\mathtt{DGA}_S^+$ and $\mathtt{CDGA}_S^+$

Proposition 42. *$\mathtt{DGA}_k^+$ and $\mathtt{CDGA}_k^+$ are complete.*

Proof. The category $\mathtt{DGA}_k^+$ has all equalizers of pairs of maps. Indeed, the equalizer of a pair of morphisms f, g is given by

$$E \xrightarrow{h} R_1 \underset{g}{\overset{f}{\rightrightarrows}} R_2,$$

where E is the difference kernel $E = \{x \in R_1 \ : \ f(x) - g(x) = 0\}$. This is a (unital) DG subalgebra of R_1, and the morphism h is the inclusion $E \hookrightarrow R_1$.

$\mathtt{DGA}_k$ also has all small products: given a set $S \subset \mathrm{Ob}(\mathtt{DGA}_k^+)$, the product is given as the set-theoretic product with component-wise operations (addition, scalar multiplication, ring multiplication, and differential).

By Corollary V.2.1 in MacLane [McL], any category with all equalizers of pairs of morphisms and all small products has all small limits. Thus, $\mathtt{DGA}_k^+$ is complete.

The same argument works for $\mathtt{CDGA}_k^+$, since the difference kernel of a commutative DG algebra is itself commutative. $\square$

Recall that a **filtered category** $\mathcal{J}$ is a non-empty category such that:

(a) For any two objects j, j' of $\mathcal{J}$, there exists an object $r \in \mathcal{J}$ and morphisms $j \to r, j' \to r$:

$$\begin{tikzcd} j \arrow[dr] & & j' \arrow[dl] \\ & r & \end{tikzcd}$$

(b) For any two morphisms $u, v : i \to j$ there exists an object $r \in \mathcal{J}$ and a morphism $w : j \to r$ such that $wu = wv$.

This generalizes the notion of a directed preorder.[2] A functor $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ from a filtered category $\mathcal{J}$ to a category $\mathcal{C}$ is called a **filtered functor** and generalizes the notion of an inverse system in $\mathcal{C}$. The colimit of such a functor is called a **filtered colimit**.

Lemma 43. *$\mathtt{DGA}_k^+$ and $\mathtt{CDGA}_k^+$ have all small filtered colimits.*

Proof. Recall that, as is usual for categories of algebraic objects, the colimit of a filtered functor $\mathcal{F} : \mathcal{J} \to \mathcal{C}$, where $\mathcal{C}$ can be taken to be either $\mathtt{DGA}_k^+$ or $\mathtt{CDGA}_k^+$, is given by

$$\bigsqcup_{j \in \mathcal{J}} \mathcal{F}(j) \ / \ \sim,$$

where $\bigsqcup$ is the set-theoretic disjoint union and $\sim$ is the equivalence relation generated by all equivalences of the form $x \sim f(x)$, where $x \in \mathcal{F}(j), j \in \mathcal{J}$, and f is a morphism in the image of $\mathcal{J}$ with domain $\mathcal{F}(j)$.

For $x \in \mathcal{F}(j)$ and $x' \in \mathcal{F}(j')$, we define $x + x'$ to be $g(x) + g'(x')$, where $g : j \to k$ and $g' : j' \to k$ are morphisms in $\mathcal{J}$ whose existence is guaranteed

[2]Recall that a directed preorder P is a set with a transitive, reflexive relation $\leq$ such that for any two elements $j, j' \in P$ there exists a (not necessarily unique) element $r \in P$ such $j \leq r$ and $j' \leq r$.

by part (a) of the definition of a filtered category. Multiplication is defined analogously. That these operations are well-defined is a consequence of part (b) of the definition of a filtered category. $\square$

Next, recall that the coproduct of two objects R_1, R_2 in $\mathtt{DGA}_k^+$ is given by $R_1 *_k R_2$, the algebra of sums of words over k in R_1 and R_2, with the degree of each word given by summing the degrees of the letters. The differential is given via the (graded) Leibniz rule.

Meanwhile, the coproduct of two objects R_1, R_2 in $\mathtt{CDGA}_k^+$ is given by $R_1 \otimes_k R_2$.

Proposition 44. $\mathtt{DGA}_k^+$ *and* $\mathtt{CDGA}_k^+$ *are cocomplete.*

Proof. By Lemma 43, the categories $\mathtt{DGA}_k^+$ and $\mathtt{CDGA}_k^+$ have all small filtered colimits, and in particular all colimits over small directed preorders. They also have finite coproducts (as we just saw), and this implies (by IX.1.1 of [McL]) that they has all small coproducts.

In addition, $\mathtt{DGA}_k^+$ and $\mathtt{CDGA}_k^+$ have all coequalizers of pairs of maps. Indeed, the coequalizer of a pair of morphisms f, g is given by

$$R_1 \overset{f}{\underset{g}{\rightrightarrows}} R_2 \xrightarrow{h} C,$$

where C is simply the quotient of R_2 by the ideal generated by all elements of the form $f(x) - g(x)$ for $x \in R_1$ (which is a DG ideal). The map h is simply the projection onto the quotient.

Therefore, since equalizers and small products yield all small limits (again, this is Corollary V.2.1 of [McL]), it is true by duality that coequalizers and small coproducts yield all small colimits, and thus $\mathtt{DGA}_k^+$ and $\mathtt{CDGA}_k^+$ are cocomplete. $\square$

Proposition 45. $\mathtt{DGA}^+_S$ *and* $\mathtt{CDGA}^+_S$ *are complete and cocomplete.*

Proof. In general, if a category $\mathcal{C}$ is complete (or cocomplete), then the under category $S \downarrow \mathcal{C}$ is also complete (respectively, cocomplete). This is because any diagram in $S \downarrow \mathcal{C}$ can be regarded as a diagram in $\mathcal{C}$ (by adding a vertex for S and the corresponding structure maps). $\square$

4.2.3 Proving the model structures on $\mathtt{DGA}^+_S$ and $\mathtt{CDGA}^+_S$

Since both $\mathtt{DGA}^+_S$ and $\mathtt{CDGA}^+_S$ are complete and cocomplete, $\mathrm{MC1}$ is satisfied. Note that we could have proven $\mathrm{MC1}$ more easily, since it requires only finite limits and colimits (which follow from finite products & equalizers and finite coproducts & coequalizers, respectively). However, it will be helpful for certain constructions to have all small limits and colimits.

In both cases, the axiom $\mathrm{MC2}$ is clear, as is $\mathrm{MC3}$ for fibrations and weak equivalences. As for $\mathrm{MC3}$ for cofibrations, simply combine the lifting diagram with the retraction diagram; for a map f which is a retract of a cofibration g, we have:

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d, "f"] & R_1' \arrow[r] \arrow[d, "g"] & R_1 \arrow[r] \arrow[d, "f"] & X \arrow[d, "\wr"] \\ R_2 \arrow[r, "i"] & R_2' \arrow[r] & R_2 \arrow[r] & Y \end{tikzcd}$$

Then, by the definition of cofibration there exists a lifting $h : R_2' \to X$, and then $h \circ i$ gives us the desired lifting, proving that f is a cofibration.

For the remaining two axioms, we will need some additional results. Our general approach will be to prove the results for $\mathtt{DGA}^+_k$ first, and then use the adjunction involving the abelianization functor to obtain analogous results for $\mathtt{CDGA}_k^+$.

Let R be a DG algebra and T a DG algebra with underlying graded algebra $T = R *_k F$, where the graded algebra F is free. If the differential on $R \subset T$ coincides with that on R, then the natural inclusion

$$i : R \to T, \qquad r \mapsto r,$$

which is a map of DG algebras, is called a **free extension**.

Note that this means that the differential on T is completely determined (via linearity and the Leibniz rule) by the differential on R and the differential of each free generator of F. Note also that F is not defined as a DG algebra, but just as a free algebra: differentials of elements in $F \subset T$ needn't land inside F.

Analogously, a free extension in $\mathtt{CDGA}_k^+$ is a DG algebra morphism $i : R \to T$ such that on the underlying graded algebras, i has the form

$$R \to T = R \otimes F, \qquad a \mapsto a \otimes 1,$$

where the graded algebra F is free and the differential on $R \subset T$ is inherited from that on R.

Any free extension in $\mathtt{CDGA}_k^+$ is the abelianization of a corresponding free extension in $\mathtt{DGA}_k^+$.

Proposition 46. *Every free extension in* $\mathtt{DGA}_k^+$ *is a cofibration.*

Proof. Consider a free extension $i : R \to T$, where as a graded algebra $T = R *_k F$, and a commutative square

$$\begin{tikzcd} R \arrow[r] \arrow[d, "i"] & B \arrow[d, "\wr"] \\ T \arrow[r] & C \end{tikzcd}$$

We must construct a lifting $h : T \to B$. Now, the algebra T is generated by the elements in $R \subset T$ along with the free generators of $F \subset T$. So, we begin by defining h on elements of $R \subset T$ to correspond with the top edge of the diagram, $R \to B$. Then, we proceed by induction on degree, specifying where we will send each generator x of F. At each stage, it is sufficient to verify that for a generator $x \in F$, the following two conditions are met:

$$dh(x) = h(dx), \qquad g(x) = p(h(x)).$$

For indeed, if these conditions hold for all generators up to degree n, then they hold in general (by linearity and the Leibniz rule) for all elements of T whose summands are homogeneous terms of degree up to n (for such summands are words whose letters are either in A or are generators of F of degree at most n, and such generators satisfy the two required conditions on d by the inductive hypothesis).

Beginning with the case when $|x| = 0$, we can send x to some (any) element of the set $p^{-1}(g(x)) = \{y \in B : p(y) = g(x)\}$. We know that this set is nonempty because a map which is both a quasi-isomorphism and a surjection on all strictly positive degrees is necessarily a surjection on degree 0, too.

Now, for $|x| = n$ with $n > 0$, we know that $h(dx)$ is a cycle, since $dh(dx) = h(ddx) = 0$ by the inductive hypothesis (for we already have the relation $hd = dh$ for all degrees lower than n). Now, p is a quasi-isomorphism, which means that it can only send a cycle to a boundary if the cycle is itself a boundary; therefore,

$h(dx)$ is a boundary in B. Pick any element $z \in B$ such that $dz = h(dx)$. We have

$$\begin{aligned} d(p(z) - g(x)) &= p(dz) - d(g(x)) \\ &= p(h(dx)) - p(h(dx)) \\ &= 0, \end{aligned}$$

and thus $p(z) - g(x)$ is a cycle. Since p is a quasi-isomorphism, this means that there is a cycle of B that maps to this element up to a boundary: in other words, there exists a cycle $u \in B$ and an element $w \in C$ such that $p(u) = dw + p(z) - g(x)$. Now, picking a $t \in B$ with $p(t) = w$, we have $dw = dp(t)$, and thus

$$p(u) = dp(t) + p(z) - g(x),$$

and hence $g(x) = p(z + dt - u)$. Therefore, setting $y := z + dt - u$, we have

$$dy = dz = h(dx), \qquad p(y) = g(x),$$

which means that we can send $x \mapsto y$ under h. □

Remark. *The Adams-Hilton Lemma (see, for example, p. 226 of [BL]) is a similar result, with two essential differences: (i) the hypothesis is more general, as the DG algebras are not required to be concentrated in non-negative degrees, and (ii) the conclusion is less strong, yielding a map h that makes the diagram commute only up to homotopy.*

Let F_x be the DG algebra freely generated (as a graded algebra) by two generators, x of degree n and dx of degree $n-1$, with differential given by $d(x) = dx$ and $d(dx) = 0$. A morphism of DG algebras

$$R \to R * \left(\mathop{*}_{i \in I} F_{x_i} \right), \qquad r \mapsto r,$$

where I is an indexing set (and we do not require the degrees of the x_i to be the same for different $i \in I$), is called a **special free extension**. By the preceding proposition, this is a cofibration.

Proposition 47. *Any special free extension* $i : R \to R * \left(\underset{i \in I}{*} F_{x_i} \right)$ *is acyclic.*

Proof. Following Jardine [J], we observe that for any DG algebra R and chain complex C, there is a DG algebra structure on the chain complex

$$R[C] = R \oplus (R \otimes C \otimes R) \oplus (R \otimes C \otimes R \otimes C \otimes R) \oplus ...$$

with multiplication given by concatenating two "words" and merging the R terms via the multiplication in R. Let C be the chain complex

$$[... \leftarrow 0 \leftarrow k \overset{\text{id}}{\leftarrow} k \leftarrow 0 \leftarrow ...]$$

with non-zero terms in degrees n and $n-1$, and with the identity of the n^{th} degree term denoted as 1_n.

A morphism of DG algebras $R[C] \to B$ is uniquely determined by its behavior on $R \subset R[C]$ and the set of elements of the form

$$1 \otimes c \otimes 1 \in R \otimes C \otimes R \subset R[C].$$

Letting $|x| = n$, the map $R[C] \to R * F_x$ determined uniquely by the assignment

$$r \mapsto r, \qquad 1 \otimes 1_n \otimes 1 \mapsto x$$

and the map $R * F_x \to R[C]$ determined uniquely (via the universal property of coproducts) by the assignment

$$r \mapsto r, \qquad x \mapsto 1 \otimes 1_n \otimes 1$$

are inverses. Thus, as DG algebras,

$$R[C] \cong R * F_x.$$

By the Knneth formula, the natural map

$$R \to R[C], \qquad r \mapsto r$$

is a quasi-isomorphism, and thus so is the natural inclusion $R \to R * F_x$. Thus, such a special free extension is acyclic.

More generally, an arbitrary special free extension $R \to R * \left(\mathop{*}_{i \in I} F_{x_i} \right)$ is a filtered colimit of such maps, and thus (since taking filtered colimits is exact in the category of k-vector spaces – see, for example, [IM]) is itself a quasi-isomorphism. $\square$

Proposition 48. *Any morphism $f : R_1 \to R_2$ in $\mathtt{DGA}_k^+$ can be factored as $R_1 \overset{i}{\hookrightarrow} T \underset{\sim}{\overset{p}{\twoheadrightarrow}} R_2$, where i is a free extension and p is an acyclic fibration.*

Proof. We will define T inductively by specifying at each degree (starting at 0) which generators we would like to have for F (in the decomposition $T = R_1 * F$) and what value the differential is to take on each generator.

At degree 0, we add a generator x for each element of the degree 0 component of R_2, setting $d(x) = 0$. We add nothing else.

At degree 1, we add a generator x for each cycle of the degree 1 component of R_2, setting $d(x) = 0$. We also add a generator y for every boundary in the zero-degree component of R_2, and set $d(y)$ to be the generator x corresponding to that boundary (which is possible since every boundary is a cycle and for every cycle, we had introduced a generator x at the previous step). Finally, we add a generator z for every cycle in the zero-degree term of our so-far constructed DG algebra T *except* for those cycles we had created ourselves by adding generators x at the previous step. (In particular, this last deed removes any cycles coming from R_1.)

At each degree $n > 1$, we proceed just as in degree 1, adding three types of generators: generators x to give us cycles corresponding to those of R_2 at degree n, generators y to allow us to kill all boundaries corresponding to those of R_2 at degree $n-1$, and finally generators z to get rid of any "extra" cycles we might have in our T that don't correspond to those of R_2 (and, in particular, which might have come from R_1). $\square$

Proposition 49. *Any morphism $f : R_1 \to R_2$ in $\mathtt{DGA}_k$ can be factored as $R_1 \underset{\sim}{\overset{i}{\hookrightarrow}} T \overset{p}{\twoheadrightarrow} R_2$, where i is a special free extension and p is a fibration.*

Proof. Take T to be $R_1 * \left(\underset{r \in R_2, |r| \geq 1}{*} F_{x_r} \right)$, and define p by sending each generator $x_r \mapsto r$ and $dx_r \mapsto d_{R_2}(r)$. Being a surjection on all terms of degree 1 or more, p is a fibration; being a special free extension, i is an acyclic cofibration.[3] $\square$

Proposition 50. *In $\mathtt{DGA}_k^+$, any cofibration is a retract of a free extension and any acyclic cofibration is a retract of a special free extension.*

Proof. If $f : R_1 \to R_2$ is a cofibration, then we factor it as $R_1 \overset{i}{\hookrightarrow} T \underset{\sim}{\overset{p}{\twoheadrightarrow}} R_2$, where i is a free extension and p is an acyclic fibration. We get a commutative square:

$$\begin{tikzcd} R_1 \arrow[r, "i"] \arrow[d, "f"] & T \arrow[d, "p", "\wr"'] \\ R_2 \arrow[r, "\mathrm{id}"] & R_2 \end{tikzcd}$$

Because f is a cofibration, we get a lifting $h : R_2 \to T$. This gives us the diagram

$$\begin{tikzcd} R_1 \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & R_1 \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & R_1 \arrow[d, "f"'] \\ R_2 \arrow[r, "h"] & T \arrow[r, "p"] & R_2 \end{tikzcd}$$

[3] We considered only $r \in R_2$ of degree 1 or more because the algebra F_{x_r} is defined only in this case. This is what prevents p from being a surjection in general.

Because h is a lifting, the bottom row composes to the identity and the left square commutes; the right square commutes by definition. Thus, this is a retraction diagram, and our cofibration f is a retract of the free extension i.

The statement that any acyclic cofibration is a retract of a special free extension is proven completely analogously (but this time using the factorization $R_1 \underset{\sim}{\overset{i}{\hookrightarrow}} T \overset{p}{\twoheadrightarrow} R_2$). □

Theorem 51. *With the model structure defined above,* $\mathtt{DGA}_k^+$ *satisfies the model category axioms.*

Proof. $\mathrm{MC1}$, $\mathrm{MC2}$, and $\mathrm{MC3}$ were already proven above.

Parts (i) and (ii) of axiom $\mathrm{MC5}$ are simply Propositions 48 and 49. Part (i) of $\mathrm{MC4}$ is true by the definition of cofibration. As for part (ii) of $\mathrm{MC4}$, this is true because acyclic cofibrations are retracts of special cofibrations, and if a morphism f has the left lifting property with respect to another morphism g, then any retract of f also has the left lifting property with respect to g. □

Theorem 52. *With the model structure defined above,* $\mathtt{CDGA}_k^+$ *satisfies the model category axioms.*

Proof. Given a map $f : R_1 \to R_2$ in $\mathtt{CDGA}_k^+$, we can factor it (via Proposition 48) as $R_1 \overset{i}{\hookrightarrow} T \underset{\sim}{\overset{p}{\twoheadrightarrow}} R_2$, where i is a free extension and p is an acyclic fibration in $\mathtt{DGA}_k^+$. Now, we can apply the abelianization functor $(-)_{\mathrm{ab}}$ to this decomposition; i_{ab} is then a free extension in the commutative sense. To see that p_{ab} is acyclic, simply walk through the steps of the induction in the proof of Proposition 48. This proves part (i) of axiom $\mathrm{MC5}$. The proof of part (ii) is analogous.

Axiom $\mathrm{MC4}$ is proven just as in the case of $\mathtt{DGA}_k^+$. □

Corollary 53. *$\mathtt{DGA}_S^+$ and $\mathtt{CDGA}_S^+$ are both model categories, with weak equivalences being quasi-isomorphisms, fibrations being surjections on positive-degree terms, and cofibrations being maps having the left lifting property with respect to acyclic fibrations.*

Proof. This is an application to the preceding theorem of Proposition 35. $\square$

4.2.4 $\mathtt{DGBA}_S^+$, $\mathtt{DGMA}_S^+$, and their model structures

A **differential graded (DG) module** (M, d) over a DG chain algebra (R, d) is a graded module over the underlying graded algebra of R equipped with a linear map d of degree -1 satisfying the Leibniz rule

$$d(r \cdot m) = d(r) \cdot m + (-1)^{|r|} r \cdot d(m) \quad \forall r \in R, m \in M.$$

There is an analogous notion of a DG module over a DG cochain algebra.

A DG bimodule (M, d) over a DG chain algebra (R, d) is a graded bimodule over the underlying graded algebra of R equipped with a linear map d of degree -1 satisfying the Leibniz rule

$$d(r_1 \cdot m \cdot r_2) = d(r_1) \cdot m \cdot r_2 + (-1)^{|r_1|} r_1 \cdot d(m) \cdot r_2 + (-1)^{|r_1|+|m|} r_1 \cdot m \cdot d(r_2)$$

for all $r_1, r_2 \in R, m \in M$.

As usual, DG bimodules over a DG algebra R can be regarded as DG modules over the algebra $R \otimes R^{\mathrm{op}}$.

Define $\mathtt{DGMA}_S^+$ to be the category whose objects are pairs (R, M), where $R \in \mathtt{CDGA}_S^+$ and M is a DG module over R, with morphisms $(R, M) \to (R', M')$

defined as pairs of maps (f,g), where $f : R \to R'$, $f \in \mathrm{Mor}(\mathtt{CDGA}_S^+)$ and $g : M \to M'$ is a morphism of abelian groups such that

$$f(r) \cdot g(m) = g(r \cdot m) \quad \forall r \in R, m \in M.$$

Define $\mathtt{DGBA}_S^+$ to be the category of pairs (R,M) where R is a DG algebra and M a DG bimodule over R, with morphisms defined as pairs (f,g) with $f : R \to R'$ a morphism of DG algebras and $g : M \to M'$ is a morphism of abelian groups such that

$$f(r_1) \cdot g(m) \cdot f(r_2) = g(r_1 \cdot m \cdot r_2) \quad \forall r_1, r_2 \in R, m \in M.$$

Definition 54. *Define a model structure on* $\mathtt{DGBA}_S^+$ *by defining a morphism* $(f,g) : (R,M) \to (R',M')$ *to be*

(i) a weak equivalence *if f and g both induce isomorphisms on homology,*

(ii) a fibration *if f and g are both surjective on all strictly positive degrees, and*

(iii) a cofibration *if (f,g) has the left lifting property with respect to all acyclic fibrations.*

Define a completely analogous model structure on $\mathtt{DGMA}_S^+$*, where the weak equivalences are pairs each of which induces isomorphism on homology, the fibrations are pairs each of which is surjective on all strictly positive degrees, and the cofibrations are morphisms with the left lifting property with respect to all acyclic fibrations.*

We will skip the proof of these model structures. The structures can be proven using similar arguments to those employed above for $\mathtt{DGA}_S^+$ and $\mathtt{CDGA}_S^+$. Alternatively, these results are consequences of more general theorems from [BM].

4.2.5 Model structures for $\mathtt{DGA}_S$, $\mathtt{CDGA}_S$, $\mathtt{DGBA}_S$, and $\mathtt{DGMA}_S$

There are also model structures on the categories corresponding to those above, but without the requirement for all objects to be concentrated in non-negative degrees. As we will see later, these model structures are less convenient to describe explicitly (at least in terms of the cofibrations and cofibrant objects), but will allow for more general results.

