Discrete Hedging Under Piecewise Linear Risk Minimization

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Abstract

In an incomplete market it is usually impossible to eliminate the intrinsic risk of an option. In this case quadratic risk-minimization is often used to determine a hedging strategy. However, it may be more natural to use piecewise linear risk-minimization since in this case the risk is measured in actual dollars (not dollars squared).
We investigate hedging strategies using piecewise linear risk-minimization. We illustrate that piecewise linear risk-minimization often leads to smaller expected total hedging cost and significantly different, possibly more desirable, hedging strategies from those of quadratic risk minimization. The distributions of the total hedging cost and risk show that hedging strategies obtained by piecewise linear risk-minimization have a larger probability of small cost and risk, though they also have a very small probability of larger cost and risk. Comparative numerical results are provided.

1. Introduction

Investors try to reduce the sensitivity of their portfolios to the fluctuations of the market by hedging. In particular, in option hedging, one tries to construct a trading strategy that replicates the option payoff and does not require any inflow or outflow of capital other than initial costs. In the Black-Scholes framework, the option can be replicated by using only the underlying asset and a bond. However, the investor's position is only instantaneously risk-free and therefore, it must be adjusted continuously. In practice, a natural problem that occurs is the impossibility of hedging continuously in time coupled with the need to hedge as little as possible due to the impact of transaction costs. If only discrete hedging times are allowed, achieving a risk-free position at each time is no longer appropriate since this instantaneous hedging will not last till the next rebalancing time. Moreover, the market becomes incomplete and the Black-Scholes framework cannot be applied. Under these conditions, there is much uncertainty regarding the choice of an optimal hedging strategy and in defining the fair price of an option. It is not possible to totally hedge the intrinsic risk carried by options that cannot be exactly replicated. An "optimal" hedging strategy can be chosen to minimize a particular measure of this risk.

Different criteria for quadratic risk minimization can be found in the literature. We mention, for example, Föllmer, and Schweizer (1989), Schäl (1994), Schweizer (1995, 2001), Mercurio, and Vorst (1996), Heath, Platen, and Schweizer (2001a, 2001b). We only briefly describe them here, but they are presented in more detail in Section 2.

Suppose we want to hedge an option with maturity T and we can only hedge at discrete times: $0 = t_0 < t_1 < ... < t_M = T$. Suppose also that the discounted underlying asset price is a square integrable process on a probability space $(\Omega, \mathcal{F}, P)$, with a filtration $(\mathcal{F}_k)_{k=\overline{0,M}}$, where $\mathcal{F}_k$ corresponds to the hedging time t_k and, w.l.o.g., $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial. Denote by C_k the cumulative cost of the hedging strategy up to time k (this includes the initial cost for setting up the hedging portfolio and the cost for rebalancing it at the hedging times $t_0, ..., t_k$).

Currently, there are two main quadratic hedging approaches for choosing an optimal strategy. One possibility is to control the total risk by minimizing the L^2-norm $E((C_M - C_0)^2)$, where $E(\cdot)$ denotes the expected value with respect to the probability measure P. This is the total risk-minimization criterion. An optimal strategy for this criterion is self-financing, that is, its cumulative cost process is constant. However, a total risk-minimization strategy may not exist in general. The additional assumption that the discounted underlying asset price has a bounded mean-variance tradeoff is required. In this case, there exists an explicit strategy. The existence and the uniqueness of a total risk-minimizing strategy have been extensively studied by Schweizer (1995).

Another possibility is to control the local incremental risk, by minimizing $E((C_{k+1} - C_k)^2|\mathcal{F}_k)$ for all $0 \leq k \leq M-1$. This is the local quadratic risk-minimizing criterion. An explicit local risk-minimizing strategy always exists under the assumption that the discounted underlying asset price is a square-integrable random process and the option payoff is a square-integrable random variable (see Schäl 1994). This strategy is no longer self-financing, but it is mean-self-financing, i.e., the cumulative cost process is a martingale. In general, the initial costs for the local risk-minimizing and total risk-minimizing strategies are different. As Schäl noticed, the initial costs agree in the case when the discounted underlying asset price has a deterministic mean-variance tradeoff. He then suggests the interpretation of this initial cost as a *fair hedging price* for the option. However, as shown by Schweizer (1995), this is not always appropriate.

In order to justify the optimal hedging strategy of the quadratic risk minimization and to ascertain the fair value of the option, it is important to analyze the dependence of the optimal hedging strategy on the choice of the quadratic risk measure. We remark that the optimal hedging strategy hinges on the subjective criterion for measuring the risk.