The structures are defined in the same way, except the fibrations must be surjective on all degrees.

Definition 55. *Define a model structure on* $\mathtt{DGA}_S$ *by defining a morphism* $f : R_1 \to R_2$ *to be*

(i) a weak equivalence *if f is a quasi-isomorphism,*

(ii) a fibration *if f is surjective on all degrees, and*

(iii) a cofibration *if f has the left lifting property with respect to all acyclic fibrations.*

Define a completely analogous model structure on $\mathtt{CDGA}_S$, *where the weak equivalences and fibrations are simply those morphisms which are in the corresponding classes as morphisms in* $\mathtt{DGA}_k$, *while cofibrations are morphisms with the left lifting property with respect to all acyclic fibrations in* $\mathtt{CDGA}_S$.

Define a model structure on $\mathtt{DGBA}_S$ *by defining a morphism* $(f, g) : (R, M) \to (R', M')$ *to be*

(i) a weak equivalence *if f and g both induce isomorphisms on homology,*

(ii) a fibration *if f and g are both surjective on all degrees, and*

(iii) a cofibration *if* (f, g) *has the left lifting property with respect to all acyclic fibrations.*

Define a completely analogous model structure on $\mathtt{DGMA}_S$, *where the weak equivalences are pairs each of which induces isomorphism on homology, the fibrations are pairs each of which is surjective on all degrees, and the cofibrations are morphisms with the left lifting property with respect to all acyclic fibrations.*

The proofs of these structures are analogous to the ones for their non-positively-graded brethren, and thus will be omitted.

4.3 Resolving Associative Algebras

Almost free resolutions of associative algebras constitute a particularly useful class of cofibrant replacements. They play a similar role in the model structures on $\mathtt{DGA}_S^+$ and $\mathtt{CDGA}_S^+$ to that played by free resolutions of modules in classical homological algebra. In this section, we define almost free resolutions and then discuss several types of such resolutions.

In particular, we discuss the bar-cobar resolution, a closed-form resolution that always exists and is important to many theoretical results. However, while the cobar-bar resolution is explicitly defined, it is rather large, and this makes it less suitable to concrete calculations than it is to theory. In special cases (which include many interesting algebras), much smaller resolutions can be found, allowing one to do concrete calculations; these resolutions tend to be sub-resolutions of the bar-cobar resolution.

4.3.1 Almost free resolutions

A DG algebra $S\backslash F \in \mathtt{DGA}_S^+$ is called **almost free** if the structure morphism $i : S \to F$ is a free extension (as defined in the preceding section on p. 65). Given a DG algebra $S\backslash R \in \mathtt{DGA}_S^+$, a quasi-isomorphism $f : S\backslash F \to S\backslash R$ is called an **almost free resolution** of R. (By abuse of terminology, we can also call F itself an almost free resolution of R.) By Proposition 48, every object in $\mathtt{DGA}_S^+$ has an almost free resolution (and, in fact, infinitely many).

In the absolute case, when $S = k$, an almost free algebra is simply one whose underlying graded algebra is free. This will be the main situation we will consider in our examples.

Analogous notions exist in the category $\mathtt{CDGA}_S^+$. Here, an algebra is also defined as **almost free** if its structure morphism is a free extension, and an almost free resolution is again a quasi-isomorphism from an almost free algebra. Every object in $\mathtt{CDGA}_S^+$ has (infinitely many) almost free resolutions. Any almost free commutative DG algebra is simply the abelianization of a corresponding almost free (associative) DG algebra.

The case that is of greatest interest to us is when the algebra R is concentrated in degree 0, i.e. is just an associative algebra.

4.3.2 The bar-cobar resolution

Let R be a DG algebra. Following the notation of [CK2] (which summarizes some results of [HMS]), define $D(R)$ to be the graded algebra (without unit) freely generated by the vector space $\bigoplus_{n=1}^{\infty} R^{\otimes n}[1-n]$. The shift $[1-n]$ means that

$$|r_1 \otimes ... \otimes r_n|_{D(R)} \ := \ |r_1|_R + ... + |r_n|_R + n - 1.$$

Note that $D(R)$ is a much larger algebra than the reduced tensor algebra $\bar{T}(R)$, since it is actually the free algebra on the underlying vector space of $\bar{T}(R)$. We denote the product of $D(R)$ by $*$.

We equip $D(R)$ with two differentials. The first, d', is given by the differential on R, extended to $\bar{T}(R)$ and thence to $D(R)$ by the Leibniz rule and linearity. The second, d'', is given on generators by the formula

$$\begin{aligned} d''(a_0 \otimes ... \otimes a_n) = & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n \\ & - \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes ... \otimes a_i) * (a_{i+1} \otimes ... \otimes a_n). \end{aligned}$$

Then, we set $d = d' + d''$. This satisfies $d^2 = 0$ and makes $D(R)$ into a DG algebra (which is almost free).[4]

Let $F(R)$ be the graded algebra freely generated (without unit) by the underlying vector space of R. We denote multiplication here by $*$, too.

There is a natural projection $p : D(R) \to F(R)$ defined on $\bigoplus_{n=1}^{\infty} R^{\otimes n}[1-n] \subset D(R)$ by sending

$$\begin{aligned} a_1 & \mapsto a_1 \\ a_1 \otimes a_2 & \mapsto 0 \\ a_1 \otimes a_2 \otimes a_3 & \mapsto 0 \\ & \vdots \end{aligned}$$

[4]In fact, we could have defined $D(R)$ as the total complex of a bicomplex with differentials (d', d'') and bigrading given in one coordinate by the grading coming from R and in the other component by the one generated by the grading coming from the shift $[1-n]$.

and extending to the rest of $\bigoplus_{n=1}^{\infty} R^{\otimes n}[1-n]$ by linearity. Since $D(R)$ is freely generated by this subset, this determines a projection $p : D(R) \to F(R)$.

There is also a natural multiplication map $m : F(R) \to R$ sending $a_1 * ... * a_n \mapsto a_1 ... a_n$. This is a morphism of DG algebras.

Proposition 56. *The composition $D(R) \xrightarrow{p} F(R) \xrightarrow{m} R$ is a quasi-isomorphism of DG algebras.*

Proof. See [HMS]. $\square$

In the special case when R is an associative algebra A (regarded as a DG algebra concentrated in degree 0), this gives us an almost free resolution $D(A) \to A$, which we call the **bar-cobar resolution**.

This is based on the bar and cobar constructions, which are so called because a standard notation uses bars – writing $[a_1|...|a_n]$ for $a_1 \otimes ... \otimes a_n$ – to avoid confusion between the multiple free/tensor products that occur. (We avoid this confusion by using $*$ in addition to $\otimes$.)

4.4 M-Homotopies and Polynomial Homotopies

4.4.1 Smooth homotopies, M-homotopies, and polynomial homotopies

Definition 57. *Let C_1, C_2 be chain complexes of vector spaces. A family of chain maps $(f_t : C_1 \to C_2)_{t \in [0,1]}$ is* ***smooth*** *if for every $x \in C_1$, the function $f_t(x) : [0,1] \to C_2$ is* *differentiable at each $t \in (0,1)$.*

Remark. *The notion of the morphisms f_t being "differentiable" is, of course, problematic in the case when C_2 is infinite-dimensional. The problem can be remedied by fixing an exhaustive filtration on C_2 with each component of the associated graded complex being finite-dimensional. This can indeed be done in the examples to which we will apply this machinery, and is left to the reader.*

Definition 58. *Let C_1, C_2 be chain complexes,*

$$(f_t : C_1 \to C_2)_{t \in [0,1]}$$

a smooth family of chain complex morphisms, and

$$(s_t : C_1 \to C_2[1])_{t \in [0,1]}$$

*a smooth family of degree 1 linear maps. The pair $(f_t, s_t)_{t \in [0,1]}$ is a **smooth homotopy** if*

$$f_t' = ds_t + s_t d.$$

Proposition 59. *Given a smooth homotopy $(f_t, s_t)_{t \in [0,1]} : C_1 \to C_2$, the maps f_0 and f_1 induce the same morphism on homology $\mathrm{H}(C_1) \to \mathrm{H}(C_2)$.*

Proof. Each map f_t' induces the zero map on homology, and thus so does their integral with respect to t over the interval $[0,1]$. $\square$

Definition 60. *Let R_1, R_2 be DG algebras, and*

$$(f_t, s_t)_{t \in [0,1]} : R_1 \to R_2$$

*a smooth homotopy with each f_t a DG algebra morphism. The pair $(f_t, s_t)_{t \in [0,1]}$ is an **M-homotopy** if the maps s_t are (graded) derivations with respect to the bimodule structures given by the f_t, i.e.*

$$s_t(ab) = s_t(a) f_t(b) + (-1)^{|a|} f_t(a) s_t(b).$$

Remark. *For any smooth family of DG algebra morphisms $(f_t : R_1 \to R_2)_{t\in[0,1]}$, we have that for all t, the map f_t' is a degree 0 derivation satisfying $[f_t', d] = 0$. Therefore, we'd expect the maps s_t to often be derivations even if we didn't explicitly impose this requirement in the definition of an M-homotopy. However, imposing this requirement on s_t, while it does not seem at first glance to make a big difference, actually makes sure that M-homotopies have their most important quality: they pass through functorial constructions on algebras.*

The "M" in the term "M-homotopy," which we adopt from [CK2], stands for "multiplicative." M-homotopies are an intermediate notion between that of smooth homotopies and polynomial homotopies.

As it turns out, M-homotopies from cofibrant objects in $\mathtt{DGA}_k$ or $\mathtt{DGA}_k^+$ correspond to homotopies in the sense of cylinder objects, and thus are simply a new costume for a familiar notion (see, for example, [BKR2]). Their more explicit formulation, however, makes them useful for calculations.

The notion of an M-homotopy (along with the results we will discuss in this section) extends naturally to the relative setting, i.e. the category $\mathtt{DGA}_S$. In that case, we simply add the requirement that the maps f_t respect the S-algebra structure.

Definition 61. *A **polynomial homotopy** from a DG algebra R_1 to a DG algebra R_2 is a DG algebra homomorphism*

$$f : R_1 \to R_2 \otimes \frac{k[t,\epsilon]}{(\epsilon^2)}, \qquad |t| = 0,\ |\epsilon| = -1,\ dt = \epsilon,\ d\epsilon = 0.$$

Proposition 62. *A polynomial homotopy determines an M-homotopy, with f_r given by setting $t = r$ and $\epsilon = 0$.*

Proof. For a given $x \in R_1$, there is an $n \in \mathbb{N}$ such that

$$f(x) = \sum_{1 \leq i \leq n} b_i \otimes (p_i(t) + q_i(t)\epsilon), \qquad b_i \in R_2,$$

where the elements of $\{p_i(t)\}_{1 \leq i \leq n}$ and $\{q_i(t)\}_{1 \leq i \leq n}$ are polynomials in t with coefficients in k.

Then, we set

$$\begin{aligned} f_r(x) &= \sum_{1 \leq i \leq n} b_i \cdot p_i(r), \\ f'_r(x) &= \sum_{1 \leq i \leq n} b_i \cdot p'_i(r), \\ s_r(x) &= (-1)^{|x|-1} \sum_{1 \leq i \leq n} b_i \cdot q_i(r). \end{aligned}$$

Correspondingly, for another element $\tilde{x} \in R_1$, there is an $m \in \mathbb{N}$ and polynomials $\tilde{p}_j(t)$ and $\tilde{q}_j(t)$, for $1 \leq j \leq m$, such that

$$f(\tilde{x}) = \sum_{1 \leq j \leq m} \tilde{b}_j \otimes (\tilde{p}_j(t) + \tilde{q}_j(t)\epsilon), \qquad \tilde{b}_j \in R_2.$$

Now that we have fixed the notation, we can proceed to the proof. First, we verify that s_r is a derivation. We have:

$$\begin{aligned} f(x \cdot \tilde{x}) &= f(x) \cdot f(\tilde{x}) \\ &= \left(\sum_{1 \leq i \leq n} b_i \otimes (p_i(t) + q_i(t)\epsilon) \right) \left(\sum_{1 \leq j \leq m} \tilde{b}_j \otimes (\tilde{p}_j(t) + \tilde{q}_j(t)\epsilon) \right) \\ &= \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} b_i \tilde{b}_j \otimes \left(p_i(t)\tilde{p}_j(t) + p_i(t)\tilde{q}_j(t)\epsilon + (-1)^{|\tilde{x}|} q_i(t)\tilde{p}_j(t)\epsilon \right), \end{aligned}$$

and thus

$$
\begin{aligned}
s_r(x \cdot \tilde{x}) &= (-1)^{|x|+|\tilde{x}|-1} \sum_{\substack{1\le i\le n\\ 1\le j\le m}} b_i\tilde{b}_j \cdot \big(p_i(r)\tilde{q}_j(r) + (-1)^{|\tilde{x}|} q_i(r)\tilde{p}_j(r)\big) \\
&= (-1)^{|x|} \left(\sum_{1\le i\le n} b_i \cdot p_i(r) \right) \left((-1)^{|\tilde{x}|-1} \sum_{1\le j\le m} \tilde{b}_j \cdot \tilde{q}_j(r) \right) \\
&\qquad + \left((-1)^{|x|-1} \sum_{1\le i\le n} b_i \cdot q_i(r) \right) \left(\sum_{1\le j\le m} \tilde{b}_j \cdot \tilde{p}_j(r) \right) \\
&= (-1)^{|x|} f_r(x) s_r(\tilde{x}) + s_r(x) f_r(\tilde{x}).
\end{aligned}
$$

To conclude, we must show that $s_r d + d s_r = f'_r$.

Now, for any $x \in R_1$,

$$
\begin{aligned}
f(dx) &= df(x) \\
&= d\left(\sum_{1\le i\le n} b_i \otimes (p_i(t) + q_i(t)\epsilon) \right) \\
&= \sum_{1\le i\le n} db_i \otimes (p_i(t) + q_i(t)\epsilon) + \sum_{1\le i\le n} (-1)^{|b_i|} b_i \otimes d(p_i(t) + q_i(t)\epsilon) \\
&= \sum_{1\le i\le n} db_i \otimes (p_i(t) + q_i(t)\epsilon) + \sum_{1\le i\le n} (-1)^{|x|} b_i \otimes d(p_i(t)) \\
&= \sum_{1\le i\le n} db_i \otimes (p_i(t) + q_i(t)\epsilon) + \sum_{1\le i\le n} (-1)^{|x|} b_i \otimes p'_i(t)\epsilon.
\end{aligned}
$$

So,

$$
\begin{aligned}
(s_r d + d s_r)(x) &= \sum_{1\le i\le n} (-1)^{|x|-2} db_i \cdot q_i(r) \\
&\qquad + (-1)^{|x|-2} \sum_{1\le i\le n} (-1)^{|x|} b_i \cdot p'_i(r) \\
&\qquad\quad + d\left((-1)^{|x|-1} \sum_{1\le i\le n} b_i \cdot q_i(r) \right) \\
&= \sum_{1\le i\le n} b_i \cdot p'_i(r) \\
&= f'_r(x).
\end{aligned}
$$

□

4.4.2 Polynomial homotopies for almost free resolutions

The following proposition was presented by Kontsevich in his 1994 course on deformation theory at Berkeley.

Proposition 63. *Let R_1, R_2 be DG algebras concentrated in non-negative degrees such that R_1 is almost free and R_2 is acyclic in all degrees greater than 0. Let $f_{0,} f_1 : R_1 \to R_2$ be two morphisms of DG algebras inducing the same morphism $H^0(R_1) \to H^0(R_2)$. Then, there exists a polynomial homotopy between f_1 and f_2.*

Proof. See Proposition 3.6.4 in [CK2]. □

This proposition is stated and proved in the absolute case. However, the relative case is also true: if $R_1, R_2 \in \mathtt{DGA}_S$ and f_0, f_1 are S-algebra morphisms, then there exists a polynomial homotopy such that the members of the family $\{f_t\}$ are S-algebra morphisms. The reader can verify that the proof given in [CK2] generalizes without any obstacles.

Let $f : A \to B$ be a morphism of associative algebras and $p_1 : R_1 \to A, p_2 : R_2 \to B$ two almost free resolutions. Then, by Proposition 46, there exists a morphism of DG algebras $\tilde{f} : R_1 \to R_2$ lifting f, i.e. making the following diagram commute:

$$\begin{tikzcd} R_1 \arrow[r, dotted, "\tilde{f}"] \arrow[d, "p_1"] & R_2 \arrow[d, "p_2"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

By Proposition 63, any two such liftings of f are polynomially homotopic.

Considering the special case where $B \cong A$ and f is the identity, we get that for any two almost free resolutions R_1, R_2 of A, there exist morphisms $h : R_1 \to R_2$ and $h' : R_2 \to R_1$ and a polynomial homotopy from $h'h$ to id_{R_1}.

4.4.3 L-homotopies

Let L_1, L_2 be DG Lie algebras,

$$(f_t, s_t)_{t\in[0,1]} : L_1 \longrightarrow L_2$$

a smooth homotopy with each f_t a DG Lie algebra morphism. The pair $(f_t, s_t)_{t\in[0,1]}$ is an **L-homotopy** if the maps s_t are (graded) Lie derivations with respect to the bimodule structures given by the f_t, i.e.

$$s_t[x, y] = [s_t(x), f_t(y)] + (-1)^{|x|}[f_t(x), s_t(y)].$$

Recall that the Lie algebraization functor $\mathcal{L}$ is the functor which sends an associative DG algebra R to the DG Lie algebra $\mathcal{L}(R)$ with the same underlying chain complex as R and with bracket given by the graded commutator.

Proposition 64. *Let R_1, R_2 be (associative) DG algebras, and $(f_t, s_t)_{t\in[0,1]}$ an M-homotopy $R_1 \to R_2$. Then, (f_t, s_t) induces an L-homotopy*

$$\mathcal{L}(f_t, s_t) : \mathcal{L}(R_1) \to \mathcal{L}(R_2)$$

between $\mathcal{L}(f_0)$ and $\mathcal{L}(f_1)$.

Proof. Let $\mathcal{L}(f_t, s_t) = (f_t, s_t)$. It follows from the definition of $\mathcal{L}$ that for every t, the map f_t is a morphism of DG Lie algebras. And of course, we still have the identity

$$f_t' = ds_t + s_t d.$$

So, it remains only to verify that s_t is a (graded) Lie derivation. We calculate:

$$
\begin{aligned}
s_t[x,y] = & \\
&= s_t(xy - (-1)^{|y||x|}yx) \\
&= s_t(x)f_t(y) + (-1)^{|x|}f_t(x)s_t(y) - (-1)^{|x||y|}s_t(y)f_t(x) - (-1)^{|x||y|+|y|}f_t(y)s_t(x) \\
&= s_t(x)f_t(y) - (-1)^{|y||x+1|}f_t(y)s_t(x) + (-1)^{|x|}\left(f_t(x)s_t(y) - (-1)^{|y+1||x|}s_t(y)f_t(x)\right) \\
&= [s_t(x), f_t(y)] + (-1)^{|x|}[f_t(x), s_t(y)].
\end{aligned}
$$

□

Remark. *As the preceding proposition suggests, L-homotopies are the natural Lie analog of M-homotopies. Besides each M-homotopy $R_1 \to R_2$ descending naturally to an L-homotopy $\mathcal{L}(R_1) \to \mathcal{L}(R_2)$, an L-homotopy $L_1 \to L_2$ lifts naturally to a M-homotopy $\mathcal{U}(L_1) \to \mathcal{U}(L_2)$ (where $\mathcal{U}$ is the universal enveloping algebra functor), and these operations correspond under the adjunction between the functors $\mathcal{U}$ and $\mathcal{L}$.*

CHAPTER 5

DEFINITION AND BASIC PROPERTIES OF DERIVED REPRESENTATION SCHEMES

Prior work on deriving representation schemes includes [TV] and – most significantly for the present work – [CK2]. In the latter work, I. Ciocan-Fontanine and M. Kapranov define the derived representation scheme of an associative algebra A geometrically. They use the bar-cobar resolution – along with two spectral sequences – to prove that their definition is independent of the choice of almost free resolution of A.

In the present chapter, we give a simple and purely algebraic definition of derived representation schemes. This has some significant advantages (which will be exploited in the subsequent chapters):

1. This definition applies not just to vector spaces, but more generally to chain complexes V of finite total dimension.
2. The definition readily generalizes to a relative form, yielding a close relationship with cyclic homology.
3. Because of the construction's explicitness, we can apply Quillen's theorem on adjunctions to prove that this is a derived functor in the sense of homotopical algebra; moreover, this proof turns out to be not only more conceptual, but also simpler than the proof (using spectral sequences) of independence of choice of resolution given in [CK2].

5.1 The DG Representation Scheme

In this section, we generalize to the DG setting the classical construction of the coordinate algebra of the representation scheme of an associative algebra. This is done by first generalizing the non-commutative case, and then applying the abelianization functor.

5.1.1 The non-commutative DG representation scheme

Let (V, d) be a chain complex of k-vector spaces of finite total dimension. Define $\underline{\mathrm{End}}\, V$ to be the graded algebra whose homogeneous elements of degree n are the degree n linear maps $V \to V$. We equip $\underline{\mathrm{End}}\, V$ with differential $d_{\underline{\mathrm{End}}\, V}$ given on homogeneous elements f by the graded commutator with d,

$$d_{\underline{\mathrm{End}}\, V} f = [f, d] = fd - (-1)^{|f|} df.$$

This makes $\underline{\mathrm{End}}\, V$ into a DG algebra. We use the underline notation to distinguish this object from $\mathrm{End}\, V$, which is the associative algebra of degree-zero chain maps $V \to V$.

Let $S \in \mathtt{DGA}_k$ and $S \to \underline{\mathrm{End}}\, V$ be a morphism of DG algebras. Let $S\backslash R \in \mathtt{DGA}_S$.

The graded algebra $\underline{\mathrm{End}}\, V *_S R$ can be given a natural DG algebra structure by inheriting the differential on words of length 1 from d_R and $d_{\underline{\mathrm{End}}\, V}$ and then extending to all elements by the Leibniz rule.

Definition 65. *Define*

$$\begin{aligned} \sqrt[V]{S\backslash R} &:= (\underline{\mathrm{End}}\, V *_S R)^{\underline{\mathrm{End}}\, V} \\ &= \{w \in \underline{\mathrm{End}}\, V *_S R \quad | \quad [w, m] = 0 \textit{ for every } m \in \underline{\mathrm{End}}\, V\}, \end{aligned}$$

where $[w,m] = w \cdot m - (-1)^{|w|\cdot|m|} m \cdot w.$

In other words, $\sqrt[V]{S\backslash R}$ is the subalgebra of graded $\underline{\mathrm{End}}\, V$-**invariants** in $\underline{\mathrm{End}}\, V *_S R$. We will prove shortly (Proposition 67) that it is a DG algebra. Note that in taking invariants, we do not retain the S-algebra structure; thus, $\sqrt[V]{S\backslash R}$ is an element of $\mathtt{DGA}_k$, not $\mathtt{DGA}_S$. We can consider it to be a **non-commutative DG representation scheme**.

Lemma 66. *Let T be any DG algebra, and $x, y \in T$. Then,*

$$d([x,y]) = [dx, y] + (-1)^{|x|}[x, dy].$$

Proof. We calculate:

$$\begin{aligned}
d([x,y]) &= d\left(x \cdot y - (-1)^{|x|\cdot|y|} y \cdot x\right) \\
&= dx \cdot y + (-1)^{|x|} x \cdot dy - (-1)^{|x|\cdot|y|}\left(dy \cdot x + (-1)^{|y|} y \cdot dx\right) \\
&= \left(dx \cdot y - (-1)^{(|x|-1)\cdot|y|} y \cdot dx\right) + \left((-1)^{|x|} x \cdot dy - (-1)^{|x|\cdot|y|} dy \cdot x\right) \\
&= [dx, y] + (-1)^{|x|}[x, dy].
\end{aligned}$$

□

Note that in particular, this lemma implies that the abelianization of any DG algebra T is again a well-defined DG algebra, since the commutator ideal $T[T,T]T$ is closed under the differential.

Proposition 67. $\sqrt[V]{S\backslash R}$ *is a DG algebra.*

Proof. $\sqrt[V]{S\backslash R}$ is an associative subalgebra of $\underline{\mathrm{End}}\, V *_S R$, so it remains to show only that it is a subcomplex.

For $w \in \sqrt[V]{S\backslash R}$, we have $[w,m]=0$ for all $m \in \underline{\mathrm{End}}(V)$. By Lemma 66,

$$\begin{aligned} 0 &= d([w,m]) \\ &= [dw,m]+(-1)^{|w|}[w,dm] \\ &= [dw,m]. \end{aligned}$$

Therefore, $dw \in \sqrt[V]{S\backslash R}$. $\square$

5.1.2 The adjunction between $\sqrt[V]{-}$ and $\underline{\mathbf{End}}$ $V\otimes_S$

Lemma 68. *The algebra map*

$$\phi : \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R} \to \underline{\mathrm{End}}\, V *_S R, \qquad x \otimes y \longmapsto xy$$

is an isomorphism.

Proof. In all sums in this proof, we will write the index (or indices) below the sum symbol $\sum$, leaving implicit that the range over which we are summing is from 1 to n for every index.

Picking a graded basis for V, which we call $\beta_1, ..., \beta_n$, we have elements $\{e_{ij}\}$ spanning $\underline{\mathrm{End}}\, V$ such that the following hold:

$$|e_{ij}| = |\beta_i| - |\beta_j|, \qquad \sum_i e_{ii} = 1, \qquad e_{ij}e_{kl} = \delta_{jk}e_{il}.$$

We'll define a map which we will prove is an inverse to ϕ,

$$\begin{aligned} \psi : \underline{\mathrm{End}}\, V *_S R &\to \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R}, \\ w &\mapsto \sum_{i,j} \left(e_{ij} \otimes \sum_k (-1)^{|e_{jk}|(|w|+|e_{ij}|)} e_{ki} w e_{jk} \right). \end{aligned}$$

To show that ψ is a well-defined map, let's verify that every $w_{ij} := \sum_{k=1}^{n}(-1)^{|e_{jk}|(|w|+|e_{ij}|)}e_{ki}we_{jk}$ is indeed an invariant of $\underline{\mathrm{End}}\, V$. By linearity, it is sufficient to check this for every $e_{i'j'}$:

$$\begin{aligned} w_{ij}e_{i'j'} &= \sum_k (-1)^{|e_{jk}|(|w|+|e_{ij}|)}e_{ki}we_{jk}e_{i'j'} \\ &= (-1)^{|e_{ji'}|(|w|+|e_{ij}|)}e_{i'i}we_{jj'}, \end{aligned}$$

while

$$\begin{aligned} e_{i'j'}w_{ij} &= \sum_k (-1)^{|e_{jk}|(|w|+|e_{ij}|)}e_{i'j'}e_{ki}we_{jk} \\ &= (-1)^{|e_{jj'}|(|w|+|e_{ij}|)}e_{i'i}we_{jj'}, \end{aligned}$$

which implies

$$[w_{ij}, e_{i'j'}] = w_{ij}e_{i'j'} - (-1)^{|e_{i'j'}|(|w|+|e_{ij}|)}e_{i'j'}w_{ij} = 0,$$

as desired.

Next, observe that ψ is a linear map. Thus, if we prove that ψ and ϕ are inverses, the fact that ψ is a homomorphism (i.e. is multiplicative) follows as a formal consequence.

So, we calculate to show that ϕ and ψ are mutual inverses:

$$\begin{aligned} \phi \circ \psi(w) &= \phi\left(\sum_{i,j} e_{ij} \otimes \sum_k (-1)^{|e_{jk}|(|w|+|e_{ij}|)}e_{ki}we_{jk}\right) \\ &= \sum_{i,j,k} (-1)^{|e_{jk}|(|w|+|e_{ij}|)}e_{ij}e_{ki}we_{jk} \\ &= \sum_{i,j,k} (-1)^{|e_{jk}|(|w|+|e_{ij}|)+|w||e_{jk}|}e_{ij}e_{ki}e_{jk}w \\ &= (-1)^{|e_{jj}||e_{ij}|}w \\ &= w, \end{aligned}$$

and in the other direction, for an element of the form $e_{i'j'} \otimes w$ (since all others are linear combinations of such elements), we have

$$\begin{aligned}
\psi \circ \phi(e_{i'j'} \otimes w) &= \sum_{i,j} e_{ij} \otimes \sum_k (-1)^{|e_{jk}|(|w|+|e_{i'j'}|+|e_{ij}|)} e_{ki} e_{i'j'} w e_{jk} \\
&= \sum_{i,j} e_{ij} \otimes \sum_k (-1)^{|e_{jk}|(|e_{ij}|+|e_{i'j'}|)} e_{ki} e_{i'j'} e_{jk} w \\
&= e_{i'j'} \otimes \sum_k e_{kk} w \\
&= e_{i'j'} \otimes w.
\end{aligned}$$

$\square$

Remark. *One may wonder how the sign rule (i.e., the power to which -1 is taken) in the definition of ψ given in the preceding proof was picked. In fact, selecting sign rules in such situations can sometimes be difficult, even when the correct choice seems "obvious"* post factum *(as one might argue is the case here). One approach is to determine all of the properties that the rule must satisfy; in this way, one can write a system of equations whose unknowns are sign rules, and then find a solution to the system. In the present case, there is one sign rule we must chose, namely τ in*

$$\psi(w) = \sum_{i,j} \left(e_{ij} \otimes \sum_k (-1)^{\tau(i,j,k,w)} e_{ki} w e_{jk} \right).$$

There are two conditions that impose limitations upon τ (since all other verifications follow from linearity and do not involve the sign rule): (1) ϕ and ψ must be inverses, and (2) the image of ψ – which a priori *is only inside $\underline{\mathrm{End}}\, V \otimes (\underline{\mathrm{End}}\, V *_S R)$ – must be in its subalgebra $\underline{\mathrm{End}}\, V \otimes \sqrt[V]{S \backslash R}$. This yields the following system of equations for τ :*

$$\begin{aligned}
\tau(i,i,i,w) &= 0 \\
\tau(i,j,i',w) + \tau(i,j,j',w) &= |e_{i'j'}|(|w| + |e_{ij}|)
\end{aligned}$$

From this system, it follows that $\tau(i,j,k,w) = |e_{jk}|(|w| + |e_{ij}|)$.

The reader interested in determining how the system of equations was found can reconstruct this from the verifications in the proof.

Theorem 69. *Let* $R \in \mathtt{DGA}_S$, $B \in \mathtt{DGA}_k$, *and* (V, d) *a chain complex of vector spaces of finite total dimension. Then, we have an adjunction*

$$\mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R}, B) = \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes_k B).$$

Proof. Define a map F as the composition of maps, $F = F_3 F_2 F_1$:

$$\begin{aligned} \mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R}, B) &\xrightarrow{F_1 = \mathrm{id} \otimes -} \mathrm{Hom}_{\mathtt{DGA}_S}(\underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R}, \underline{\mathrm{End}}\, V \otimes B) \\ &\xrightarrow{F_2 = (\phi^{-1})^*} \mathrm{Hom}_{\mathtt{DGA}_S}(\underline{\mathrm{End}}\, V *_S R, \underline{\mathrm{End}}\, V \otimes B) \\ &\xrightarrow{F_3 = |_R} \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes B). \end{aligned}$$

F_1 is tensoring on the left with the identity of $\underline{\mathrm{End}}\, V$, so that $F(f) = \mathrm{id} \otimes f$. F_2 is the map induced by the inverse of the isomorphism ϕ described in the preceding Lemma 68. F_3 is the restriction to $R \subset \underline{\mathrm{End}}\, V *_S R$.

In the opposite direction, define $G = G_3 G_2 G_1$:

$$\begin{aligned} \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes B) &\xrightarrow{G_1 = j *_S -} \mathrm{Hom}_{\mathtt{DGA}_S}(\underline{\mathrm{End}}\, V *_S R, \underline{\mathrm{End}}\, V \otimes B) \\ &\xrightarrow{G_2 = \phi^*} \mathrm{Hom}_{\mathtt{DGA}_S}(\underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R}, \underline{\mathrm{End}}\, V \otimes B) \\ &\overset{G_3 = |_{\sqrt[V]{S\backslash R}}}{\dashrightarrow} \mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R}, B). \end{aligned}$$

G_1 is the map produced by the universal property of the free product with the map

$$j : \underline{\mathrm{End}}\, V \to \underline{\mathrm{End}}\, V \otimes B$$

defined by $j(m) = m \otimes 1$. We write this as $G_1(f) = j *_S f$.

G_2 is induced by the isomorphism ϕ described in the preceding Lemma 68.

Note that, as signified by the dotted arrow, the map G_3 is only defined on elements g of the form $g = G_2G_1(f)$ for some f. It is given by restricting to the (graded) $\underline{\mathrm{End}}\, V$-invariants, which are exactly $\sqrt[V]{S\backslash R}$. By construction of G_1 and G_2, any map $g = G_2G_1(f)$ sends $\underline{\mathrm{End}}\, V \subset \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R}$ identically to $\underline{\mathrm{End}}\, V \otimes 1 \subset \underline{\mathrm{End}}\, V \otimes B$. Thus, for any such g, if we restrict the domain of g to (graded) $\underline{\mathrm{End}}\, V$-invariants, then the range of the restriction also consists of (graded) $\underline{\mathrm{End}}\, V$-invariants, which is precisely the subalgebra $B = 1 \otimes B \subset \underline{\mathrm{End}}\, V \otimes B$. Thus, $G_3(g) \in \mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R}, B)$.

To obtain the desired result, it remains to prove that $FG = \mathrm{id}$ and $GF = \mathrm{id}$. Now, $F_3G_1 = \mathrm{id}$, and on the image of F_2F_1, we also have that $G_1F_3 = \mathrm{id}$. Similarly, $G_2F_2 = \mathrm{id}$ and $F_2G_2 = \mathrm{id}$. Finally, $G_3F_1 = \mathrm{id}$, and on the image of G_2G_1, we also have that $F_1G_3 = \mathrm{id}$. Therefore:

$$\begin{aligned} FG &= F_3F_2F_1G_3G_2G_1 &= \mathrm{id}, \\ GF &= G_3G_2G_1F_3F_2F_1 &= \mathrm{id}. \end{aligned}$$

□

Remark. *The maps $F_1, F_2, F_3, G_1, G_2, G_3$ of the preceding proof will be used in some of the subsequent sections.*

Proposition 70. *Let R be a DG algebra and V a chain complex of finite total dimension. The morphism $\tilde{\pi} \in \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R})$ corresponding via the adjunction of Theorem 69 to $\mathrm{id}_{\sqrt[V]{S\backslash R}} \in \mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R}, \sqrt[V]{S\backslash R})$ is given by*

$$\tilde{\pi} = \left(\phi^{-1}\right)|_R,$$

where $\phi : \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R} \xrightarrow{\sim} \underline{\mathrm{End}}\, V *_S R$ is the isomorphism of Lemma 68.

Proof. We simply pass the identity map through $F = F_3F_2F_1$ to find $\tilde{\pi}$. We obtain:

$$\begin{array}{ccc}
\mathrm{id}_{\sqrt[V]{S\backslash R}} & \in & \mathrm{Hom}_{\mathtt{DGA}_k}(\sqrt[V]{S\backslash R},\ \sqrt[V]{S\backslash R}) \\
\Big\downarrow{\scriptstyle F_1=(\mathrm{id}\otimes -)} & & \\
\mathrm{id}_{\underline{\mathrm{End}}\,V\otimes\sqrt[V]{S\backslash R}} & \in & \mathrm{Hom}_{\mathtt{DGA}_S}(\underline{\mathrm{End}}\,V\otimes\ \sqrt[V]{S\backslash R}, \underline{\mathrm{End}}\,V\otimes\ \sqrt[V]{S\backslash R}) \\
\Big\downarrow{\scriptstyle F_2=(\phi^{-1})^*} & & \\
\phi^{-1} & \in & \mathrm{Hom}_{\mathtt{DGA}_S}(\underline{\mathrm{End}}\,V *_S R, \underline{\mathrm{End}}\,V\otimes\ \sqrt[V]{S\backslash R}) \\
\Big\downarrow{\scriptstyle F_3=|_R} & & \\
(\phi^{-1})\,|_R & \in & \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\,V\otimes\ \sqrt[V]{S\backslash R})
\end{array}$$

□

The map $\tilde{\pi}$ is a generalization to the DG setting of the eponymous map defined on p. 30. Thus, it can be regarded as a **non-commutative universal DG representation**.

5.1.3 An alternative proof of the adjunction

The proof given in the preceding section is a generalization of proofs given by Bergman and Cohn (and reviewed in the preceding chapters of the present thesis). However, another proof can be given, generalizing an argument given in V. Ginzburg's notes [G1] attributed to M. Boyarchenko. This proof relies on the classification of $\mathrm{End}\,V$-bimodules, which are (possibly infinite) direct sums of V. This section can be skipped on first reading, as it is provided solely for additional context.

One might hope that classifying graded $\mathrm{End}\,V$-bimodules (with V a graded vector space) would be quite simple once one has classified (not graded) $\mathrm{End}\,V$-

bimodules. The problem, however, is that establishing that there must be a (non-graded) isomorphism between such a bimodule and a bimodule of the form $V \otimes L$ (where L is a bimodule on which $\operatorname{End} V$ acts trivially) isn't enough, as we must also show that the elements $v \otimes l$ are homogeneous when v and l are. To get around this barrier, we prove the result from scratch, generalizing the proof in the ungraded case (which can be found, for example, in [E]).

Definition 71. *For A a graded k-algebra, the opposite algebra A^{op} is the k-vector space A with multiplication $x * y = yx$ and opposite grading, i.e. homogeneous $x \in A$ satisfies $|x|_{A^{\mathrm{op}}} = -|x|_A$.*

One can verify that for a graded left A-module M, we have a natural dual graded left A^{op}-module M^*, with the action given by $xf(v) = f(xv)$ for $f \in M^*$, $v \in M$, $x \in A^{\mathrm{op}}$.

Lemma 72. *Let V be a graded k-vector space of finite total dimension. Then, $(\operatorname{End}(V)^{\mathrm{op}})^*$ is isomorphic to $\operatorname{End}(V)$ as a graded $\operatorname{End}(V)$-module.*

Proof. Pick a homogeneous basis $\{x_1, ..., x_n\}$ for V. Then, $\operatorname{End}(V)$ can be written as matrices with respect to this basis, and as a vector space it has the canonical basis $E = \{e_{ij} : 1 \leq i, j \leq n\}$ of single-entry matrices. The elements of E are homogeneous, with

$$|e_{ij}|_{\operatorname{End}(V)} = |x_i|_V - |x_j|_V.$$

Now, the vector space $\operatorname{End}(V)^*$ has dual basis E^*, where e_{ij}^* has degree opposite to that of e_{ij}. Thus, both $\operatorname{End}(V)^*$ and $\operatorname{End}(V)^{\mathrm{op}}$ can be represented as matrices with the same grading:

$$|e_{ij}^*|_{\operatorname{End}(V)^*} = |e_{ij}|_{\operatorname{End}(V)^{\mathrm{op}}}.$$

The natural (dual) left action of $\mathrm{End}(V)^{\mathrm{op}}$ on $\mathrm{End}(V)^*$ is given by right matrix multiplication. Similarly, right matrix multiplication gives a left action of $\mathrm{End}(V)^{\mathrm{op}}$ on itself. Thus, the mapping $\phi : \mathrm{End}(V)^* \to \mathrm{End}(V)^{\mathrm{op}}$ given by $\phi(e_{ij}^*) = e_{ij}$ is an isomorphism of graded $\mathrm{End}(V)^{\mathrm{op}}$-modules:

$$_{\mathrm{End}(V)^{\mathrm{op}}}\mathrm{End}(V)^* \cong_{\mathrm{End}(V)^{\mathrm{op}}} \mathrm{End}(V)^{\mathrm{op}}.$$

Dualizing both sides, we obtain an isomorphism of graded $\mathrm{End}(V)$-modules:

$$(\mathrm{End}(V)^{\mathrm{op}})^* \cong \mathrm{End}(V).$$

□

Proposition 73. $\mathrm{End}(V)$ *and* $\mathrm{End}(V)^{\mathrm{op}}$ *are isomorphic as graded k-algebras.*

Proof. Fix a basis for V and express the two algebras as spaces of matrices. The isomorphism is then given by matrix transposition; it is graded because

$$|e_{ij}|_{\mathrm{End}(V)} = -|\tau(e_{ij})|_{\mathrm{End}(V)} = |\tau(e_{ij})|_{\mathrm{End}(V)^{\mathrm{op}}},$$

where τ is the transposition map. □

Lemma 74. *Let V be a graded k-vector space of finite total dimension. Any finite-dimensional graded $\mathrm{End}(V)$-module M is isomorphic to a graded submodule of a finite direct sum of the form $\bigoplus_{i=1}^{t} V[\lambda_i]$, with $\lambda_i \in \mathbb{Z}$.*

Proof. Pick a homogeneous basis $\{b_1, ..., b_m\}$ for M, with dual basis $\{b_1^*, ..., b_m^*\}$ for M^*. Now we define an $\mathrm{End}(V)^{\mathrm{op}}$-module map

$$\begin{aligned} \phi : (\mathrm{End}(V)^{\mathrm{op}}[-|b_1^*|]) \oplus ... \oplus (\mathrm{End}(V)^{\mathrm{op}}[-|b_m^*|]) &\longrightarrow M^* \\ (a_1, ..., a_m) &\longmapsto a_1 b_1^* + ... + a_m b_m^*. \end{aligned}$$

The map ϕ preserves grading. We get a dual $\mathrm{End}(V)$-module map:

$$\phi^* : M \to (\mathrm{End}(V)^{\mathrm{op}}[-|b_1^*|])^* \oplus ... \oplus (\mathrm{End}(V)^{\mathrm{op}}[-|b_m^*|])^* .$$

Because $(\mathrm{End}(V)^{\mathrm{op}})^*$ is isomorphic to $\mathrm{End}(V)$ as a graded $\mathrm{End}(V)$-module by Lemma 72, we can write

$$\phi^* : M \to (\mathrm{End}(V)[|b_1^*|]) \oplus ... \oplus (\mathrm{End}(V)[|b_m^*|]) .$$

The map ϕ is surjective, so ϕ^* is injective. Now each $\mathrm{End}(V)[|b_i^*|]$ is a shift of a direct sum of modules V, and therefore we conclude that M is a submodule of a direct sum of the form $\bigoplus_{i=1}^{t} V[\lambda_i]$, where $t = m \cdot n$ and $\lambda_i \in \mathbb{Z}$. $\square$

Theorem 75. *Let V be a graded k-vector space of finite total dimension. Any finite-dimensional graded $\mathrm{End}(V)$-module M is isomorphic to a direct sum of the form $\bigoplus_{i=1}^{t} V[\lambda_i]$, with $\lambda_i \in \mathbb{Z}$.*

Proof. By Lemma 74, for some t' the module M is isomorphic to a graded submodule of $\bigoplus_{i=1}^{t'} V[\lambda_i]$, with $\lambda_i \in \mathbb{Z}$. We induct on t'.

For $t' = 1$, either $M = 0$ or $M = V[\lambda_1]$, since $M \subseteq V$ and V is a simple module.

Now assume the theorem holds for $t' - 1$, and M is a submodule of $\bigoplus_{i=1}^{t'} V[\lambda_i]$. Define two natural projections:

$$\begin{aligned} \pi_1 : M &\to V[\lambda_1] \\ \pi_2 : M &\to \bigoplus_{i=2}^{t'} V[\lambda_i]. \end{aligned}$$

These projections are graded homomorphisms, so $ker(\pi_1)$ and $ker(\pi_2)$ are graded submodules of M. Because any $x \in M$ equals $\pi_1(x) + \pi_2(x)$, we get

$$\begin{aligned} \ker(\pi_1) \cap \ker(\pi_2) &= 0, \\ \ker(\pi_1) + \ker(\pi_2) &= M. \end{aligned}$$

Therefore,

$$M = \ker(\pi_1) \oplus \ker(\pi_2).$$

Now, as $\ker(\pi_1) \subseteq \bigoplus_{i=2}^{t'} V[\lambda_i]$ and $\ker(\pi_2) \subseteq V[\lambda_1]$, by the inductive hypothesis these modules are direct sums of copies of V, up to shift. Therefore, so is their direct sum, M. $\square$

Next, we generalize this result to infinite-dimensional $\mathrm{End}(V)$-modules.

Corollary 76. *Let V be a graded k-vector space of finite total dimension. Any graded $\mathrm{End}(V)$-module M has the form $V \otimes L$, where L is a graded vector space and the action of $\mathrm{End}(V)$ on L is trivial.*

Proof. We define a set $\mathcal{I}$ consisting of all sets of homogeneous elements $\{x_i\}_I$, $x_i \in M$ satisfying:

(i) There is no $j \in I$ such that x_j can be expressed as

$$x_j = \sum_{k=1}^{r} f_{i_k} x_{i_k}$$

for some $r \in \mathbb{N}$, $f_{i_k} \in \mathrm{End}(V)$.

(ii) The module $\langle x_i \rangle$ is simple for each $i \in I$.

We have an ordering by inclusion on the elements of $\mathcal{I}$, and this collection is non-empty (because it contains the empty set). Any increasing chain is contained in its union, which is also in $\mathcal{I}$. Therefore, by Zorn's Lemma, we have a maximal set $Y = \{y_i\}_I$ in $\mathcal{I}$.