Since we are trying to minimize the risk of a trading strategy valued in monetary units, it seems more natural from a financial point of view to consider as measures of the risk the L^1-norm $E(|C_M - C_0|)$ and, respectively, $E(|C_{k+1} - C_k|\ |\mathcal{F}_k)$. Further support for preferring these criteria is given by the (well-known) observation that a quadratic measure tends to overemphasize large values, even if these values occur infrequently.

To illustrate our preference for the new risk-minimizing criteria, consider the following comparison between the piecewise linear risk-minimization with respect to the local risk measure $E(|C_{k+1} - C_k|\ |\mathcal{F}_k)$, and the quadratic risk-minimization with respect to $E((C_{k+1} - C_k)^2|\mathcal{F}_k)$. Suppose the price of the underlying asset follows the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t, \tag{1}$$

where Z_t is a Wiener process. Let the initial value of the asset $S_0 = 100$, the instantaneous expected return $\mu = .2$, the volatility $\sigma = .2$ and the riskless rate of return $r = .1$. Suppose we want to statically hedge a deep in-the-money and a deep out-of-money put option with maturity $T = 1$; we only have one hedging opportunity, at time 0. At the maturity T we compare the payoff of the options with the hedging portfolio values of the strategies obtained by the piecewise linear and quadratic local risk-minimization. The payoff and the hedging portfolio values at time T are multiplied by the density function of the asset price and are discounted to time 0. The first plot in Figure 1 shows the weighted payoff and the weighted values of the hedging portfolios at the maturity T for the in-the-money put option. The second plot presents the corresponding data for the out-of-money put option.

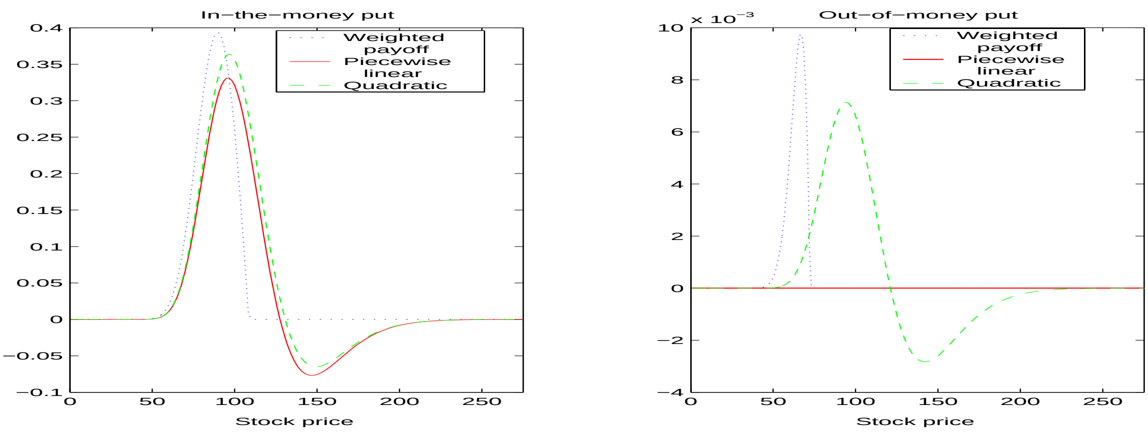

Figure 1: Best fitting of the option payoff

In the case of the in-the-money put option, the weighted payoff, closer to normal, is much easier to fit. We remark that in this case both criteria generate similar plots of the hedging strategy values and these fit relatively well the option payoff. However, the weighted payoff for the out-of-money put option seems more difficult to match. The L^2-norm (i.e., quadratic) attempts to penalize large residuals excessively and this leads to a worse fit. When the put expires out of money, the L^2 hedging strategy either over or under replicates the option payoff which is exactly hedged by the L^1 strategy.

Unfortunately, there are no known analytic expressions for the optimal hedging strategies in the case of the risk measures $E(|C_M - C_0|)$ and $E(|C_{k+1} - C_k|\ |\mathcal{F}_k)$. In this paper we concentrate on describing alternative hedging strategies for local risk-minimization. We compare the effectiveness of the hedging strategies based on piecewise linear risk minimization to those based on quadratic risk minimization. We first illustrate that, by generating synthetic paths for the asset price, the piecewise linear risk minimization often leads to smaller average total hedging cost and risk. We then confirm these results by computing the expected total cost and risk in the binomial view of the asset price. Finally we provide numerical results which emphasize the fact that the optimal hedging strategies with respect to piecewise linear risk-minimization are significantly different, possibly more desirable, than the traditional strategies, having a larger probability of small cost and risk, although a very small probability of larger cost and risk.