Now assume that $\langle Y \rangle$, the module generated by Y, is not all of M. Because $\langle Y \rangle$ is a graded submodule of M, we may select a homogeneous element $z \in M \setminus \langle Y \rangle$. The module $\langle z \rangle$ is finite-dimensional, and thus by Theorem 75 it is a finite direct sum of simple graded modules, which we call $Z_1, Z_2, ..., Z_t$; pick a homogeneous generator z_j for each Z_j. Now, since $z \notin Y$, there must be at least one $j \in \{1, ..., t\}$ such that z_j not contained in $\langle Y \rangle$; for the corresponding Z_j, the intersection $\langle Y \rangle \cap \langle Z_j \rangle$ must be empty, for otherwise it would be a finite-dimensional graded $\mathrm{End}(V)$-module of dimension strictly less than n, which isn't possible. Therefore, the set $Y \cup \{z_j\}$ is in $\mathcal{I}$; but this contradicts the maximality of Y. Therefore, $\langle Y \rangle = M$, and thus M is a direct sum (indexed by the elements of Y) of graded modules of the form V up to shift, which is the desired result. $\square$

Corollary 77. *Let V be a graded k-vector space of finite total dimension. Any graded $\mathrm{End}(V) \otimes \mathrm{End}(V)^{\mathrm{op}}$-module M has the form $\mathrm{End}(V) \otimes L$, where L is a graded vector space and the action of $\mathrm{End}(V) \otimes \mathrm{End}(V)^{\mathrm{op}}$ on L is trivial.*

Proof. By Proposition 73,

$$\mathrm{End}(V) \otimes \mathrm{End}(V)^{\mathrm{op}} \cong \mathrm{End}(V) \otimes \mathrm{End}(V).$$

We also have an isomorphism of graded algebras

$$\psi : \mathrm{End}(V) \otimes \mathrm{End}(V) \to \mathrm{End}(V \otimes V)$$

given by

$$\psi(f \otimes g)(\sum_{i=1}^{r} v_i \otimes v_i') = \sum_{i=1}^{r} f(v_i) \otimes g(v_i').$$

Therefore, by Corollary 76, $\mathrm{End}(V) \otimes \mathrm{End}(V)^{\mathrm{op}}$ has (up to shift) only one simple graded module M, and any graded $\mathrm{End}(V) \otimes \mathrm{End}(V)^{\mathrm{op}}$-module is isomorphic to the module $M \otimes L$ for some graded vector space L.

Because $\mathrm{End}(V)$ is a graded $\mathrm{End}(V) \otimes \mathrm{End}(V)^{\mathrm{op}}$-module of dimension n^2, where $n = \dim(V)$, it follows that $M \cong \mathrm{End}(V)$. $\square$

Corollary. *The algebra map*

$$\phi : \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R} \to \underline{\mathrm{End}}\, V *_S R, \qquad x \otimes y \longmapsto xy$$

is an isomorphism.

Proof. The fact that ϕ is a morphism of DG algebras is straightforward (for details, see the corresponding proof in the previous section). The harder part is to prove that it is a bijection.

To this end, we can forget about the differential on V (and $\underline{\mathrm{End}}\, V$), and work simply with the underlying graded vector space V and the corresponding graded algebra $\mathrm{End}(V)$.

To show that ϕ is a bijection, we will use the preceding Corollary 77. By restricting ϕ to $\mathrm{End}(V) \otimes 1$, we obtain a graded $\mathrm{End}(V)$-bimodule structure on $\mathrm{End}(V) *_S R$.

By Corollary 77, we have an isomorphism of graded $\mathrm{End}(V)$-bimodules ψ, giving:

$$\mathrm{End}(V) \otimes \sqrt[V]{S\backslash R} \xrightarrow{\phi} \mathrm{End}(V) *_S R \underset{\sim}{\xrightarrow{\psi}} \mathrm{End}(V) \otimes L$$

for some graded k-vector space L (on which $\mathrm{End}(V)$ acts trivially on both sides). Now $\sqrt[V]{S\backslash R}$ is a vector subspace of $\mathrm{End}(V) *_S R$, and under ψ it corresponds bijectively to $1 \otimes L$, since these two subspaces are exactly what is centralized (in their respective bimodules) by the $\mathrm{End}(V)$ action.

Consequently, the composition (regarded as a left $\mathrm{End}(V)$-module map) $\psi \circ \phi$ sends $1 \otimes \sqrt[V]{R}$ bijectively to $1 \otimes L$, and thus is a bijection. Therefore, ϕ is a bijection. $\square$

The adjunction of Theorem 69 follows from this just as in the other proof.

5.1.4 The DG representation scheme and its adjunction with $\underline{\mathbf{End}}$ $V\otimes$

By analogy with the classical case, we define the (coordinate algebra of the) **DG representation scheme**

$$(S\backslash R)_V = \left(\sqrt[V]{S\backslash R}\right)_{\mathrm{ab}}.$$

Recall that in the DG setting, one abelianizes by quotienting by the (graded) commutator ideal (which is a DG ideal).

The mapping

$$(-)_V : \mathtt{DGA}_S \to \mathtt{CDGA}_k, \qquad S\backslash R \mapsto (S\backslash R)_V\,,$$

being a composition of functors (free product, taking invariants, and abelianization), is itself a functor. We call it the **DG representation scheme functor**. It associates to every DG algebra (the coordinate algebra of) its DG representation scheme, and to every morphism between DG algebras an induced morphism between (the coordinate algebras of) their DG representation schemes.

The following adjunction is a corollary of Theorem 69.

Corollary 78. *Let $B \in \mathtt{CDGA}_k, R \in \mathtt{DGA}_S$, and (V, d) a complex of vector spaces of finite total dimension. Then, we have an adjunction*

$$\mathrm{Hom}_{\mathtt{CDGA}_k}\left((S\backslash R)_V, B\right) = \mathrm{Hom}_{\mathtt{DGA}_S}(R, \underline{\mathrm{End}}\, V \otimes B).$$

Proof. Because B is commutative, morphisms $\sqrt[V]{S\backslash R} \to B$ necessarily send

$$[\sqrt[V]{S\backslash R}, \sqrt[V]{S\backslash R}] \mapsto 0,$$

and thus correspond bijectively to morphisms $\left(\sqrt[V]{S\backslash R}\right)_{\mathrm{ab}} \to B$. □

Remark. *Specializing to the case when $S = k$, B and R are associative algebras (regarded as DG algebras concentrated in degree zero), and V is an n-dimensional vector space (regarded as a complex concentrated in degree 0), we recover the classical adjunction for representation varieties:*

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}(R_V, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(R, \mathrm{End}\, V \otimes B).$$

Proposition 79. *Let $R \in \mathtt{DGA}_S$ and V a chain complex of finite total dimension. The morphism $\pi : R \to \underline{\mathrm{End}}\, V \otimes (S\backslash R)_V$ corresponding via the adjunction of Corollary 78 to $\mathrm{id}_{(S\backslash R)_V}$ is given by*

$$\pi = (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}|_R\right),$$

where $\phi : \underline{\mathrm{End}}\, V \otimes \sqrt[V]{S\backslash R} \xrightarrow{\sim} \underline{\mathrm{End}}\, V *_S R$ is the isomorphism of Lemma 68 and $(-)_{\mathrm{ab}} : \sqrt[V]{S\backslash R} \to (S\backslash R)_V$ is the abelianization map.

Proof. The adjunction of Corollary 78 was obtained simply by abelianizing a term in the adjunction of Theorem 69. Thus, we can find π by taking

$$(-)_{\mathrm{ab}} : \sqrt[V]{S\backslash R} \to (S\backslash R)_V$$

(which is the non-commutative map corresponding to $\mathrm{id}_{(S\backslash R)_V}$) and passing it through the adjunction

$$\mathrm{Hom}_{\mathtt{DGA}_k}\left(\sqrt[V]{S\backslash R}, (S\backslash R)_V\right) = \mathrm{Hom}_{\mathtt{DGA}_S}\left(R, \underline{\mathrm{End}}\, V \otimes (S\backslash R)_V\right).$$

Proceeding just as we did in the proof of Proposition 70 (i.e., passing $(-)_{\mathrm{ab}}$ through F_1, F_2, and F_3), we obtain

$$\pi = (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}|_R\right).$$

$\square$

Remark 80. *The map π plays an important role, as it is the generalization to the DG algebra setting of the universal representation (which, naturally, we also called π). We call it the* ***universal DG representation****. For details on the geometric meaning of the classical π, see Subsection 3.1.1.*

5.1.5 Generalizing the $GL(V)$ action

Let $G \subset \mathrm{End}\, V$ be the group of invertible chain complex endomorphisms of V. We can generalize the classical action of $GL(V)$ on the representation scheme to an action of G on the DG analog we defined in the preceding section.

The natural left action by conjugation of G on $\underline{\mathrm{End}}\, V$ induces a left action on $\underline{\mathrm{End}}\, V \otimes (S\backslash R)_V$ given by

$$g \cdot (\phi \otimes x) = g\phi g^{-1} \otimes x, \qquad g \in G,\ \phi \in \underline{\mathrm{End}}\, V,\ x \in (S\backslash R)_V$$

and extended by linearity. Now for $g \in G$, define

$$\pi_g : R \to \underline{\mathrm{End}}\, V \otimes (S\backslash R)_V \qquad r \mapsto g \cdot \pi(r),$$

where π is the universal representation. This corresponds via the adjunction of the preceding section to a morphism $f_g^* : (S\backslash R)_V \to (S\backslash R)_V$. The assignment $g \mapsto f_g^* \in \mathrm{Aut}\,(S\backslash R)_V$ defines a group homomorphism $G^{\mathrm{op}} \to \mathrm{Aut}\,(S\backslash R)_V$, i.e. a natural right action of G on $(S\backslash R)_V$.

To explain the notation f_g^*, recall that in the classical case, the group $GL(V)$ acted on the left on $\mathrm{Rep}_n A$, and this corresponded (via the main adjunction) to a right action on the coordinate algebra A_V. We labeled the left action by $g \mapsto f_g$, and the right action by $g \mapsto f_g^*$. In the DG case, we haven't defined a geometric representation scheme, but we have constructed an algebraic generalization of A_V, namely $(S\backslash R)_V$. Thus, for consistency of notation, we denote the automorphism corresponding to g by f_g^*, even though we have not defined any f_g in this setting.[1]

5.2 The Derived Representation Scheme

This section is the culmination of all our efforts to this point. Here, we apply the preceding chapter's results about model categories along with the preceding section's generalization of classical facts from the theory of representation schemes to the DG setting to define and prove the existence of a derived representation scheme functor. The proof is quite simple, being an immediate consequence of Quillen's theorem on adjunctions. In the second subsection, we

[1]Actually, it is possible to define such a geometric DG analog for $\mathrm{Rep}_V A$. This involves the notion of a DG scheme, and is done in [CK2]. We don't discuss this geometric approach in the present work since our definitions and their consequences do not require this.

establish some basic facts about this functor. Finally, in the third subsection, we reprove some of these results using a completely different approach – one that circumvents the need for model categories, instead relying on M-homotopies.

5.2.1 Existence

In this subsection, we prove the existence of a derived functor (in the sense of Quillen) to the DG representation scheme functor.

Lemma 81. *The pair*

$$(-)_V : \mathtt{DGA}_S \rightleftarrows \mathtt{CDGA}_k : \underline{\mathrm{End}}\, V \otimes$$

is a Quillen adjunction.

Proof. We already proved that these functors are adjoint (Corollary 78). To have a Quillen adjunction (see Theorem 39), we must prove that $(-)_V$ preserves cofibrations and acyclic cofibrations.

As for cofibrations, let $f : S\backslash R_1 \to S\backslash R_2$ be a cofibration in $\mathtt{DGA}_S$. We must prove that there exists a lifting h for every commutative square

$$\begin{tikzcd} (S\backslash R_1)_V \arrow[r] \arrow[d, "f_V"'] & S\backslash X \arrow[d, "g", "\wr"'] \\ (S\backslash R_2)_V \arrow[r] \arrow[ur, dotted, "h"] & S\backslash Y \end{tikzcd}$$

where $X, Y \in \mathtt{CDGA}_k$ and g is an acyclic fibration.

To construct this lifting, observe that passing (each horizontal morphism of) this diagram through the adjunction for $(-)_V$, we obtain the diagram

$$\begin{tikzcd} R_1 \arrow[r] \arrow[d, "f"'] & \underline{\mathrm{End}}\, V \otimes X \arrow[d] \\ R_2 \arrow[r] & \underline{\mathrm{End}}\, V \otimes Y \end{tikzcd}$$

The right edge is a surjection (since it is simply the right-exact functor $\underline{\mathrm{End}}\, V\otimes$ applied to the surjection $X \to Y$) as well as a quasi-isomorphism (by the Knneth formula), so it is an acyclic fibration in $\mathtt{CDGA}_k$. Thus, since f is a cofibration, there exists a lifting $h' : R_2 \to \underline{\mathrm{End}}\, V \otimes X$. Passing this lifting through the adjunction (this time in the opposite direction) gives us the desired h.

The argument showing that $(-)_V$ preserves acyclic cofibrations is identical.

□

Applying Quillen's theorem (Part (i) of Theorem 39) to this lemma, we obtain:

Theorem 82. *The total derived functors of $(-)_V$ and $\underline{\mathrm{End}}\, V\otimes$, which we will call $\mathsf{D}\,(-)_V$ and $\mathcal{E}$ (respectively), exist and form an adjoint pair*

$$\mathsf{D}\,(-)_V : \mathcal{H}o(\mathtt{DGA}_S) \rightleftarrows \mathcal{H}o(\mathtt{CDGA}_k) : \mathcal{E}.$$

We call the functor $\mathsf{D}\,(-)_V$ the **derived representation scheme** functor, and the object $\mathsf{D}\,(S\backslash R)_V \in \mathcal{H}o(\mathtt{CDGA}_k)$ the (coordinate algebra of the) derived representation scheme.

From Part (ii) of Theorem 39, we see that the functor $\mathsf{D}\,(-)_V$ is given on objects $R \in \mathcal{H}o(\mathtt{DGA}_S)$ by considering R as an object of $\mathtt{DGA}_S$, taking an almost free[2] resolution $F \to R$, and concluding with $\gamma_{\mathtt{CDGA}_S}((F)_V)$. For R_1 and R_2 with

[2]Recall that an almost free resolution is a cofibrant replacement.

almost free resolutions F_1 and F_2 (respectively), the functor is given on morphisms $[f] \in \pi(F_1, F_2)$ by picking a representative f of $[f]$ and concluding with $\gamma_{\mathtt{CDGA}_k}(f_V)$.

Remark 83. *Let's specialize to the case when V is concentrated in degree 0 and A is an associative algebra over another associative algebra S. In this case, we can use the model structures on $\mathtt{DGA}_S^+$ and $\mathtt{CDGA}_k$, and $\mathsf{D}\,(S\backslash R)_V$ is given by $(S\backslash F)_V$, where F is any almost free resolution $S\backslash F \to S\backslash A$.*

Once consequence of the results of this section is that $(S\backslash F)_V$ is independent (up to quasi-isomorphism) of the choice of almost free resolution. In particular, its homology (which is a graded algebra) is an invariant of A.

5.2.2 Some basic properties

In this subsection, we gather some important properties of $\mathsf{D}\,(-)_V$.

Given that $\mathsf{D}\,(-)_V$ is the derived functor of $(-)_V$, it is natural to expect the following theorem to hold.

Theorem 84. *Let A be an associative algebra over S and V be concentrated in degree 0. Then,*

$$\mathrm{H}_0\mathsf{D}\,(S\backslash A)_V = (S\backslash A)_V.$$

Proof. Let B be a commutative DG algebra concentrated in degree 0. Pick an almost free resolution $F \to A$ (over S). Then we have a chain of natural isomorphisms:

$$
\begin{aligned}
\mathrm{Hom}_{\mathtt{CommAlg}_k}\left(A_V, B\right) &\overset{(1)}{=} \mathrm{Hom}_{\mathtt{Alg}_S}(A, \mathrm{End}(V)\otimes B)\\
&\overset{(2)}{=} \mathrm{Hom}_{\mathtt{DGA}_S}(F, \mathrm{End}(V)\otimes B)\\
&\overset{(3)}{=} \mathrm{Hom}_{\mathtt{CDGA}_k}((S\backslash F)_V, B)\\
&\overset{(4)}{=} \mathrm{Hom}_{\mathtt{CommAlg}_k}\left(\mathrm{H}_0\left((S\backslash F)_V\right), B\right).
\end{aligned}
$$

Considering each natural isomorphism separately:

- Equality (1) simply comes from the main adjunction; in the case when $S = k$, this is simply Corollary 6, while for general S it is a special case of Corollary 78.
- Equality (2) comes about because $\mathrm{End}(V)\otimes B$ is concentrated in degree 0, and therefore any algebra homomorphism $A \to \mathrm{End}(V)\otimes B$ lifts uniquely to a DG algebra morphism $F \to \mathrm{End}(V)\otimes B$.
- Equality (3) is the main adjunction again (this time, in the DG version).
- Equality (4) holds for the same reason as equality (2).

Thus, recalling that F_V is a representative of the quasi-isomorphism class $\mathsf{D}\left(S\backslash A\right)_V$ and comparing the leftmost and rightmost term of the chain, we see by the Yoneda lemma that

$$\mathrm{H}_0\mathsf{D}\left(S\backslash A\right)_V = (S\backslash A)_V.$$

$\square$

Remark. *This argument does not apply in the case when V is not concentrated in degree 0. We know from the preceding subsection that the degree-zero homology in that* *case gives a well-defined invariant of associative algebras, but it is not known to the author if it is a familiar object.*

Proposition 85. *Let $S\backslash A$ be an associative algebra and V concentrated in degree 0. Then, $\mathrm{H}_i\mathsf{D}\left(S\backslash A\right)_V = 0$ for all $i > 0$.*

Proof. Pick an almost free resolution $F \to A$ (over S). The algebra F is concentrated in non-positive degree, and thus the same is true of $\mathrm{End}(V) *_S F$. Taking invariants (i.e., forming $\sqrt[V]{F}$) and abelianizing don't change this. So, *a fortiori*, the homology of $(S\backslash F)_V = \left(\sqrt[V]{S\backslash F}\right)_{\mathrm{ab}}$ – which is a representative of the quasi-isomorphism class $\mathsf{D}\left(S\backslash A\right)_V$ – in non-positive degrees is 0. □

Remark. *Again, this argument does not work for the case when V is not concentrated in degree 0. The question of whether the positive homologies are 0 in that case is open.*

In the preceding subsection, we saw that the derived representation scheme is an invariant of associative algebras. It is natural to consider using it to study algebras that are difficult to approach using classical means, such as the Weyl algebras A_n. The next proposition shows that, at least in the case of the Weyl algebra (which has no finite-dimensional representations), this is impossible.

Proposition 86. *Let V be concentrated in degree 0 and let A be a unital associative algebra with no finite-dimensional representations. Then, $\mathrm{H}_\bullet\mathsf{D}\left(A\right)_V = 0$.*

Proof. Because A has no finite-dimensional representations, its (classical) representation scheme in V is empty, and thus has coordinate algebra $A_V = 0$. Therefore, by Theorem 84, we have $\mathrm{H}_0\mathsf{D}\left(A\right)_V = 0$.

Now $\mathrm{H}_\bullet\mathsf{D}\left(A\right)_V$ is unital (since all of the categories involved in the constructions of this thesis are categories of unital algebras). The unit 1 must equal 0, and therefore the entire algebra is zero. $\square$

As it turns out, $\mathsf{D}\,(S\backslash -)_V$ is also an invariant of the quasi-isomorphism class of V. We will state the result, but postpone the proof to a forthcoming publication:

Theorem 87. *Let R be a DG algebra over S. Then, for quasi-isomorphic complexes V_1, V_2 (of finite total dimension), we have $(S\backslash R)_{V_1} \cong (S\backslash R)_{V_2}$. In particular, $\mathsf{D}\,(S\backslash A)_V$ depends only on the associative algebra $S\backslash A$ and on the quasi-isomorphism class of V.*

5.2.3 A different approach (via M-homotopies)

In this subsection, we present an alternative approach to the results of Subsection 5.2.1. Namely, we define the derived representation scheme in terms of almost free resolutions, and then prove (using the machinery of M-homotopies) that the resulting DG algebra does not depend (up to quasi-isomorphism) on the choice of resolution.

This approach is in one way less conceptual, since it does not use Quillen's axiomatics or the notion of homotopy categories. It is also less robust, since it does not prove that the resulting "derived" functor is party to an adjunction. However, it is easier (since no model structures must be proved), and in some ways closer in spirit to classical homological algebra.

Proposition 88. *Let $R_{1,}R_2 \in \mathtt{DGA}_S$ and (f_t, s_t) an M-homotopy between them. Then, for any $C \in \mathtt{DGA}_S$, the M-homotopy (f_t, s_t) induces an M-homotopy*

$$(f_t, s_t) *_S \mathrm{id}_C : R_1 *_S C \to R_2 *_S C$$

between $f_0 *_S \mathrm{id}_C$ and $f_1 *_S \mathrm{id}_C$.

Proof. By the definition of an M-homotopy,

$$f_t' = s_t d + d s_t \tag{5.1}$$

Now, we set $\tilde{f}_t := f_t *_S \mathrm{id}_C$ and we define $\tilde{s}_t : R_1 *_S C \to (R_2 *_S C)[1]$ by letting $\tilde{s}_t|_{R_1} = s_t$, $\tilde{s}_t|_C = 0$, and extending by the (graded) Leibniz rule. The map $\tilde{s}_t$ is a derivation. We claim that $(\tilde{f}_t, \tilde{s}_t)$ is the desired M-homotopy.

We must show that

$$\tilde{f}_t' = \tilde{s}_t d + d\tilde{s}_t.$$

It is enough to show this identity on words in $R_1 *_S C$, since then it extends by linearity. We will argue by induction on the word length.

For one-letter words coming from R_1, it follows from the equation 5.1. For one-letter words coming from C, it follows because both sides are 0.

Now assume it holds for all words of length m. Let W be a word of length $m+1$. Write $W = wx$ for w of length k and x in R_1 or C. Then:

$$\begin{aligned} \tilde{f}_t'(wx) &= \tilde{f}_t'(w)\tilde{f}_t(x) + \tilde{f}_t(w)\tilde{f}_t'(x) \\ &= \left(\tilde{s}_t d + d\tilde{s}_t\right)(w)\tilde{f}_t(x) + \tilde{f}_t(w)\left(\tilde{s}_t d + d\tilde{s}_t\right)(x). \end{aligned}$$

Since $\tilde{s}_t$ is a derivation (with respect to $\tilde{f}_t$), so is $\left(\tilde{s}_t d + d\tilde{s}_t\right)$. Therefore, the line above equals

$$\left(\tilde{s}_t d + d\tilde{s}_t\right)(wx),$$

as desired. □

Proposition 89. *Let $R_1, R_2 \in \mathtt{DGA}_S$, let (f_t, s_t) be an M-homotopy between them, and let V be a chain complex of finite total dimension. Then, (f_t, s_t) induces an M-homotopy*

$$\sqrt[V]{(f_t, s_t)} : \sqrt[V]{S\backslash R_1} \to \sqrt[V]{S\backslash R_2}$$

between $\sqrt[V]{f_0}$ and $\sqrt[V]{f_1}$.

Proof. Applying the preceding Proposition with $C = \underline{\mathrm{End}}\,V$, we get an M-homotopy

$$\mathrm{id}_{\underline{\mathrm{End}}\,V} *_S (f_t, s_t) : \underline{\mathrm{End}}\,V *_S R_1 \to \underline{\mathrm{End}}\,V *_S R_2.$$

The family $\tilde{f}_t$ restricts to a smooth family of morphisms on the DG subalgebras of $\underline{\mathrm{End}}\,V$ invariants, $\sqrt[V]{S\backslash R_1}$ and $\sqrt[V]{S\backslash R_2}$. It remains to prove only that $\tilde{s}_t$ restricts to these DG subalgebras for every t (in other words, that $\tilde{s}_t(x)$ is an $\underline{\mathrm{End}}\,V$ invariant for every $\underline{\mathrm{End}}\,V$ invariant x).

So, let x satisfy $[x, m] = 0$ for all $m \in \underline{\mathrm{End}}\,V$. Using the facts that $|\tilde{s}_t(x)| = |x| + 1$ and that $\tilde{s}_t(m) = 0$ for all $m \in \underline{\mathrm{End}}\,V$, we calculate:

$$\begin{aligned} 0 &= \tilde{s}_t[x, m] \\ &= \tilde{s}_t(xm - (-1)^{|x||m|}mx) \\ &= \tilde{s}_t(x)m + (-1)^{|x|}x\tilde{s}_t(m) - (-1)^{|x||m|}\left(\tilde{s}_t(m)x + (-1)^{|m|}m\tilde{s}_t(x)\right) \\ &= \tilde{s}_t(x)m - (-1)^{|x-1||m|}m\tilde{s}_t(x) \\ &= [\tilde{s}_t(x), m]. \end{aligned}$$

□

Lemma 90. *Let R_1,R_2 be DG algebras, $f : R_1 \to R_2$ a morphism of DG algebras, and $s : R_1 \to R_2[n]$ a derivation of degree n with respect to the R_1-bimodule structure given by f on R_2. Then, s descends (through the quotient by the commutator ideal) to $s_{\mathrm{ab}} : (R_1)_{\mathrm{ab}} \to (R_2)_{\mathrm{ab}}[n]$, a degree n derivation with respect to the $(R_1)_{\mathrm{ab}}$-bimodule structure on $(R_2)_{\mathrm{ab}}$ given by f_{ab}.*

Proof. The lemma follows immediately from the verification that

$$s\left(R_1[R_1, R_1]R_1\right) \subseteq R_2[R_2, R_2]R_2.$$

To see this, we first note that for any $r_1, r_2 \in R_1$, we have $s[r_1, r_2] \in [R_2, R_2]$:

$$\begin{aligned}
s[r_1, r_2] &= s\left(r_1 r_2 - (-1)^{|r_1||r_2|} r_2 r_1\right) \\
&= s(r_1) f(r_2) + (-1)^{n \cdot |r_1|} f(r_1) s(r_2) \\
&\quad - (-1)^{|r_1||r_2|} \left(s(r_2) f(r_1) + (-1)^{n \cdot |r_2|} f(r_2) s(r_1)\right) \\
&= s(r_1) f(r_2) - (-1)^{|r_1 + n||r_2|} f(r_2) s(r_1) \\
&\quad + (-1)^{n \cdot |r_1|} \left(f(r_1) s(r_2) - (-1)^{|r_1||r_2 + n|} s(r_2) f(r_1)\right) \\
&= [s(r_1), f(r_2)] + (-1)^{n \cdot |r_1|} [f(r_1), s(r_2)].
\end{aligned}$$

Now, we calculate:

$$\begin{aligned}
s(r_1[r_2, r_3]r_4) &= s(r_1) f([r_2, r_3]r_4) \pm f(r_1) s([r_2, r_3]r_4) \\
&= s(r_1) f[r_2, r_3] f(r_4) \pm f(r_1) s[r_2, r_3] f(r_4) \pm f(r_1) f[r_2, r_3] s(r_4).
\end{aligned}$$

Because f sends commutators to commutators and because $s[r_2, r_3] \in [R_2, R_2]$, each of the three terms of this last expression is in $R_2[R_2, R_2]R_2$, so we're done. □

Proposition 91. *Let* $R_1, R_2 \in \mathtt{DGA}_S$ *and* (f_t, s_t) *an M-homotopy between them. Then,* (f_t, s_t) *induces an M-homotopy*

$$(f_t, s_t)_{\mathrm{ab}} : (R_1)_{\mathrm{ab}} \to (R_2)_{\mathrm{ab}}$$

between $(f_0)_{\mathrm{ab}}$ and $(f_1)_{\mathrm{ab}}$.

Proof. The family f_t descends to a smooth family of morphisms on the quotient DG algebras $(R_1)_{\mathrm{ab}}$ and $(R_2)_{\mathrm{ab}}$. It remains to show only that s_t descends to these quotient DG algebras for every t, and this follows from Lemma 90. □

Theorem 92. *Let $R_{1,}R_2 \in \mathtt{DGA}_S$ and (f_t, s_t) an M-homotopy between them. Then, (f_t, s_t) induces an M-homotopy*

$$(f_t, s_t)_V : (S\backslash R_1)_V \to (S\backslash R_2)_V$$

between $(f_0)_V$ and $(f_1)_V$.

Proof. This follows from the preceding three propositions. $\square$

Corollary 93. *Let $S\backslash F_1, S\backslash F_2$ be two almost free resolutions of an associative algebra $A \in \mathtt{Alg}_S$. Then, there exists a quasi-isomorphism of DG S-algebras $f : F_1 \to F_2$, and for any such f, the map*

$$f_V : (S\backslash F_1)_V \to (S\backslash F_2)_V$$

is a quasi-isomorphism.

Proof. By Proposition 63, there exist morphisms of DG S-algebras $f : F_1 \to F_2$ and $g : F_2 \to F_1$, and from this we get polynomial homotopies from gf to id_{R_1} and from fg to id_{R_2}. By the preceding Theorem 92, we get an M-homotopy from $(gf)_V$ to $(\mathrm{id}_{R_1})_V$.

But because we have

$$\begin{aligned}
\left(\mathrm{id}_{S\backslash R_1}\right)_V &= \mathrm{id}_{(S\backslash R_1)_V}, \\
(gf)_V &= g_V f_V,
\end{aligned}$$

this is an M-homotopy from $g_V f_V$ to $\mathrm{id}_{(R_1)_V}$.

We have a similar M-homotopy in the other direction, and therefore g_V and f_V both induce isomorphism on homology, and in particular f_V is a quasi-isomorphism. $\square$

Theorem 94. *Let $\mathcal{C}$ be the localization of $\mathtt{DGA}_S$ at the class of quasi-isomorphisms, with $\gamma : \mathtt{DGA}_S \to \mathcal{C}$ the functor sending each algebra to its quasi-isomorphism class in $\mathcal{C}$. Consider the assignment*

$$D_V : \mathtt{Alg}_S \to \mathcal{C}$$

taking an associative algebra $S\backslash A$ to $\gamma\left((S\backslash F)_V\right)$, where $F \to A$ is an almost free resolution of A over S, and taking a morphism of DG S-algebras $f : A \to B$ to $\gamma\left(\left(\tilde{f}\right)_V\right)$, where $\tilde{f} : S\backslash F_1 \to S\backslash F_2$ is a lifting of f to almost free resolutions $F_1 \to A, F_2 \to B$. This assignment is well-defined (i.e., independent of the choices of almost free resolution). Moreover, it is a functor.

Proof. On objects, independence of the choice of resolution follows from Corollary 93.

Now, for any morphism of DG S-algebras $f : A \to B$ and resolutions $F_1 \to A, F_2 \to B$, there exists (by Proposition 46[3]) a lifting $\tilde{f} : F_1 \to F_2$, and any two such liftings are M-homotopic (as a consequence of Proposition 63). This M-homotopy passes through the functor $(S\backslash -)_V$ (by previous results), and therefore the two liftings give the same map $D_V(A) \to D_V(B)$.

We must also show that this is independent of the choice of resolutions F_1, F_2. Indeed, for another choice F_1' and F_2' we have morphisms of DG S-algebras $i_A : F_1' \to F_1$ and $i_B : F_2 \to F_2'$ lifting the identities on A and B respectively. Further, the map $(F_1')_V \to (F_1)_V$ induced by i_A is a quasi-isomorphism (by the results on polynomial homotopies in this context) and similarly for i_B.

[3]While this proposition occurs in the section on model categories, it can be restated without recourse to the notions of fibrations and cofibrations. The proof also is self-contained. Thus, the present approach (using M-homotopies) is not simply the same model-categorical approach in new clothing.

Also, while the proposition is stated in the absolute case (i.e., in $\mathtt{DGA}_k$), one can readily verify that the proof generalizes without problems to the relative case ($\mathtt{DGA}_S$).

This shows that lifting f to a map of DG algebras between F_1' and F_2' cannot yield a different map between $D_V(A)$ and $D_V(B)$.

This means that an assignment $D_V : \mathrm{Mor}(\mathtt{Alg}_S) \to \mathrm{Mor}(\mathcal{C})$ is well defined. That this assignment is functorial follows immediately from the fact that all of the composite assignments of $(S\backslash -)_V = \left(\sqrt[V]{S\backslash -}\right)_{\mathrm{ab}}$ are functors, as is γ. $\square$

CHAPTER 6

THE DERIVED KONTSEVICH-ROSENBERG PRINCIPLE

Having defined the derived representation scheme, the next step is to develop a derived version of the Kontsevich-Rosenberg principle. Such a derived principle would posit that every derived non-commutative geometric structure on an associative algebra A would induce corresponding derived commutative structures on the (coordinate algebra of the) representation scheme of A.

As it turns out, the *ad hoc* approach (several examples of which were discussed in Chapter 3) transfers to the derived setting with no major conceptual differences, despite a considerable increase in the complexity of the calculations involved.

Our main interest is in developing a derived version of the Van den Bergh functor, and it turns out that this is possible. Moreover, the proof of the functor's existence is – just as in the case of the derived representation scheme – a straightforward consequence of Quillen's theorem on adjunctions; we also provide a second, more concrete, proof.

6.1 The Principle Illustrated with Non-commutative Vector Fields

In Subsection 3.3.3, we saw a natural (albeit *ad hoc*) construction associating to every non-commutative vector field (i.e., derivation) on an associative algebra A a corresponding commutative vector field on $\mathrm{Rep}_V A$. In this section, we give a derived version of this result; as we will see, this amounts to constructing a natural map from the Hochschild cohomology of A to the space of derived vector fields on $\mathrm{Rep}_V A$. Throughout the section, we work in an absolute setting ($S = k$) and with V concentrated in degree 0.

This section has a number of cumbersome constructions and calculations. It serves primarily to demonstrate the extent to which the machinery developed in the preceding chapter allows classical results to be carried into the derived setting. There are no results in this thesis that depend on this section, and the reader is thus advised to skip it on first reading (and, quite frankly, on all subsequent readings as well).

6.1.1 Derivations

Recall that for R a DG algebra and M a DG R-bimodule, $\mathrm{Der}_m(R, M)$ is the space of all degree m linear maps $\phi : R \to M$ satisfying

$$\phi(xy) = \phi(x)y + (-1)^{m|x|}x\phi(y)$$

for all homogeneous $x, y \in R$.

Define the space of **graded derivations** (which we will usually call just "derivations")

$$\mathrm{Der}(R, M) := \bigoplus_m \mathrm{Der}_m(R, M).$$

The graded vector space $\mathrm{Der}(R, M)$ comes equipped with a differential given by

$$d_{\mathrm{Der}(R,M)}(\phi) := [d, \phi] = d_M \circ \phi - (-1)^{|\phi|}\phi \circ d_R$$

on homogeneous ϕ.

Now, let R and M be DG algebras, and $f : R \to M$ be a DG algebra morphism making M into a DG R-algebra. Form $M(\epsilon_m)$, where ϵ_m is graded central,[1] $|\epsilon_m| = -m$, $\epsilon_m^2 = 0$, and $d(\epsilon_m) = 0$. We have a natural projection map

$$\begin{aligned} p_{(\epsilon_m=0)} : M(\epsilon_m) &\to M, \\ x + y\epsilon_m &\mapsto x. \end{aligned}$$

Define $\mathrm{Hom}_{\mathtt{GrAlg}_k, f}(R, M(\epsilon_m))$ be the set of graded algebra homomorphisms $\psi : R \to M(\epsilon_m)$ satisfying $p\psi = f$.

Proposition 95. *When R, M are DG algebras and $f : R \to M$ a morphism, we have a natural identification*

$$\Psi : \mathrm{Der}_m(R, M) \to \mathrm{Hom}_{\mathtt{GrAlg}_k, f}(R, M(\epsilon_m))$$

given by

$$\Psi(\phi)(x) = f(x) + \epsilon_m \cdot \phi(x).$$

Proof. Let $\phi \in \mathrm{Der}_m(R, M)$. Clearly, $\Psi(\phi)$ is a graded linear map. To see that it is an algebra homomorphism, we calculate:

$$\begin{aligned} \Psi(\phi)(xy) &= f(xy) + \epsilon_m \cdot \phi(xy) \\ &= f(x)f(y) + \epsilon_m \cdot \big(\phi(x)f(y) + (-1)^{m|x|} f(x)\phi(y)\big) \\ &= f(x)f(y) + \epsilon_m \cdot \phi(x)f(y) + (-1)^{m|x|}(-1)^{m|x|} f(x)\epsilon_m\phi(y) \\ &= \left(f(x) + \epsilon_m \cdot \phi(x)\right)\left(f(y) + \epsilon_m \cdot \phi(y)\right) \\ &= \left(\Psi(\phi)(x)\right)\left(\Psi(\phi)(y)\right). \end{aligned}$$

The other direction is similar. $\square$

[1] I.e., $\epsilon_m x = (-1)^{m|x|} x\epsilon_m$ for all homogeneous $x \in M$.

Remark 96. *Under the identification of* $\mathrm{Der}_m(R, M)$ *with* $\mathrm{Hom}_{\mathtt{GrAlg}_k, f}(R, M(\epsilon_m))$*, the differential* $d_{\mathrm{Der}(R,M)}$ *is given on elements* $\psi : x \mapsto f(x) + \epsilon_m \cdot \phi(x)$ *by*

$$d(\psi) : x \longmapsto f(x) + \epsilon_{m-1} \cdot (d_M \phi(x) - (-1)^{|\phi|} \phi d_R(x)).$$

6.1.2 Derived derivations (Hochschild cohomology)

Let R_1, R_2 be DG algebras, $f : R_1 \to R_2$ a DG algebra morphism, and M at the same time an R_2-bimodule and an R_1-bimodule. Define

$$\Psi_f : \mathrm{Der}(R_2, M) \to \mathrm{Der}(R_1, M)$$

by

$$\Psi_f(\phi) = \phi \circ f.$$

Proposition 97. *Let* R_1, R_2 *be DG algebras,* $(f_t, s_t) : R_1 \to R_2$ *a smooth homotopy, and* M *at the same time an* R_2*-bimodule and an* R_1*-bimodule. Then, we have a smooth homotopy* (Ψ_{f_t}, S_t) *between* Ψ_{f_0} *and* Ψ_{f_1}*, where* S_t *is given by*

$$S_t(\phi) = (-1)^{|\phi|} \phi \circ s_t.$$

Proof. Ψ_{f_t} and S_t are both smooth families. We have

$$\Psi'_{f_t}(\phi) = (\phi \circ f_t)' = \phi \circ f'_t.$$

So, we calculate:

$$
\begin{aligned}
[d, S_t](\phi) &= d_{\mathrm{Der}(R_1,M)} S_t(\phi) + S_t\left(d_{\mathrm{Der}(R_2,M)}\phi\right) \\
&= d_{\mathrm{Der}(R_1,M)}\left((-1)^{|\phi|}\phi s_t\right) + S_t([d,\phi]) \\
&= [d, (-1)^{|\phi|}\phi s_t] + (-1)^{|\phi|+1}\left(d_M\phi - (-1)^{|\phi|}\phi d_{R_2}\right) s_t \\
&= (-1)^{|\phi|} d_M \phi s_t - (-1)^{|\phi|+1}(-1)^{|\phi|}\phi s_t d_{R_1} \\
&\quad + (-1)^{|\phi|+1} d_M \phi s_t - (-1)^{|\phi|+1}(-1)^{|\phi|}\phi d_{R_2} s_t \\
&= \phi s_t d_{R_1} + \phi d_{R_2} s_t \\
&= \phi[d, s_t] \\
&= \phi \circ f_t' \\
&= \Psi'_{f_t}(\phi).
\end{aligned}
$$

$\square$

Proposition 98. *Let R_1, R_2 be two almost free resolutions of an associative algebra A. Then, there exists a quasi-isomorphism of DG algebras $f : R_1 \to R_2$, and for any such f, the map*

$$\Psi_f : \mathrm{Der}(R_2, A) \to \mathrm{Der}(R_1, A)$$

is a quasi-isomorphism of chain complexes.

Proof. By Proposition 63, there exist quasi-isomorphisms $f : R_1 \to R_2$ and $g : R_2 \to R_1$, and this gives us M-homotopies between gf and id_{R_1} and between fg and id_{R_2}. The maps f and g induce maps

$$
\begin{aligned}
\Psi_f : \mathrm{Der}(R_2, A) &\to \mathrm{Der}(R_1, A) \\
\Psi_g : \mathrm{Der}(R_1, A) &\to \mathrm{Der}(R_2, A)
\end{aligned}
$$

and by Proposition 97 we end up with smooth homotopies from $\Psi_f\Psi_g$ to

$\mathrm{id}_{\mathrm{Der}(R_1,A)}$ and from $\Psi_g\Psi_f$ to $\mathrm{id}_{\mathrm{Der}(R_2,A)}$. From this it follows that Ψ_f and Ψ_g induce isomorphism on homology. □

Definition 99. *Let A be an associative algebra. The* ***derived derivations*** $\mathrm{R}_\bullet\mathrm{Der}(A, A)$ *are defined as the graded vector space* $\mathrm{H}_\bullet\mathrm{Der}(F, A)$*, where* $F \xrightarrow{\rho} A$ *is an almost free resolution and* ρ *determines the bimodule structure of* A *over* F*.*

As a consequence of Proposition 98, this definition is independent of the choice of almost free resolution $F \xrightarrow{\rho} A$. In fact, this is just the Hochschild cohomology of A, shifted by one degree (see Lemma 4.2.1 of [BP]).

6.1.3 Constructing the map $\tilde{\Gamma}$

In this subsection, we construct a natural map $\tilde{\Gamma}$ whose homology will give us a map from derived derivations on A to derived vector fields on the representation variety.

Let $F \xrightarrow{\rho} A$ be an almost free resolution of an associative algebra A, and V a finite-dimensional k-vector space. Consider the following maps:

1. The natural map of graded algebras (concentrated in degree 0)

$$\pi : A \to \mathrm{End}(V) \otimes A_V$$

corresponding in the adjunction of Corollary 78 to $\mathrm{id} : A_V \to A_V$.

2. The extension of π by scalars,

$$\pi^{\epsilon_m} : A(\epsilon_m) \to \mathrm{End}(V) \otimes (A_V(\epsilon_m))\,,$$

where ϵ_m is graded central, $|\epsilon_m| = -m$, $\epsilon_m^2 = 0$, and $d(\epsilon_m) = 0$.

3. The map

$$G : \mathrm{Hom}_{\mathtt{GrAlg}_k}\left(F, \mathrm{End}(V) \otimes (A_V(\epsilon_n))\right) \to \mathrm{Hom}_{\mathtt{GrAlg}_k}\left(F_V, (A_V(\epsilon_n))\right)$$

obtained by forgetting the differential on F and then following the proof of Theorem 69. (Note that because we have forgotten the differential on F, this is a map between sets of graded algebra morphisms, not DG algebra morphisms.)

Define the map of vector spaces

$$\begin{aligned} \Gamma_m : \mathrm{Hom}_{\mathtt{GrAlg}_k}\left(F, A(\epsilon_m)\right) &\rightarrow \mathrm{Hom}_{\mathtt{GrAlg}_k}\left(F_V, (A_V(\epsilon_m))\right), \\ \phi &\mapsto G \circ \pi^{\epsilon_m} \circ \phi. \end{aligned}$$

The map G factors, as in the proof of Theorem 69, as $G = G_3 G_2 G_1$. Thus, the map Γ_m factors into five maps, which we will call $\Gamma_m^{(1)}, \ldots, \Gamma_m^{(5)}$, as shown in the diagram below. To save space, $\mathrm{Hom}_{\mathtt{GrAlg}_k}(-,-)$ is written in the diagram simply as $(-,-)$.

$$\begin{array}{rlc} D_m^{(0)} & = & (F, A(\epsilon_m)) \\ & & \Big\downarrow {\scriptstyle \pi^{\epsilon_m}\circ -} \; {\scriptstyle \Gamma_m^{(1)}} \\ D_m^{(1)} & = & (F, \mathrm{End}(V) \otimes (A_V(\epsilon_m))) \\ & & \Big\downarrow {\scriptstyle G_3 = j * -} \; {\scriptstyle \Gamma_m^{(2)}} \\ D_m^{(2)} & = & (\mathrm{End}(V) * F, \mathrm{End}(V) \otimes (A_V(\epsilon_m))) \\ & & \Big\downarrow {\scriptstyle G_2 = \phi^*} \; {\scriptstyle \Gamma_m^{(3)}} \\ D_m^{(3)} & = & \left(\mathrm{End}(V) \otimes \sqrt[V]{F}, \mathrm{End}(V) \otimes (A_V(\epsilon_m))\right) \\ & & \Big\downarrow {\scriptstyle G_3 = |_{\sqrt[V]{F}}} \; {\scriptstyle \Gamma_m^{(4)}} \\ D_m^{(4)} & = & \left(\sqrt[V]{F}, (A_V(\epsilon_m))\right) \\ & & \Big\downarrow {\scriptstyle (-)_{\mathrm{ab}}} \; {\scriptstyle \Gamma_m^{(5)}} \\ D_m^{(5)} & = & (F_V, (A_V(\epsilon_m))) \end{array} \tag{6.1}$$

Define $\rho_0, ..., \rho_5$ by

$$(\rho = \rho_0) \overset{\Gamma_m^{(1)}}{\mapsto} \rho_1 \overset{\Gamma_m^{(2)}}{\mapsto} \rho_2 \overset{\Gamma_m^{(3)}}{\mapsto} \rho_3 \overset{\Gamma_m^{(4)}}{\mapsto} \rho_4 \overset{\Gamma_m^{(5)}}{\mapsto} \rho_5.$$

Each ρ_i satisfies $p_{(\epsilon_m=0)} \circ \rho_i = p_{(\epsilon_m=1)} \circ \rho$, where p is the same as in Proposition 95. In words, ρ_i has no ϵ_m component. (This is easy to see directly by passing $\rho = \rho_0$ through each of the $\Gamma_m^{(i)}$ in sequence and noticing that the result still has no ϵ_m component.)