2. Quadratic risk-minimization

Consider a financial market where a risky asset (called stock) and a risk-free asset (called bond) are traded. Let $T > 0$ and let $0 = t_0 < t_1 < ... < t_M = T$ be discrete hedging dates. Suppose $(\Omega, \mathcal{F}, P)$ is a filtered probability space with filtration $(\mathcal{F}_k)_{k=\overline{0,M}}$, where $\mathcal{F}_k$ corresponds to the hedging time t_k and w.l.o.g. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial. Assume the stock price follows a stochastic process $S = (S_k)_{k=\overline{0,M}}$, with S_k being $\mathcal{F}_k$-measurable for all $0 \leq k \leq M$. We can set the bond price $B \equiv 1$ by assuming the discounted stock price process $X = (X_k)_{k=\overline{0,M}}$, where $X_k = \frac{S_k}{B_k}, \forall 0 \leq k \leq M$.

Suppose we want to hedge a European option with maturity T whose payoff is given by an $\mathcal{F}_M$-measurable random variable H. For example, in the case of a European put with maturity T and discounted strike price K, we have $H = (K - X_M)^+$.

A *trading strategy* is given by two stochastic processes $(\xi_k)_{k=\overline{0,M}}$ and $(\eta_k)_{k=\overline{0,M}}$, where ξ_k is the number of shares held at time t_k and η_k is the amount invested in the bond at time t_k. We assume ξ_k, η_k are $\mathcal{F}_k$-measurable, for all $0 \leq k \leq M$ and $\xi_M = 0$. Consider the portfolio consisting of the combination of the stock and bond given by the trading strategy. The condition $\xi_M = 0$ corresponds to the fact that at time M we liquidate the portfolio in order to cover for the payoff of the option. The value of the portfolio at any time t_k, $0 \leq k \leq M$ is given by:

$$V_k = \xi_k X_k + \eta_k.$$

For all $0 \leq j \leq M-1$, $\xi_j(X_{j+1} - X_j)$ represents the change in value due to the change in the stock price at time t_{j+1} before any changes in the portfolio. Therefore, the *accumulated gain* G_k is given by:

$$G_k = \sum_{j=0}^{k-1} \xi_j(X_{j+1} - X_j), \quad 1 \leq k \leq M$$

and $G_0 = 0$.

The *cumulative cost* at time t_k, C_k, is defined by:

$$C_k = V_k - G_k, \quad 0 \leq k \leq M.$$

A strategy is called *self-financing* if its cumulative cost process $(C_k)_{k=\overline{0,M}}$ is constant over time, i.e. $C_0 = C_1 = ... = C_M$. This is equivalent to $(\xi_{k+1} - \xi_k)X_{k+1} + \eta_{k+1} - \eta_k = 0$ (a.s.), for all $0 \leq k \leq M-1$. In other words, any fluctuations in the stock price can be neutralized by rebalancing ξ and η with no inflow or outflow of capital. The value of the portfolio for a self-financing strategy is then given by $V_k = V_0 + G_k$ at any time $0 \leq k \leq M$.

A market is complete if any claim H is attainable, that is, there exists a self-financing strategy with $V_M = H$ (a.s.). If the market is incomplete, for instance in the case of discrete hedging, a claim is, in general, non-attainable and a hedging strategy has to be chosen based on some optimality criterion.

One approach to hedging in an incomplete market is to consider only self-financing strategies. An optimal self-financing strategy is then chosen which best approximates H by its terminal value V_M. The quadratic criterion for this *total risk-minimization* is given by minimizing the L^2-norm:

$$E((H - V_M)^2) = E((H - V_0 - G_M)^2). \tag{1}$$

By solving the total risk-minimization problem (1), we obtain the initial value of the portfolio, V_0, and the number of shares, $(\xi_0, ..., \xi_{M-1})$. The amount invested in the bond, $(\eta_0, ..., \eta_M)$, is then uniquely determined since the strategy is self-financing. Unfortunately, this problem does not have a solution in general. Schweizer (1995) proves the existence of a total risk-minimizing strategy when the discounted stock price has a bounded mean-variance tradeoff, that is:

$$\frac{(E(X_k - X_{k-1}|\mathcal{F}_k))^2}{\mathrm{Var}(X_k - X_{k-1}|\mathcal{F}_k)} \text{ is P-a.s. uniformly bounded.}$$