Each $D_m^{(i)}$ is of the form $\mathrm{Hom}_{\mathtt{GrAlg}_k}(C_1, C_2(\epsilon_m))$ for graded algebras C_1, C_2. This is of course immediately apparent for $i = 0, 4, 5$. For $i = 1, 2, 3$ it is true because $(\epsilon_m \otimes \mathrm{id}_{\mathrm{End}(V)})$ is in the graded center (by a simple calculation). Thus, we're in a position to talk about derivations in each $D_m^{(i)}$.

Under the correspondence of Proposition 95, a derivation with respect to ρ_i is a morphism of graded algebras $\psi \in D_m^{(i)}$ of the form $\psi(x) = \rho_i(x) + \epsilon_m \cdot g(x)$, where g has no ϵ_m component, i.e. $p_{(\epsilon_m=0)} \circ g = p_{(\epsilon_m=1)} \circ g$.

Lemma 100. *For each $1 \leq i \leq 5$, if $\psi \in D_m^{(i-1)}$ is a derivation with respect to ρ_{i-1}, then $\Gamma_m^{(i)}(\psi)$ is a derivation with respect to ρ_i.*

Proof. First, we observe that for each $\Gamma_m^{(i)}$ and morphisms of graded algebras $f, g \in D_m^{(i-1)}$ we have

$$\Gamma_m^{(i)}(f + g) = \Gamma_m^{(i)}(f) + \Gamma_m^{(i)}(g).$$

This holds for $i = 1$ and $i = 3$ because addition of functions distributes over function composition, for $i = 2$ because addition distributes over free product, for $i = 4$ because $\Gamma_m^{(4)}$ is simply a restriction, and for $i = 5$ because addition of functions commutes with quotients (such as abelianization).

Next, we observe that if a map g with $(\epsilon_m \cdot g) \in D_m^{(i-1)}$ has no ϵ_m component (i.e., $p_{(\epsilon_m=0)} \circ g = p_{(\epsilon_m=1)} \circ g$), then $\Gamma_m^{(i)}(\epsilon_m \cdot g)$ has the form $(\epsilon_m \cdot f) \in D_m^{(i)}$ for some f with no ϵ_m component. This holds for $i = 5$ because ϵ_m is in the graded center (and thus passes untouched through abelianization), and it's similarly easy to see it for each of $i = 1, 2, 3, 4$. $\square$

Lemma 101. The map $\rho_5 = \rho_V$.

Proof. We pass $\rho = \rho_0$ through the maps $\Gamma_m^{(i)}$:

$$\begin{array}{ccc}
\rho_0 = \rho & \in & (F, A) \\
\Big\downarrow {\scriptstyle (\pi\circ -)} \ {\scriptstyle \Gamma_m^{(1)}} & & \\
\rho_1 = \left(\mathrm{id}_{\mathrm{End}(V)} \otimes (-)_{\mathrm{ab}}\right) \circ ((\phi^{-1})\,|_A) \circ \rho & \in & (F, \mathrm{End}(V) \otimes A_V) \\
\Big\downarrow {\scriptstyle G_1 = (\mathrm{id}_{\mathrm{End}(V)} * -)} \ {\scriptstyle \Gamma_m^{(2)}} & & \\
\rho_2 = \left(\left(\mathrm{id}_{\mathrm{End}(V)} \otimes (-)_{\mathrm{ab}}\right) \circ \phi^{-1}\right) \circ \left(\mathrm{id}_{\mathrm{End}(V)} * \rho\right) & \in & (\mathrm{End}(V) * F, \mathrm{End}(V) \otimes A_V) \\
\Big\downarrow {\scriptstyle G_2 = \phi^*} \ {\scriptstyle \Gamma_m^{(3)}} & & \\
\rho_3 = \left(\mathrm{id}_{\mathrm{End}(V)} \otimes (-)_{\mathrm{ab}}\right) \circ \left(\mathrm{id}_{\mathrm{End}(V)} \otimes \sqrt[V]{\rho}\right) & \in & \left(\mathrm{End}(V) \otimes \sqrt[V]{F}, \mathrm{End}(V) \otimes A_V\right) \\
\Big\downarrow {\scriptstyle G_3 = |_{\sqrt[V]{F}}} \ {\scriptstyle \Gamma_m^{(4)}} & & \\
\rho_4 = (-)_{\mathrm{ab}} \circ \sqrt[V]{\rho} & \in & \left(\sqrt[V]{F}, A_V\right) \\
\Big\downarrow {\scriptstyle (-)_{\mathrm{ab}}} \ {\scriptstyle \Gamma_m^{(5)}} & & \\
\rho_5 = \rho_V & \in & (F_V, A_V)
\end{array}$$

The value of ρ_1 follows from the calculation of π in Proposition 79. The morphism ρ_2 has the given form because $\mathrm{id}_{\mathrm{End}(V)} * (\phi^{-1})\,|_A = \phi^{-1}$. The morphism ρ_3 has the given form because the following square commutes (as can be seen by tracking through both branches a sample element $\sum_j w_j \otimes m_j$, where w_j is a word in $\sqrt[V]{F}$ and $m_j \in \mathrm{End}(V)$ for each j):

$$\begin{array}{ccc} \mathrm{End}(V) \otimes \sqrt[V]{F} & \xrightarrow{\phi} & \mathrm{End}(V) * F \\ \Big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{End}(V)} \otimes \sqrt[V]{\rho}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{End}(V)} * \rho} \\ \mathrm{End}(V) \otimes \sqrt[V]{A} & \xrightarrow{\phi} & \mathrm{End}(V) * A \end{array}$$

So, one can apply map $\Gamma_m^{(3)}$ to obtain

$$\begin{aligned} \left(\mathrm{id}_{\mathrm{End}(V)} \otimes (-)_{\mathrm{ab}}\right) \circ \phi^{-1} \quad \circ \quad & \left(\mathrm{id}_{\mathrm{End}(V)} * \rho\right) \circ \phi \\ = \quad & \left(\mathrm{id}_{\mathrm{End}(V)} \otimes (-)_{\mathrm{ab}}\right) \circ \phi^{-1} \circ \phi \circ \left(\mathrm{id}_{\mathrm{End}(V)} \otimes \sqrt[V]{\rho}\right) \\ = \quad & \left(\mathrm{id}_{\mathrm{End}(V)} \otimes (-)_{\mathrm{ab}}\right) \circ \left(\mathrm{id}_{\mathrm{End}(V)} \otimes \sqrt[V]{\rho}\right) . \end{aligned}$$

The map $\Gamma_m^{(4)}$, which is the restriction to the $\mathrm{End}(V)$-invariants $\sqrt[V]{F} \subset \mathrm{End}(V) \otimes \sqrt[V]{F}$, then gives us

$$(-)_{\mathrm{ab}} \circ (\sqrt[V]{\rho}) ,$$

which passes through the abelianization $\Gamma_m^{(5)}$ to become

$$(\sqrt[V]{\rho})_{\mathrm{ab}} = \rho_V .$$

$\square$

Using the notation of Diagram 6.1, for each $0 \leq i \leq 5$, define the space $\tilde{D}_m^{(i)} \subset D_m^{(i)}$ as the collection of ρ_i-derivations in $D_m^{(i)}$. Define for each $1 \leq i \leq 5$ the map $\tilde{\Gamma}_m^{(i)}$ as the restriction of $\Gamma_m^{(i)}$ to $\tilde{D}_m^{(i-1)}$.

Remark. *A pedantic (but important) point to make is that, just like $\Gamma_m^{(4)}$, the map $\tilde{\Gamma}_m^{(4)}$ isn't defined on the entirety of its domain $D_m^{(3)}$. However, $\tilde{\Gamma}_m^{(4)}$* is *defined on the subset of its domain that is in the image of $\tilde{\Gamma}_m^{(3)} \circ \tilde{\Gamma}_m^{(2)}$.*

Proposition 102. *Restricting the map*

$$\Gamma_m : \mathrm{Hom}_{\texttt{GrAlg}_k}\left(F, A(\epsilon_m)\right) \to \mathrm{Hom}_{\texttt{GrAlg}_k}\left(F_V, (A_V(\epsilon_m))\right)$$

to derivations (with respect to ρ) gives a map

$$\tilde{\Gamma}_m : \mathrm{Der}_m(F, A) \to \mathrm{Der}_m\left(F_V, A_V\right).$$

Proof. We will show that the composition of the maps $\tilde{\Gamma}_m^{(i)}$ gives a map from $\tilde{D}_m^{(0)}$ to $\tilde{D}_m^{(5)}$. Indeed, if we take a $\psi \in \mathrm{Der}_m(F, A) = \tilde{D}_m^{(0)}$, a derivation with respect to $\rho = \rho_0$, and pass it through each of the $\tilde{\Gamma}_m^{(i)}$, we'll get (by Lemma 90) at each successive step a derivation with respect to ρ_i, i.e. an element of $\tilde{D}_m^{(i)}$.

In the end, we end up with a derivation with respect to ρ_5, which by Lemma 101 is equal to ρ_V. Therefore, $\tilde{D}_m^{(5)} = \mathrm{Der}_m\left(F_V, A_V\right)$.

Thus, $\tilde{\Gamma}_m$ indeed gives a map of the required form. □

Define the map of graded vector spaces

$$\tilde{\Gamma} : \mathrm{Der}(F, A) \to \mathrm{Der}\left(F_V, A_V\right)$$

as the map obtained by gathering together the $\tilde{\Gamma}_m$.

6.1.4 Constructing the map from derived derivations to derived vector fields on the representation scheme

Theorem 103. *The map $\tilde{\Gamma}$ is a morphism of chain complexes.*

Proof. We defined $\tilde{\Gamma}_n$ as the composition of the maps $\tilde{\Gamma}_n^{(i)}$, which are the restrictions of the maps $\Gamma_n^{(i)}$ the spaces of ρ_i-derivations of degree n, the $\tilde{D}_n^{(i)}$. For each i, the spaces $\tilde{D}_n^{(i)}$ form a complex of derivations

$$\tilde{D}^{(i)} = \bigoplus_{n \in \mathbb{Z}} \tilde{D}_n^{(i)},$$

with differential as described in Remark 96. Calling this differential d_i, writing out the full form of each $\tilde{D}_n^{(i)}$, and abbreviating $\mathrm{Hom}_{\mathtt{GrAlg},f}(-,-)$ to $(-,-)_{[f]}$, we get the following picture:

$$\begin{tikzcd}
(F, A(\epsilon_n))_{[\rho=\rho_0]} \arrow[r, "d_0"] \arrow[d, "-\circ\pi^{\epsilon_n}"', "\tilde{\Gamma}_n^{(1)}"] & (F, A(\epsilon_{n+1}))_{[\rho=\rho_0]} \arrow[d, "-\circ\pi^{\epsilon_{n+1}}"', "\tilde{\Gamma}_{n+1}^{(1)}"] \\
(F, \mathrm{End}(V) \otimes (A_V(\epsilon_n)))_{[\rho_1]} \arrow[r, "d_1"] \arrow[d, "G_3 = j * -"', "\tilde{\Gamma}_n^{(2)}"] & (F, (\mathrm{End}(V) \otimes A_V(\epsilon_{n+1})))_{[\rho_1]} \arrow[d, "G_3 = j * -"', "\tilde{\Gamma}_{n+1}^{(2)}"] \\
(\mathrm{End}(V) * F, (\mathrm{End}(V) \otimes A_V(\epsilon_n)))_{[\rho_2]} \arrow[r, "d_2"] \arrow[d, "G_2 = \phi^*"', "\tilde{\Gamma}_n^{(3)}"] & (\mathrm{End}(V) * F, (\mathrm{End}(V) \otimes A_V(\epsilon_{n+1})))_{[\rho_2]} \arrow[d, "G_2 = \phi^*"', "\tilde{\Gamma}_{n+1}^{(3)}"] \\
\left(\mathrm{End}(V) \otimes \sqrt[V]{F}, (\mathrm{End}(V) \otimes A_V(\epsilon_n))\right)_{[\rho_3]} \arrow[r, "d_3"] \arrow[d, dashed, "G_3 = |_{\sqrt[V]{F}}"', "\tilde{\Gamma}_n^{(4)}"] & \left(\mathrm{End}(V) \otimes \sqrt[V]{F}, (\mathrm{End}(V) \otimes A_V(\epsilon_{n+1}))\right)_{[\rho_3]} \arrow[d, dashed, "G_3 = |_{\sqrt[V]{F}}"', "\tilde{\Gamma}_{n+1}^{(4)}"] \\
\left(\sqrt[V]{F}, (A_V(\epsilon_n))\right)_{[\rho_4]} \arrow[r, "d_4"] \arrow[d, "(-)_{\mathrm{ab}}"', "\tilde{\Gamma}_n^{(5)}"] & \left(\sqrt[V]{F}, (A_V(\epsilon_{n+1}))\right)_{[\rho_4]} \arrow[d, "(-)_{\mathrm{ab}}"', "\tilde{\Gamma}_{n+1}^{(5)}"] \\
(F_V, (A_V(\epsilon_n)))_{[\rho_5]} \arrow[r, "d_5"] & (F_V, (A_V(\epsilon_{n+1})))_{[\rho_5]}
\end{tikzcd}$$

We must verify that the four outer edges of the diagram commute, i.e. that

$$d_5 \tilde{\Gamma}_n = \tilde{\Gamma}_{n+1} d_0.$$

We'll show this by checking that each of the five squares making up the diagram commutes.

Beginning with the first square, say we have $f \in \tilde{D}_n^{(0)}$, with $f(x) = \rho_0(x) + \epsilon_n \cdot \psi(x)$. We get for $d_1 \tilde{\Gamma}_n^{(1)}(f)$:

$$
\begin{aligned}
f &\xmapsto{\tilde{\Gamma}_n^{(1)}} \rho_1 + \epsilon_n \cdot (\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A \circ \psi \\
&\xmapsto{d_1} \rho_1 + \epsilon_{n+1} \cdot \big(d_{\mathrm{End}(V)\otimes A_V} \circ (\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A \circ \psi \\
&\qquad -(-1)^n (\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A \circ \psi \circ d_R\big).
\end{aligned}
$$

On the other hand, we put f through $\tilde{\Gamma}_{n+1}^{(1)} d_0$ to get:

$$
\begin{aligned}
f &\xmapsto{d_0} \rho_0 + \epsilon_{n+1} \cdot (d_A \circ \psi - (-1)^n \psi \circ d_R) \\
&\xmapsto{\tilde{\Gamma}_{n+1}^{(1)}} \rho_1 + \epsilon_{n+1} \cdot \big((\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A \circ d_A \circ \psi \\
&\qquad -(-1)^n (\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A \circ \psi \circ d_R\big).
\end{aligned}
$$

But this is the same thing, since

$$
(\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A \circ d_A = d_{\mathrm{End}(V)\otimes A_V} \circ (\mathrm{id} \otimes (-)_{\mathrm{ab}}) \circ \left(\phi^{-1}\right)|_A,
$$

as $\phi^{-1}|_A$ is a morphism of DG algebras.

The remaining squares are calculated similarly. □

6.1.5 $\tilde{\Gamma}$ induces a natural map on derived derivations (i.e. Hochschild cohomology)

Remark 104. *Let $\rho : F \to A$ and $\rho' : F' \to A$ be two almost free resolutions of A. The maps $\tilde{\Gamma}_n$ were defined with reference to a choice of almost free resolution, so to avoid confusion, throughout this subsection we will call the two maps corresponding to these resolutions $\tilde{\Gamma}_{n,F}$ and $\tilde{\Gamma}_{n,F'}$. Accordingly, for the F- and F'-versions of the map $\tilde{\Gamma}$, we will use $\tilde{\Gamma}_F$ and $\tilde{\Gamma}_{F'}$.*

Lemma 105. *If $\rho : F \to A$ and $\rho' : F' \to A$ are two almost free resolutions of A and $g : F' \to F$ is a morphism of DG algebras, then for any $\psi \in \mathrm{Der}^n_\rho(F, A)$, we have*

$$\tilde{\Gamma}_{n,R'} \circ (g \circ \psi) = g_V \circ \left(\tilde{\Gamma}_{n,R} \circ \psi\right).$$

Proof. Simply verify each of the five squares of the corresponding diagram. □

Proposition 106. *The map $\mathrm{H}_\bullet\tilde{\Gamma}$ is independent of the choice of resolution $\rho : F \to A$.*

Proof. Let $\rho : F \to A$ and $\rho' : F' \to A$ be two almost free resolutions of A. We have by Corollary 93 a quasi-isomorphism $g : F' \to F$ such that g_V is also a quasi-isomorphism. By Proposition 98, we have quasi-isomorphisms of cochain complexes

$$\Psi_g : \mathrm{Der}_\rho(F, A) \to \mathrm{Der}_{\rho'}(F', A),$$

$$\Psi_{g_V} : \mathrm{Der}_{\rho_V}(F_V, A_V) \to \mathrm{Der}_{(\rho')_V}(F'_V, A_V).$$

By Lemma 105, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Der}_\rho(F, A) \arrow[r, "\sim", "\Psi_g"'] \arrow[d, "\tilde{\Gamma}_R"'] & \mathrm{Der}_{\rho'}(F', A) \arrow[d, "\tilde{\Gamma}_{R'}"'] \\ \mathrm{Der}_{\rho_V}(F_V, A_V) \arrow[r, "\sim", "\Psi_{g_V}"'] & \mathrm{Der}_{(\rho')_V}(F'_V, A_V) \end{tikzcd}$$

Therefore, the maps $\tilde{\Gamma}_R$ and $\tilde{\Gamma}_{R'}$ induce the same morphism on homology.

□

6.2 The Derived Van den Bergh Functor

In this section, we introduce the derived functor of the Van den Bergh functor, and give two methods for showing that it is well-defined.

6.2.1 The DG Van den Bergh functor

First, let's generalize the Van den Bergh functor to the DG setting (and also to the non-commutative setting).

Fix $R \in \mathtt{DGA}_k$ and a chain complex V of finite total dimension, and let $\pi : R \to \underline{\mathrm{End}}\, V \otimes_k \sqrt[V]{R}$ be the DG universal representation. The complex $V \otimes_k \sqrt[V]{R}$ is naturally a left DG module over $\underline{\mathrm{End}}\, V \otimes_k \sqrt[V]{R}$ and right DG module over $\sqrt[V]{R}$, so restricting the left action via π we can regard $V \otimes_k \sqrt[V]{R}$ as a DG bimodule over R and $\sqrt[V]{R}$. Similarly, for $V^* = \mathrm{Hom}(V, k)$, we make $\sqrt[V]{R} \otimes_k V^*$ a $\sqrt[V]{R}$-R-bimodule. Using these bimodules, we define the functors

$$\begin{aligned} \sqrt[V]{-} \quad &: \quad \mathtt{DGBimod}_R \to \mathtt{DGBimod}_{\sqrt[V]{R}}, \quad M \mapsto \left(\sqrt[V]{R} \otimes_k V^*\right) \otimes_R M \otimes_R \left(V \otimes_k \sqrt[V]{R}\right), \\ (-)_V \quad &: \quad \mathtt{DGBimod}_R \to \mathtt{DGMod}_{R_V}, \quad M \mapsto M_V := \sqrt[V]{M} \otimes_{\left(\sqrt[V]{R}\right)^e} R_V. \end{aligned}$$

The second of these functors is a DG generalization of the Van den Bergh functor. The first, meanwhile, is a DG non-commutative version.

Recall that if R is a DG algebra and M, N are DG modules over R, then $\underline{\mathrm{Hom}}_R(M, N)$ carries a natural chain complex structure, where the homogeneous elements of degree n are the degree n R-linear maps $M \to N$, and we equip $\underline{\mathrm{Hom}}_R(M, N)$ with differential $d_{\underline{\mathrm{Hom}}_R(M,N)}$ given on homogeneous elements f by the graded commutator with the differential on the complexes M and N,

$$d_{\underline{\mathrm{Hom}}_R(M,N)} f = [f, d] = f \circ d_M - (-1)^{|f|} d_N \circ f.$$

The functor $\sqrt[V]{-}$ satisfies an adjunction analogous to the one satisfied by the non-commutative representation scheme functor (which we also denoted by $\sqrt[V]{-}$):

Theorem 107. *There is a canonical isomorphism of complexes*

$$\underline{\mathrm{Hom}}_{\left(\sqrt[V]{R}\right)^e}\left(\sqrt[V]{M}, N\right) \cong \underline{\mathrm{Hom}}_{R^e}\left(M, (\underline{\mathrm{End}}\, V) \otimes_k N\right).$$

Proof. For a DG R-bimodule M, we have

$$\begin{aligned} \left(\sqrt[V]{R} \otimes_k V^*\right) \otimes_R M \otimes_R \left(V \otimes_k \sqrt[V]{R}\right) &\cong M \otimes_{R^e} \left(\sqrt[V]{R} \otimes_k V^* \otimes_k V \otimes_k \sqrt[V]{R}\right) \\ &\cong M \otimes_{R^e} \left(\sqrt[V]{R} \otimes_k \underline{\mathrm{End}}\, V \otimes_k \sqrt[V]{R}\right), \end{aligned}$$

via the usual identification $V^* \otimes V \cong \underline{\mathrm{End}}\, V$. From here, we calculate:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\sqrt[V]{R}^e}\left(\sqrt[V]{M}, N\right) &:= \underline{\mathrm{Hom}}_{\sqrt[V]{R}^e}\left(\sqrt[V]{R} \otimes_k V^* \otimes_R M \otimes_R V \otimes_k \sqrt[V]{R}, N\right) \\ &\cong \underline{\mathrm{Hom}}_{\sqrt[V]{R}^e}\left(M \otimes_{R^e}\left(\sqrt[V]{R} \otimes_k \underline{\mathrm{End}}\, V \otimes_k \sqrt[V]{R}\right), N\right) \\ &\cong \underline{\mathrm{Hom}}_{R^e}\left(M, \underline{\mathrm{Hom}}_{\sqrt[V]{R}^e}\left(\sqrt[V]{R} \otimes_k \underline{\mathrm{End}}\, V \otimes_k \sqrt[V]{R}, N\right)\right) \\ &\cong \underline{\mathrm{Hom}}_{R^e}\left(M, \left(\sqrt[V]{R} \otimes_k \underline{\mathrm{End}}\, V \otimes_k \sqrt[V]{R}\right) \otimes_{\sqrt[V]{R}^e} N\right) \\ &\cong \underline{\mathrm{Hom}}_{R^e}\left(M, \left(V \otimes_k \sqrt[V]{R}\right) \otimes_{\sqrt[V]{R}} N \otimes_{\sqrt[V]{R}} \left(\sqrt[V]{R} \otimes_k V^*\right)\right) \\ &\cong \underline{\mathrm{Hom}}_{R^e}\left(M, V \otimes_k N \otimes_k V^*\right) \\ &\cong \underline{\mathrm{Hom}}_{R^e}\left(M, \underline{\mathrm{End}}\, V \otimes_k N\right) \end{aligned}$$

□

From this we get the following generalization to the DG setting of a result of Van den Bergh (see [VdB1]):

Corollary 108. *There is a canonical isomorphism of complexes*

$$\underline{\mathrm{Hom}}_{R_V}(M_V, N) \cong \underline{\mathrm{Hom}}_{R^e}(M, (\underline{\mathrm{End}}\, V) \otimes_k N).$$

Proof. This follows from the preceding theorem and the standard $\otimes$-Hom adjunction:

$$\begin{aligned}
\underline{\mathrm{Hom}}_{R_V}(M_V, N) &:= \underline{\mathrm{Hom}}_{R_V}\left(\sqrt[V]{M} \otimes_{\sqrt[V]{R}^e} R_V, N\right) \\
&\cong \underline{\mathrm{Hom}}_{\sqrt[V]{R}^e}\left(\sqrt[V]{M}, \underline{\mathrm{Hom}}_{R_V}(R_V, N)\right) \\
&\cong \underline{\mathrm{Hom}}_{\sqrt[V]{R}^e}\left(\sqrt[V]{M}, N\right) \\
&\cong \underline{\mathrm{Hom}}_{R^e}(M, \underline{\mathrm{End}}\, V \otimes N)
\end{aligned}$$

□

6.2.2 Explicit approach to deriving the Van den Bergh functor

To construct the (non-abelian) derived functors of Van den Bergh's functor, we follow the standard procedure in differential homological algebra. Recall that a DG module M over a DG algebra R has a **semi-free** resolution $L \to M$, which is a generalization of a free resolution for ordinary modules over ordinary algebras (see [FHT2]).

Given an algebra $A \in \mathtt{Alg}_k$ and a complex M of bimodules over A, we first choose an almost free resolution $f : F \to A$ in $\mathtt{DGA}_k$ and consider M as a DG bimodule over F via f. Then, we choose a semi-free resolution $L(F, M) \to M$ in the category $\mathtt{DGBimod}_F$ and apply to $L(F, M)$ the functor $(-)_V$. The result of this construction is described by the following theorem.

Theorem 109. *Let A be an associative k-algebra, and let M be a complex of bimodules over A. The assignment $M \mapsto L(R, M)_V$ induces a well-defined functor between the derived categories*

$$\mathsf{D}(-)_V : \mathsf{D}(\mathtt{Bimod}_A) \to \mathsf{D}(\mathtt{DGMod}_{F_V}),$$

and this functor is independent of the choice of resolutions $F \to A$ and $L \to M$, up to auto-equivalence of $\mathsf{D}(\mathtt{DGMod}_{F_V})$ inducing the identity on homology.

Proof. The proof is standard differential homological algebra. Given an almost free resolution F of A and two semi-free resolutions $L_1 \to M$ and $L_2 \to M$ in $\mathsf{D}(\mathtt{DGBimod}_F)$, we have a quasi-isomorphism $L_1 \to L_2$ such that the composite map $L_1 \to L_2 \to M$ is homotopic to the resolution $L_1 \to M$ by an F-linear homotopy (see Proposition 2.1(ii) of [FHT1]). By Proposition 2.3(i) of [FHT1], $(L_1)_V$ is quasi-isomorphic to $(L_2)_V$. Therefore, the construction is independent of the choice of semi-free resolution of M.

Given a morphism $M \to N$ of complexes of A-bimodules, and resolutions $L_1 \to M$ and $L_2 \to M$ in $\mathsf{D}(\mathtt{DGBimod}_F)$, one uses Proposition 2.1(ii) of [FHT1] to obtain a map $L_1 \to L_2$ (unique up to F-linear homotopy) such that the composite $L_1 \to L_2 \to N$ is homotopic to the composite $L_1 \to M \to N$ by an F-linear homotopy. Applying $(-)_V$ to $L_1 \to L_2$ yields a well-defined map $(L_1)_V \to (L_2)_V$ in $\mathsf{D}(\mathtt{DGMod}_{F_V})$ (which is a quasi-isomorphism if $M \to N$ is a quasi-isomorphism by Proposition 2.3(ii) of [FHT1] and which is compatible with compositions of morphisms of complexes of A-bimodules). These arguments show that the assignment $M \mapsto L(F, M)_V$ yields a well-defined functor from $\mathsf{D}(\mathtt{DGBimod}_F)$ to $\mathsf{D}(\mathtt{DGMod}_{F_V})$, and thus the composition of this functor with the (fully faithful) embedding $\mathsf{D}(\mathtt{DGBimod}_A) \to \mathsf{D}(\mathtt{DGBimod}_F)$ is a functor, as desired.

It remains to check that $\mathsf{D}(-)_V$ is independent of the choice of almost free resolution of A. Let F_1 and F_2 be two almost free resolutions of A. By Proposition 3.2 of [FHT1], there is a morphism $f : F_1 \to F_2$ in $\mathtt{DGA}_k$ such that $\mathrm{H}_0(f) = \mathrm{id}_A$. Note that if M is a DG F_1-bimodule, $M \otimes_{F_1^e} F_2^e$ is a DG F_2-bimodule, which is semi-free if M is semi-free. Here, F_2^e gets a left F_1^e-module structure via $f \otimes f$. It follows that $M \mapsto L(F_1, M) \otimes_{F_1^e} F_2^e$ yields a well defined functor $\mathsf{D}(\mathtt{DGBimod}_{F_1}) \to \mathsf{D}(\mathtt{DGBimod}_{F_2})$. Similarly, $N \mapsto L(F_1, N) \otimes_{(F_1)_V} (F_2)_V$ gives a well defined functor $\mathsf{D}(\mathtt{DGMod}_{(F_1)_V}) \to \mathsf{D}(\mathtt{DGMod}_{(F_2)_V})$. The following diagram commutes.

$$\begin{tikzcd}
\mathsf{D}(\mathtt{DGBimod}_{F_1}) \arrow[r, "\mathsf{D}(-)_V"] \arrow[d, "M \mapsto L(F_1,M)\otimes_{F_1^e} F_2^e"'] & \mathsf{D}(\mathtt{DGMod}_{(F_1)_V}) \arrow[d, "N \mapsto L(F_1,N)\otimes_{(F_1)_V} (F_2)_V"] \\
\mathsf{D}(\mathtt{DGBimod}_{F_2}) \arrow[r, "\mathsf{D}(-)_V"] & \mathsf{D}(\mathtt{DGMod}_{(F_2)_V})
\end{tikzcd}$$

We claim that the vertical arrows in the above diagram are equivalences of categories. Indeed, by the standard $\otimes$-Hom adjunction and the fact that semi-free F_1-modules are cofibrant in the sense of Section 8.3 of [K2], the functor induced by $M \mapsto L(F_1, M) \otimes_{F_1^e} F_2^e$ is the left adjoint of the restriction functor $\mathsf{D}(\mathtt{DGBimod}_{F_2}) \to \mathsf{D}(\mathtt{DGBimod}_{F_1})$. That the latter functor is an equivalence of categories follows from Proposition 8.4 of [K2]. Hence, $M \mapsto L(F_1, M) \otimes_{F_1^e} F_2^e$ induces an equivalence of categories $\mathsf{D}(\mathtt{DGBimod}_{F_1}) \to \mathsf{D}(\mathtt{DGBimod}_{F_2})$. Similarly, one shows that the right vertical arrow in the above diagram is an equivalence of categories. To complete the proof, we note that the left vertical arrow commutes with the natural embeddings $\mathsf{D}(\mathtt{DGBimod}_A) \to \mathsf{D}(\mathtt{DGBimod}_{F_1})$ and $\mathsf{D}(\mathtt{DGBimod}_A) \to \mathsf{D}(\mathtt{DGBimod}_{F_2})$, respectively. $\square$

The following result, which adds further justification to regarding $\mathsf{D}(-)_V$ as the derived functor of Ven den Bergh's functor, is proven completely analogously to Theorem 84 (on p. 107).

Proposition 110. *When M is concentrated in degree zero (i.e., is simply a bimodule over A), we have* $\mathrm{H}_0\mathsf{D}(M)_V \cong M_V$.

6.2.3 Alternative approach (via model categories)

Here, we sketch an alternative approach, which parallels the approach taken to deriving the representation scheme functor.

Recall that there exist model structures on the categories $\mathtt{DGBA}_k$ and $\mathtt{DGMA}_k$, where the former is the category of pairs (R, M) where $R \in \mathtt{DGA}_k$ and M is a DG bimodule over R, and the latter is the category of pairs (R, M) where $R \in \mathtt{DGA}_k$ and M is a DG module over R. More generally, we can define a functor, which we will also denote $(-)_V$, that sends a pair $(R, M) \in \mathtt{DGBA}_k$ to the pair $(R_V, M_V) \in \mathtt{DGMA}_k$. In fact, by the same arguments as in the case of DG algebras, this functor preserves cofibrations and acyclic fibrations, and thus we have a Quillen adjunction. The consequence is the following theorem.

Theorem 111. *The total derived functors of* $(-)_V$ *and* $\underline{\mathrm{End}}\, V \otimes$, *which we will call* $\mathsf{D}\,(-)_V$ *and* $\mathcal{E}$ *(respectively), exist and form an adjoint pair*

$$\mathsf{D}\,(-)_V : \mathcal{H}o(\mathtt{DGBA}_k) \rightleftarrows \mathcal{H}o(\mathtt{DGMA}_k) : \mathcal{E}.$$

6.3 An Alternative Approach via Semidirect Products

In this section, we prove a technical result about semidirect products and the functors $\sqrt[V]{-}$ (both for algebras and modules), and then sketch how it could be used to provide an alternative proof that the derived Van den Bergh functor is well-defined.

6.3.1 Motivation

The **semidirect product** $A \ltimes M$ of an algebra A and its bimodule M is given as a set by $A \times M$, with coordinate-wise addition and with multiplication given by

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2).$$

One might hypothesize that, for a k-algebra A and an A-bimodule M, one would have

$$\sqrt[V]{A \ltimes M} \cong \sqrt[V]{A} \ltimes \sqrt[V]{M}. \tag{6.2}$$

This is significant because it would allow one to define the non-commutative Van den Bergh functor $\sqrt[V]{M}$ in terms of the non-commutative representation scheme functor (as a quotient). Indeed this, construction suggests that the Van den Bergh functor should be viewed as a "linearization" of the representation schemes functor. The suggestion of considering this approach is due to M. Kassabov.

As it turns out, 6.2 as stated is incorrect. To provide a counterexample, first note that for any algebra B, we have that $\mathrm{Hom}(A \ltimes M, B)$ can be regarded as the collection of pairs of linear maps (f, g) with $f : A \to B$ an algebra map, $g : M \to B$ a map of A-bimodules (with B given an A-bimodule structure via f), and with $xy = 0$ for all $x, y \in \mathrm{im}(g)$.

Now, take $A = M = \mathbb{C}$, and V of dimension 2. We have in this case $\sqrt[V]{A} \cong \mathbb{C}$ and $\sqrt[V]{M} \cong (V^* \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes V) \cong \mathrm{End}(V)$.

Now, we will disprove that for every algebra B,

$$\mathrm{Hom}(\sqrt[V]{A \ltimes M}, B) = \mathrm{Hom}(\sqrt[V]{A} \ltimes \sqrt[V]{M}, B).$$

For this, take $B = \mathbb{C}$.

The right-hand side is then $\mathrm{Hom}(\mathbb{C} \ltimes \mathrm{End}(V), \mathbb{C})$, which consists of all pairs (f, g) as above; the only possible f is the identity on $\mathbb{C}$. As for g, this must necessarily be 0, since $\mathbb{C}$ has no zero-divisors. So, the right-hand side has only one element.

On the other hand, the left-hand side is naturally isomorphic to

$$\mathrm{Hom}(A \ltimes M, \mathrm{End}(V) \otimes B).$$

This is $\mathrm{Hom}(\mathbb{C} \ltimes \mathbb{C}, \mathrm{End}(V))$. Now, f must be the map $1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. But as for g, it can now map 1 to any square-zero endomorphism (of which there are more then one). So, the left-hand side has larger cardinality than the right-hand side.

Fortunately, a modified version of 6.2 does indeed hold, and will be stated and proven in the next subsection.

6.3.2 Main result

Let A be a k-algebra and M an A-bimodule.

Proposition 112. *For an algebra B, any morphism $f : A \ltimes M \to B$ can be regarded as a pair (g, h), where $g : A \to B$ is an algebra morphism and $h : M \to B$ is an A-bimodule map (with B regarded as an A-bimodule via g) satisfying $h(m_1) \cdot h(m_2) = 0$ for all $m_1, m_2 \in M$.*

Proof. Straightforward calculation. $\square$

Theorem 113. *For any algebra B,*

$$\mathrm{Hom}_{\mathtt{Alg}_k}(\frac{\sqrt[V]{A \ltimes M}}{\langle M^2 \rangle}, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A} \ltimes \sqrt[V]{M}, B),$$

where $M^2 \subseteq \sqrt[V]{A \ltimes M}$ is the set of all products of pairs of elements of the form $(\mathrm{id} \otimes \phi^*)(\pi(0, m))$, where $\pi : A \ltimes M \to \mathrm{End}(V) \otimes \sqrt[V]{A \ltimes M}$, $m \in M$, and $\phi^* \in \mathrm{End}(V)^*$.

Remark. *If we fix a basis for V and let $\{e_{ij}\}$ be the canonical basis for $\mathrm{End}(V)$, then the ideal $\langle M^2 \rangle$ is generated by all elements of the form $\left(\sum_k e_{ki} m e_{jk}\right) \cdot \left(\sum_{k'} e_{k'i'} m' e_{j'k'}\right)$ for $m, m' \in M$.*

The theorem will follow from the two lemmas in this section.

Definition 114. *Define $\widetilde{\mathrm{Hom}}_{\mathtt{Alg}_k}(A \ltimes M, \mathrm{End}(V) \otimes B)$ as the set of $f \in \mathrm{Hom}(A \ltimes M, \mathrm{End}(V) \otimes B)$ such that for any $m_1, m_2 \in M \subseteq A \ltimes M$ and any $\phi^*, \psi^* \in \mathrm{End}(V)^*$,*

$$[(\phi^* \otimes \mathrm{id})(f(m_1))] \cdot [(\psi^* \otimes \mathrm{id})(f(m_2))] = 0. \tag{6.3}$$

Lemma 115. $\mathrm{Hom}_{\mathtt{Alg}_k}(\frac{\sqrt[V]{A \ltimes M}}{\langle M^2 \rangle}, B) = \widetilde{\mathrm{Hom}}_{\mathtt{Alg}_k}(A \ltimes M, \mathrm{End}(V) \otimes B).$

Proof. In the adjunction $\mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A \ltimes M}, B) = \mathrm{Hom}_{Alg}(A \ltimes M, \mathrm{End}(V) \otimes B)$, a map f on the left-hand side corresponds to a map on the right-hand side via the composition

$$A \ltimes M \overset{\pi}{\hookrightarrow} \mathrm{End}(V) \otimes \sqrt[V]{A \ltimes M} \xrightarrow{\mathrm{id} \otimes f} \mathrm{End}(V) \otimes B.$$

By the definition of M^2, the map f sends all elements of M^2 to 0 if and only if for any $m_1, m_2 \in M \subseteq A \ltimes M$ and any $\phi^*, \psi^* \in \mathrm{End}(V)^*$, we have that

$$f((\phi^* \otimes \mathrm{id})(\pi(m_1))) \cdot f((\psi^* \otimes \mathrm{id})(\pi(m_2))) = 0,$$

which, since $(\phi^* \otimes \mathrm{id})\,(\mathrm{id} \otimes f) = f \circ (\phi^* \otimes \mathrm{id})$, is equivalent to

$$(\phi^* \otimes \mathrm{id})(\mathrm{id} \otimes f)\,(\pi(m_1)) \cdot (\psi^* \otimes \mathrm{id})(\mathrm{id} \otimes f)\,(\pi(m_2)) = 0.$$

Now, defining $g = \pi \circ (\mathrm{id} \otimes f)$, this condition means that for any $m_1, m_2 \in M \subseteq A \ltimes M$ and any $\phi^*, \psi^* \in \mathrm{End}(V)^*$,

$$[(\phi^* \otimes \mathrm{id})\,(g(m_1))] \cdot [(\psi^* \otimes \mathrm{id})\,(g(m_2))] = 0.$$

And this is exactly the condition on maps g in the right-hand side of the statement of this lemma. $\square$

Lemma 116. $\widetilde{\mathrm{Hom}}_{\mathtt{Alg}_k}(A \ltimes M, B \otimes \mathrm{End}(V)) = \mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A} \ltimes \sqrt[V]{M}, B)$.

Proof. Elements of the right-hand side are pairs (g, h) with $g \in \mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A}, B)$ and $h \in \mathrm{Hom}_{(\sqrt[V]{A})^e}(\sqrt[V]{M}, B)$ such that product of any two elements in the image of h is 0. Let's see how this condition on h translates via the adjunction to $\mathrm{Hom}_{A^e}(M, \mathrm{End}(V) \otimes B)$.

Since

$$\sqrt[V]{M} = \left(\sqrt[V]{A} \otimes V^*\right) \otimes_A M \otimes_A \left(V \otimes \sqrt[V]{A}\right),$$

any such h yields (via the $\otimes$-Hom adjunction) a map

$$h' \in \mathrm{Hom}_{A^e}\left(M, \mathrm{Hom}_{(\sqrt[V]{A})^e}\left(\left(\sqrt[V]{A} \otimes V^*\right) \otimes \left(V \otimes \sqrt[V]{A}\right), B\right)\right)$$

such that any pair of elements in B that are in the image of homomorphisms in the image of h' multiply to zero. Now, the A-bimodule

$$\mathrm{Hom}_{(\sqrt[V]{A})^e}\left(\left(\sqrt[V]{A} \otimes V^*\right) \otimes \left(V \otimes \sqrt[V]{A}\right), B\right)$$

can be rewritten as $\mathrm{Hom}_{\mathbb{C}}(V^* \otimes V, B)$, with the A-bimodule structure obtained via $\pi : A \to \mathrm{End}(V) \otimes \sqrt[V]{A}$ from the bimodule structure over $\mathrm{End}(V) \otimes \sqrt[V]{A}$

given by letting elements $(a \otimes \phi)$ act by

$$\begin{aligned}(a \otimes \phi)(f) &: v^* \otimes v \mapsto a \cdot f\left((v^* \circ \phi) \otimes v\right), \\ (f)(a \otimes \phi) &: v^* \otimes v \mapsto f\left(v^* \otimes \phi v\right) \cdot a.\end{aligned}$$

Finally, $\mathrm{Hom}_{\mathbb{C}}(V^* \otimes V, B) = B \otimes V \otimes V^* = B \otimes \mathrm{End}(V)$. The action then becomes, by duality,

$$\begin{aligned}(a \otimes \phi)(b \otimes \psi) &= ab \otimes \phi\psi \\ (b \otimes \psi)(a \otimes \phi) &= ba \otimes \psi\phi,\end{aligned}$$

and a map h' becomes under this identification a map $h'' \in \mathrm{Hom}_{A^e}(M, B \otimes \mathrm{End}(V))$ such that for any pair of elements $m_1, m_2 \in M$ and any $\phi^*, \psi^* \in \mathrm{End}(V)^*$,

$$[(\phi^* \otimes \mathrm{id})(h''(m_1))] \cdot [(\psi^* \otimes \mathrm{id})(h''(m_2))] = 0.$$

But this means that if we assign to our map g a corresponding $g'' \in \mathrm{Hom}_{Alg}(A, \mathrm{End}(V) \otimes B)$, then the pair (g'', h'') exactly determines an element of $\widetilde{\mathrm{Hom}}_{\mathtt{Alg}_k}(A \ltimes M, \mathrm{End}(V) \otimes B)$. □

Corollary 117. Let A be a k-algebra and M a bimodule over A. Then,

$$\frac{\sqrt[V]{A \ltimes M}}{\langle M^2 \rangle} \cong \sqrt[V]{A} \ltimes \sqrt[V]{M}.$$

Proof. This follows from the Yoneda lemma. □

6.3.3 Application

In this section, we give a rough sketch of how the result of the preceding section could be used to give an alternative proof that the derived Van den Bergh functor is well-defined.

The first step would be to generalize the result to the DG setting (which can be done analogously to the DG generalizations carried out above). Next, select the algebra B in the statement of Theorem 113 to be commutative; then, since $(A \ltimes M)_{\mathrm{ab}} = A_{\mathrm{ab}} \ltimes M_{\mathrm{ab}}$, we get

$$\mathrm{Hom}_{\mathtt{Alg}_k}(\sqrt[V]{A} \ltimes \sqrt[V]{M}, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(A_V \ltimes M_V, B).$$

Taking $B = A_V$, we have the natural projection map $p : A_V \ltimes M_V \to A_V$ with kernel M_V.

Meanwhile, for commutative B, the left-hand side of the theorem's equality gives us

$$\mathrm{Hom}_{\mathtt{Alg}_k}(\frac{\sqrt[V]{A \ltimes M}}{\langle M^2 \rangle}, B) = \mathrm{Hom}_{\mathtt{Alg}_k}(\frac{(A \ltimes M)_V}{\langle M^2 \rangle}, B),$$

where $\langle M^2 \rangle$ is defined analogously to the non-commutative case (via the commutative version of the map π). One can verify that the map q here corresponding to p via the equality coincides with the map induced by the functor $\sqrt[V]{-}$ from the projection $A \ltimes M \to A$. Thus, $\ker(q) = M_V$.