Another approach to hedging in an incomplete market is to first impose $V_M = H$, hence $\eta_M = H$. Since such a strategy cannot be self-financing, we should then choose the optimal trading strategy to minimize the incremental cost incurred from adjusting the portfolio at each hedging time. The quadratic criterion for this *local risk-minimizing* strategy is given by minimizing:

$$E((C_{k+1} - C_k)^2|\mathcal{F}_k)\ ,\ 0 \le k \le M - 1. \tag{2}$$

When $M = 1$, the local risk-minimization and the total risk-minimization criteria coincide. Under the assumption that the above mean-variance tradeoff is deterministic, Schäl (1994) proves that the initial cost for the local risk-minimizing strategy is equal to the cost for the total risk-minimizing strategy. He then infers that this cost is a *fair hedging price*. However, this assumption is strong and in general the two initial costs are different.

These two criteria are discussed in detail in Föllmer, and Schweizer (1989), Schäl (1994), Schweizer (1995, 2001). We will concentrate here only on the local risk-minimization (2).

The problem is to minimize the local risk:

$$E((C_{k+1} - C_k)^2|\mathcal{F}_k) = E((V_{k+1} - V_k - \xi_k(X_{k+1} - X_k))^2|\mathcal{F}_k),$$

for all $0 \le k \le M - 1$, starting from the final condition $V_M = H$.

This local risk-minimization strategy has also a "global aspect": in order to determine the holdings in the hedging portfolio at a certain time, optimization problems need to be solved backward in time from the maturity of the option. Since the market is incomplete, the option values are not known at intermediate times and therefore, the hedging strategy cannot consider only the information in the current hedging period.

If H is a square integrable random variable and X is a square integrable process, then (2) is guaranteed to have a solution obtained in the following way: starting from $V_M = \eta_M = H$, for $k = M - 1, ..., 0$ we choose ξ_k, η_k recursively to minimize,

$$E((V_{k+1} - V_k - \xi_k(X_{k+1} - X_k))^2|\mathcal{F}_k) = E((X_{k+1}(\xi_{k+1} - \xi_k) + (\eta_{k+1} - \eta_k))^2|\mathcal{F}_k). \tag{3}$$

The hedging strategy constructed in this way is given explicitly by:

$$\begin{cases} \xi_M = 0, \ \eta_M = H \\ \\ \xi_k = \frac{\mathrm{Cov}(\xi_{k+1} X_{k+1} + \eta_{k+1}, X_{k+1} | \ \mathcal{F}_k)}{\mathrm{Var}(X_{k+1} | \mathcal{F}_k)}, & 0 \leq k \leq M-1 \\ \\ \eta_k = E((\xi_{k+1} - \xi_k) X_{k+1} + \eta_{k+1} | \mathcal{F}_k), & 0 \leq k \leq M-1. \end{cases} \tag{4}$$

This hedging strategy is no longer self-financing, but it has a weaker property: it is *mean-self-financing*, that is $E(C_{k+1}|\mathcal{F}_k) = C_k$, for all $0 \leq k \leq M-1$, or, in other words, the cost process is a martingale. In particular, this implies $C_0 = E(C_M)$, which seems to justify choosing $C_0 = V_0$ as a fair hedging price.

3. Piecewise linear local risk-minimization

An alternative way for choosing a local risk-minimizing strategy is to use the risk measure $E(|C_{k+1} - C_k| \ |\mathcal{F}_k)$ in the above minimization problem. The problem of choosing a piecewise linear local risk-minimizing strategy is then given by: starting from the final condition $V_M = H$, minimize,

$$E(|C_{k+1} - C_k| \ |\mathcal{F}_k) \ , \ 0 \leq k \leq M-1. \tag{5}$$

Problem (5) seems more natural than (2) from the financial point of view, since we are trying to minimize the risk measured by the incremental cost for adjusting the portfolio valued in monetary units. Moreover, as shown in the Appendix, when the risky asset price follows a binomial model, problems (2) and (5) respectively reduce to solving L^2 and L^1-minimization problems. The fact that the L^2-norm overemphasizes the large values even if these values have a very small probability of occurrence is another reason why we believe that problem (5) is more appropriate than (2). We believe the choice of the quadratic risk-minimization criterion in literature was made mainly because it allows explicit results, but we will see that using piecewise linear risk-minimization often leads to significantly different hedging strategies and possibly better hedging results.