Now, consider M_1, M_2 projective DG bimodules over almost free DG algebras A_1 and A_2 (respectively) such that $\mathrm{H}_\bullet(A_1) = \mathrm{H}_\bullet(A_2) = A$ and $\mathrm{H}_\bullet(M_1) = \mathrm{H}_\bullet(M_2) = M$. In this case, we have quasi-isomorphisms between A_1 and A_2 along with M-homotopies from their compositions to the identities on A_1 and A_2, and we also have DG-module quasi-isomorphisms between M_1 and M_2 (which can be regarded as bimodules over both A_1 and A_2 via the aforementioned maps) which induce homotopy equivalences (since two projective DG bimodules over almost free DG algebras are homotopy equivalent). This data can be used to construct an M-homotopy equivalence between $A_1 \ltimes M_1$ and $A_2 \ltimes M_2$, and consequently the projections

$$\frac{(A_1 \ltimes M_1)_V}{\langle M_1^2 \rangle} \twoheadrightarrow (A_1)_V\,, \qquad \frac{(A_2 \ltimes M_2)_V}{\langle M_2^2 \rangle} \twoheadrightarrow (A_2)_V$$

induce the same map on cohomology. Therefore, the homologies of their kernels are isomorphic, giving the desired result.

This same approach can be used without going through the theorem, provided that one can prove that a DG bimodule homotopy between two A_1-bimodules M_1 and M_2 yields a DG bimodule homotopy between $(M_1)_V$ and $(M_2)_V$.

CHAPTER 7

OBTAINING EXPLICIT PRESENTATIONS

7.1 Calculating Derived Representation Schemes

In this section, we establish the theoretical results that will enable to us to find explicit presentations for derived representation schemes.

7.1.1 Background

A special class of finitely generated algebras for which it is quite easy to describe $\sqrt[V]{A}$ is the class of path algebras of quivers. In Section 4 of [LBW], L. Le Bruyn and G. van de Weyer provide a procedure for obtaining a finite presentation of $\sqrt[V]{\mathbb{C}Q}$, where $\mathbb{C}Q$ is the path algebra (over $\mathbb{C}$) of a quiver Q (with a finite number of vertices and arrows). Generally speaking, one extends the quiver, adding a new vertex v_0 and $2n$ arrows for each old vertex (n leading to it from the new vertex, and the other n leading from it to the new vertex). Then, one imposes certain relations on the resulting quiver (obtaining a "quiver with relations"), and finally takes the spherical subalgebra of paths originating and terminating at the new vertex. The resulting algebra is written $v_0\mathbb{C}\tilde{Q}_{n\sigma}v_0$. Le Bruyn and van de Weyer prove (Theorem 4.1) that $v_0\mathbb{C}\tilde{Q}_{n\sigma}v_0 \cong \sqrt[V]{\mathbb{C}Q}$.

The departing point for us is the (trivial) observation that finitely generated free algebras are quiver path algebras, and this is all we need to calculate derived representation schemes. In other words, if we can present $\sqrt[V]{F}$ for free resolutions $F \to A$, then all that remains is to abelianize, which is quite simple once one has an explicit presentation.

This requires generalizing the result to DG algebras, i.e. accounting for differentials. Surprisingly, this turns out to be possible, even for V not in degree zero.

This turn of events is especially surprising in light of the fact that the theorem on quivers is aimed at non-commutative representation schemes – calculating a presentation for the commutative representation scheme is quite simple in general (see Subsection 3.1.2). And yet, in this case, we use the result concerning the non-commutative representation scheme to calculate something for the commutative one (i.e., the value of its derived functor) which we would not have been able to find otherwise.

7.1.2 Main theorem

Recall that throughout, we use the Koszul sign rule for graded objects. For details, see the notation conventions on p. 16.

Let R be an almost free DG algebra with generators $\{r_k\}_{k\in K}$ and an explicitly given differential d_R. Let V be a complex of finite total dimension n with homogeneous basis $\{\beta_i\}_{1\leq i\leq n}$. Recall that we regard $\underline{\mathrm{End}}\, V$ as a DG algebra with differential

$$d_{\underline{\mathrm{End}}\, V} : T \mapsto d_V \circ T - (-1)^{|T|} T \circ d_V.$$

As a vector space, $\underline{\mathrm{End}}\, V$ has a canonical homogeneous basis consisting of elements $e_{ij} : \beta_k \mapsto \delta_{kj}\beta_i$ with $|e_{ij}| = |\beta_i| - |\beta_j|$. We wish to calculate an explicit presentation of $\sqrt[V]{R}$.

Definition 118. *Define:*

1. *The algebra $\tilde{R}$ as the free graded algebra on generators $\{r_k^{ij}\}$ for $k \in K$ and $1 \leq i, j \leq n$ of degree $|r_k^{ij}| = |r_k| + |v_j| - |v_i|$, equipped with a degree $+1$ derivation $d_{\tilde{R}}$, given on generators by*

$$d_{\tilde{R}}(r_k^{ij}) = \left(\pi \circ d_R(r_k) - d_{\mathrm{End}(V)} \otimes \mathrm{id}_{\tilde{R}}(\pi(r_k))\right)_{ij}.$$

2. *The derivation $d_{\underline{\mathrm{End}}\,V \otimes \tilde{R}} : \underline{\mathrm{End}}\,V \otimes \tilde{R} \otimes \to \underline{\mathrm{End}}\,V \otimes \tilde{R}$ as*

$$(\mathrm{id}_{\underline{\mathrm{End}}\,V} \otimes d_{\tilde{R}} + d_{\underline{\mathrm{End}}\,V} \otimes \mathrm{id}_{\tilde{R}}).$$

3. *The map $\pi : R \to \underline{\mathrm{End}}\,V \otimes \tilde{R}$ as the morphism of graded algebras sending $r_k \mapsto (r_k^{ij})$, the matrix whose (i, j) entry is r_k^{ij}.*

Proposition 119. $d_{\underline{\mathrm{End}}\,V \otimes \tilde{R}} \circ \pi = \pi \circ d_R$.

Proof. By the definition of $d_{\tilde{R}}$,

$$\mathrm{id}_{\underline{\mathrm{End}}\,V} \otimes d_{\tilde{R}} \circ \pi(r_k) = \pi \circ d_R(r_k) - d_{\underline{\mathrm{End}}\,V} \otimes \mathrm{id}_{\tilde{R}}(\pi(r_k)),$$

where r_k is a generator; from this is follows immediately that the result is true for all generators r_k.

Now, the result also holds for all products of generators. For if it holds for homogeneous elements $x, y \in R$, then we have

$$\begin{aligned}
d_{\underline{\mathrm{End}}\,V \otimes \tilde{R}}(\pi(xy)) &= d_{\underline{\mathrm{End}}\,V \otimes \tilde{R}}(\pi(x)\pi(y)) \\
&= d_{\underline{\mathrm{End}}\,V \otimes \tilde{R}}(\pi(x)) \cdot \pi(y) + (-1)^{|x|}\pi(x) \cdot d_{\underline{\mathrm{End}}\,V \otimes \tilde{R}}(\pi(y)) \\
&= \pi(d_R(x)) \cdot \pi(y) + (-1)^{|x|}\pi(x) \cdot \pi(d_R(y)) \\
&= \pi\left(d_R(x) \cdot y + (-1)^{|x|} x \cdot d_R(y)\right) \\
&= \pi(d_R(xy)).
\end{aligned}$$

Thus, the result extends to all products of generators; similarly, it extends to all sums of such products, since all maps involved are linear. $\square$

Proposition 120. $d_{\tilde{R}}^2 = 0$.

Proof. On one hand,

$$d^2_{\underline{\mathrm{End}}\, V \otimes \tilde{R}} \circ \pi = \pi \circ d_R^2 = 0$$

for each generator r_k. On the other hand,

$$\begin{aligned}
& d^2_{\underline{\mathrm{End}}\, V \otimes \tilde{R}} \circ \pi \\
= \;& (d_{\tilde{R}} \otimes \mathrm{id}_{\underline{\mathrm{End}}\, V} + \mathrm{id}_{\tilde{R}} \otimes d_{\underline{\mathrm{End}}\, V}) \circ (d_{\tilde{R}} \otimes \mathrm{id}_{\underline{\mathrm{End}}\, V} + \mathrm{id}_{\tilde{R}} \otimes d_{\underline{\mathrm{End}}\, V}) \circ \pi \\
= \;& (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d_{\tilde{R}}) \circ (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d_{\tilde{R}}) \circ \pi + (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d_{\tilde{R}}) \circ (d_{\underline{\mathrm{End}}\, V} \otimes \mathrm{id}_{\tilde{R}}) \circ \pi \\
& + (d_{\underline{\mathrm{End}}\, V} \otimes \mathrm{id}_{\tilde{R}}) \circ (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d_{\tilde{R}}) \circ \pi + (d_{\underline{\mathrm{End}}\, V} \otimes \mathrm{id}_{\tilde{R}}) \circ (d_{\underline{\mathrm{End}}\, V} \otimes \mathrm{id}_{\tilde{R}}) \circ \pi \\
= \;& (-1)^{0 \cdot 1} \left(\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d^2_{\tilde{R}}\right) \circ \pi + (-1)^{0 \cdot 0} \left(d_{\underline{\mathrm{End}}\, V} \otimes d_{\tilde{R}}\right) \circ \pi \\
& + (-1)^{1 \cdot 1} \left(d_{\underline{\mathrm{End}}\, V} \otimes d_{\tilde{R}}\right) \circ \pi + (-1)^{1 \cdot 0} \left(d^2_{\underline{\mathrm{End}}\, V} \otimes \mathrm{id}_{\tilde{R}}\right) \circ \pi \\
= \;& \left(\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d^2_{\tilde{R}}\right) \circ \pi.
\end{aligned}$$

Thus, $\left(\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes d^2_{\tilde{R}}\right) \circ \pi(r_k) = 0$ for all generators r_k, and this means that $d^2_{\tilde{R}}(r_k^{ij}) = 0$ for all generators r_k^{ij}. $\square$

Therefore, $(\tilde{R}, d_{\tilde{R}})$ is an almost free DG algebra.

Theorem 121. *For every DG algebra B,*

$$\mathrm{Hom}(\tilde{R}, B) = \mathrm{Hom}(R, \underline{\mathrm{End}}\, V \otimes B).$$

Proof. We will use d (without subscript) to mean the differential, as it will be clear from the context which DG algebra d is on.

The graded (i.e., ignoring all differentials) version of the adjunction holds (by prior results), and the correspondence is given by associating to a map $\psi : \tilde{R} \to B$ the map $\phi = (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi) \circ \pi$. We claim that ϕ commutes with the differential if and only if ψ does.

Form

$$\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi : \underline{\mathrm{End}}\, V \otimes \tilde{R} \to \underline{\mathrm{End}}\, V \otimes B.$$

It is trivial to verify that $d\psi = \psi d$ if and only if $d\,(\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi) = (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi)\, d$.

For $\phi = (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi) \circ \pi$, we have

$$\begin{aligned} d\phi &= d \circ (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi) \circ \pi, \\ \phi d &= (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi) \circ \pi \circ d \;=\; (\mathrm{id}_{\underline{\mathrm{End}}\, V} \otimes \psi) \circ d \circ \pi. \end{aligned}$$

Therefore, using the fact that $\pi(r_k) = [r_k^{ij}]$, we have that $d\phi = \phi d$ if and only if $d\psi = \psi d$. $\square$

By the Yoneda lemma, we have the following.

Corollary 122. *For any almost free $R \in \mathtt{DGA}_k$, we have $\tilde{R} \cong \sqrt[V]{R}$. In particular, for any almost free resolution $F \to A$ of an associative algebra A, the algebra $(\tilde{F})_{\mathrm{ab}} \in \mathtt{CDGA}_k$ is a representative of the equivalence class $\mathsf{D}\,(A)_V \in \mathcal{H}o(\mathtt{CDGA}_k)$.*

Note the following convenient facts about the above construction:

- The underlying graded algebra of $\tilde{R}$ is free.
- If R has m generators, then $\tilde{R}$ has mn^2 generators, where $n = \dim(V)$.
- Once we have generators for $\tilde{R}$, we automatically have the generators for $(\tilde{R})_{\mathrm{ab}}$ (since we can just take the same set).

These facts make it quite simple to give quite small, explicit presentations for $\mathsf{D}(A)_V$, as we will see in the next section. Note that the results here represent a generalization of the procedure for calculating presentations for (classical) representation schemes, as described in Subsection 3.1.2.

7.2 Examples of Derived Representation Schemes

In this section, we give explicit presentations for some examples of derived representation schemes.

7.2.1 Example: $\mathbb{C}[x, y, z]$ with V in degree 0

It is possible to resolve the algebra $\mathbb{C}[x, y, z]$ with the almost free DG algebra F, with underlying graded algebra on seven generators,

$$F = \mathbb{C}\langle x, y, z, X, Y, Z, t\rangle,$$

where x, y, z are in degree 0, the elements X, Y, Z are in degree 1, and t is in degree 2. The differential acts as

$$x, y, z \mapsto 0, \quad X \mapsto yz - zy, \quad Y \mapsto xz - zx, \quad Z \mapsto xy - yx,$$

$$t \mapsto xX - Xx + yY - Yy + zZ - Zz.$$

Applying the procedure to find $\tilde{F}$ and then abelianizing, we obtain a commutative DG algebra with 28 generators: 12 in degree 0 (x_{ij}, y_{ij}, z_{ij} for $1 \leq i, j \leq 2$), 12 in degree 1 (X_{ij}, Y_{ij}, Z_{ij} for $1 \leq i, j \leq 2$), and 4 in degree 2 (t_{ij} for $1 \leq i, j \leq 2$).

The differential behaves on the generators as follows:

- Of course, $d(x_{11}) = 0$, just as for the other 11 elements of degree 0
- $d(X_{ij}) = y_{i1}z_{1j} + y_{i2}z_{2j} - z_{i1}y_{1j} - z_{i2}y_{2j}$
- $d(Y_{ij}) = z_{i1}x_{1j} + z_{i2}x_{2j} - x_{i1}z_{1j} - x_{i2}z_{2j}$
- $d(Z_{ij}) = x_{i1}y_{1j} + x_{i2}y_{2j} - y_{i1}x_{1j} - y_{i2}x_{2j}$

And finally,

$$d(t_{ij}) = (x_{i1}X_{1j} + x_{i2}X_{2j} - X_{i1}x_{1j} - X_{i2}x_{2j})$$

$$+ (y_{i1}Y_{1j} + y_{i2}Y_{2j} - Y_{i1}y_{1j} - Y_{i2}y_{2j}) + (z_{i1}Z_{1j} + z_{i2}Z_{2j} - Z_{i1}z_{1j} - Z_{i2}z_{2j})$$

7.2.2 Example: $\mathbb{C}[x, y]$ with V in two degrees

Consider the cochain complex $V = [... \leftarrow 0 \leftarrow \mathbb{C} \overset{\text{id}}{\leftarrow} \mathbb{C} \leftarrow 0 \leftarrow ...]$ concentrated in degrees 1 and 2. We then have:

$$\begin{aligned} d_{\underline{\text{End}}(V)}(e_{11}) &= e_{21} \\ d_{\underline{\text{End}}(V)}(e_{12}) &= e_{22} + e_{11} \\ d_{\underline{\text{End}}(V)}(e_{21}) &= 0 \\ d_{\underline{\text{End}}(V)}(e_{22}) &= -e_{21} \end{aligned}$$

Taking the almost free resolution $\mathbb{C}[X, Y, T]$ with $|X| = |Y| = 0$ and $|T| = 1$, and differential $d(T) = XY - YX$, we obtain a presentation for $\sqrt[V]{\mathbb{C}[X, Y, T]}$ with generators X^{ij}, Y^{ij}, T^{ij} (twelve in total). We have

$$|X^{11}| = |X^{22}| = |Y^{11}| = |Y^{22}| = 0$$

$$|X^{12}| = |Y^{12}| = -1, \qquad |X^{21}| = |Y^{21}| = 1$$

$$|T^{11}| = |T^{22}| = 1, \qquad |T^{12}| = 0, \qquad |T^{21}| = 2$$

We calculate:

$$\begin{aligned} d(X^{ij}) &= \left(\pi \circ d(x) - d_{\underline{\mathrm{End}}(V)} \otimes \mathrm{id}_{\tilde{R}}(\pi(x))\right)_{ij} \\ &= \left(e_{21} \otimes (X^{22} - X^{11}) + e_{22} \otimes X^{12} + e_{11} \otimes X^{12}\right)_{ij}. \end{aligned}$$

Analogously,

$$d(Y^{ij}) = \left(e_{21} \otimes (Y^{22} - Y^{11}) + e_{22} \otimes Y^{12} + e_{11} \otimes Y^{12}\right)_{ij}.$$

Finally,

$$d(T^{ij}) = \left(\pi \circ d(T) - d_{\underline{\mathrm{End}}(V)} \otimes \mathrm{id}_{\tilde{R}}(\pi(T))\right)_{ij},$$

from which we get

$$\begin{aligned} T^{11} &\overset{d}{\mapsto} X^{11}Y^{11} - X^{12}Y^{21} - Y^{11}X^{11} + Y^{12}X^{21} - T^{12} \\ T^{12} &\overset{d}{\mapsto} X^{11}Y^{12} + X^{12}Y^{22} - Y^{11}X^{12} - Y^{12}X^{22} \\ T^{21} &\overset{d}{\mapsto} X^{21}Y^{11} - X^{22}Y^{21} - Y^{21}X^{11} + Y^{22}X^{21} + T^{11} - T^{22} \\ T^{22} &\overset{d}{\mapsto} -X^{21}Y^{12} + X^{22}Y^{22} + Y^{21}X^{12} - Y^{22}X^{22} - T^{12} \end{aligned}$$

It is a straightforward calculation to verify that d^2 is indeed zero on all generators (as we know from the results of the preceding section).

7.2.3 Example: $\mathbb{C}[x]/(x)^2$ with V in two degrees

Consider the cochain complex $V = [... \leftarrow 0 \leftarrow \mathbb{C} \overset{\mathrm{id}}{\leftarrow} \mathbb{C} \leftarrow 0 \leftarrow ...]$ concentrated in degrees 1 and 2.

Consider the algebra $\mathbb{C}[x]/(x)^2$. Taking the almost free resolution $\mathbb{C}[X,T]$ with $|X| = 0$ and $|T| = 1$, and differential $d(T) = X^2$, we obtain a presentation for $\sqrt[V]{\mathbb{C}[X,T]}$ with generators X^{ij}, T^{ij} (eight in total). We have

$$|X^{11}| = |X^{22}| = 0$$

$$|X^{12}| = -1, \qquad |X^{21}| = 1$$

$$|T^{11}| = |T^{22}| = 1, \qquad |T^{12}| = 0, \qquad |T^{21}| = 2$$

We have

$$\begin{aligned} d(X^{ij}) &= \left(\pi \circ d(x) - d_{\underline{\mathrm{End}}(V)} \otimes \mathrm{id}_{\tilde{R}}(\pi(x))\right)_{ij} \\ &= \left(e_{21} \otimes (X^{22} - X^{11}) + e_{22} \otimes X^{12} + e_{11} \otimes X^{12}\right)_{ij}, \end{aligned}$$

since it's the same calculation as in the previous example.

Meanwhile,

$$d(T^{ij}) = \left(\pi \circ d(T) - d_{\underline{\mathrm{End}}(V)} \otimes \mathrm{id}_{\tilde{R}}(\pi(T))\right)_{ij},$$

which means that we have

$$\begin{aligned} T^{11} &\overset{d}{\mapsto} X^{11}X^{11} - X^{12}X^{21} - T^{12} \\ T^{12} &\overset{d}{\mapsto} X^{11}X^{12} - X^{12}X^{22} \\ T^{21} &\overset{d}{\mapsto} X^{21}X^{11} - X^{22}X^{21} + T^{11} - T^{22} \\ T^{22} &\overset{d}{\mapsto} -X^{21}X^{12} - X^{22}X^{22} - T^{12} \end{aligned}$$

7.3 Computer Calculations of Homology

Having obtained explicit presentations for (representatives of) the derived representation scheme, the next logical step is to calculate the homology. In fact, this is in certain special cases possible (using Grbner bases by way of a computer algebra program). We used a new package on DG algebras for Macaulay2 developed by R. Frank Moore, who also ran the actual calculations through the software.

7.3.1 The method and its limitations

The specific method used was to regard the homology as a module over its zeroth degree. In the case when V is in degree zero, this is actually a finitely generated module, and thus it is possible to do the calculations. Even here, the calculations quickly become too difficult as $\dim(V)$ increases; in some cases, such as the case for $\mathcal{U}(\mathfrak{sl}_2)$, even low-degree V was too difficult to calculate in a short period of time (within an hour) on a powerful desktop computer. The author is not aware of any method for estimating the computational difficulty of the task *a priori*.

A limitation of this method is that it doesn't allow one to treat the case when V is not concentrated in degree zero, since in that case the algebra we obtain is *not* a finitely generated module over the degree-zero component. It is possible that some workaround exists.

7.3.2 Example: $\mathbb{C}[x, y]$ with $\dim(V) = 2$

The presentation for a representative of $\mathrm{DRep}_V(\mathbb{C}[x, y])$ is straightforward to find, and this is left to the reader. (The resolution to use is given in the previous section.) A computer calculation shows that

$$\mathrm{H}_\bullet\mathsf{D}\left(\mathbb{C}[x,y]\right)_V = \frac{(\mathbb{C}[x,y])_V[r,s]}{\left(\begin{array}{c} x_{21}r - y_{21}s, \\ x_{12}r - y_{12}s, \\ (x_{11}-x_{22})r - (y_{11}-y_{22})s, \\ rs,\ r^2,\ s^2 \end{array}\right)},$$

where $|r| = |s| = 1$.

7.3.3 Further examples

The presentations for $\mathrm{H}_\bullet\mathsf{D}\left(\mathbb{C}[x,y]\right)_V$ with V of dimension 3, $\mathrm{H}_\bullet\mathsf{D}\left(\mathbb{C}[x,y,z]\right)_V$ with V of dimension 2, and $\mathrm{H}_\bullet\mathsf{D}\left(\mathbb{C}[x]/(x^2)\right)_V$ with V of dimension 2 are larger, but can still be easily handled by a computer.

For example, $\mathrm{H}_\bullet\mathsf{D}\left(\mathbb{C}[x,y]\right)_V$ (with V of dimension 3) has six generators in non-zero degrees, all of them in degree 1. It has nonzero components only in degrees 0, 1, 2, and 3.

The algebra $\mathrm{H}_\bullet\mathsf{D}\left(\mathbb{C}[x,y,z]\right)_V$ with V of dimension 2 has 16 generators in degree 1, and others in higher degrees; regarded as a $(\mathbb{C}[x,y,z])_V$-module, its minimal generating set has 16 elements in degree 1, 56 in degree 2, 128 in degree 3, 233 in degree 4, and more in lower degrees.

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