Similar to the case of the quadratic criterion, the optimal piecewise linear strategy is constructed recursively by first choosing $\eta_M = H, \ \xi_M = 0$, then for $k = M-1, ..., 0$ choosing $\xi_k, \ \eta_k$ to minimize:

$$E(|X_{k+1}(\xi_{k+1} - \xi_k) + (\eta_{k+1} - \eta_k)| \ |\mathcal{F}_k). \tag{6}$$

The strategy constructed in this way is no longer mean-self-financing. The lack of this property is not a practical drawback since we are predominantly interested in reducing the mean of the incremental costs and not in preserving the mean cost. However, if we insist on mean-self-financing strategies, we can solve instead the following local risk-minimization problem: starting from $V_M = H$, for all $k = M-1, ..., 0$, minimize

$$\begin{cases} \min \ E(|C_{k+1} - C_k| \ |\mathcal{F}_k) \\ \\ \text{subject to } E(C_{k+1}|\mathcal{F}_k) = C_k. \end{cases} \tag{7}$$

That is, first take $\eta_M = H,\ \xi_M = 0$, then recursively, for $k = M-1, ..., 0$, choose $\xi_k,\ \eta_k$ to minimize:

$$E(|\xi_{k+1}X_{k+1} + \eta_{k+1} - E(\xi_{k+1}X_{k+1} + \eta_{k+1}|\ \mathcal{F}_k) - \xi_k(X_{k+1} - E(X_{k+1}|\mathcal{F}_k))|\ |\mathcal{F}_k) \qquad (8)$$

and define $\eta_k = E(\xi_{k+1}X_{k+1} + \eta_{k+1} - \xi_k X_{k+1}|\mathcal{F}_k)$.

We note that while in problem (6) we have the liberty of choosing both η_k and ξ_k, the mean-self-financing constraint imposes a relation between these two variables.

Unfortunately, it is not possible to obtain an analytic solution for problems (6) and (8).

4. Numerical results

Assume the writer of a European option with maturity T wants to hedge his position using the underlying stock and a bond. Also assume there are only M hedging opportunities at $0 = t_0 < t_1 < ... < t_{M-1} < t_M := T$.

Suppose the price of the underlying stock follows the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t, \qquad (9)$$

where Z_t is a Wiener process.

We generate a binomial tree of possible values of the asset price choosing the parameters such that if the number of periods in the tree is increased, in the limiting case, the binomial process converges to the continuous process (9). The numerical results presented in this section refer to hedging put options with maturity $T = 1$ and different strike prices. They have been obtained for the initial value of the stock $S_0 = 100$, the instantaneous expected return $\mu = .2$, the volatility $\sigma = .2$, and the riskless rate of return $r = .1$. The number of periods in the binomial tree is 600. The detailed implementation of all the three methods when using a binomial tree is presented in the Appendix.

We compute the holdings (ξ, η) in the portfolio at each node in the binomial tree using the previously described methods of local risk-minimization:

- Method 1: Piecewise linear risk-minimization (6)
- Method 2: Quadratic risk-minimization (4)
- Method 3: Constrained piecewise linear risk-minimization (8)

We want to analyze the performance of the three methods given that the stock price satisfies equation (9). We generate synthetic paths for the stock price based on this equation. For each path and each of the three methods we determine the holdings in the portfolio used to hedge the option if the stock price had the values given by the path. The portfolio holdings are determined based on the already computed holdings in the binomial tree: at every hedging time we find the node in the binomial tree where the stock price has the closest value to the one on the path and we consider the holdings at that particular node. We then compute the following:

- End of period *cumulative cost* discounted to time 0:

$$C_M = H - \sum_{k=0}^{M-1} \xi_k(X_{k+1} - X_k).$$

  This is the discounted total amount of money necessary for the writer to carry on the hedging strategy and honor the option payoff.

- *Incremental risk* per rebalancing time discounted to time 0:

$$\frac{1}{M}\sum_{k=0}^{M-1}|C_{k+1}-C_k|.$$

  This is the average of the absolute values of all the adjustments in the portfolio. It is a measure of the unplanned intermediate costs or income of the strategy.

The numerical results will show that Method 1, the piecewise linear risk minimization, is the best in most of the cases in terms of average cumulative cost and incremental risk. Method 3, the constrained piecewise linear risk minimization, is also better than Method 2, the quadratic risk minimization. The differences between the three methods tend to increase as we rebalance less frequently.

Let us first analyze the case of the static hedging when we only hedge once, at time 0. Tables 1 and 2 show the average cumulative cost and incremental risk over 500 simulated paths.

Strike	95			100			105		
Method	1	2	3	1	2	3	1	2	3
	1.0886	1.7124	1.3267	1.8960	2.8826	2.3667	3.0545	4.5074	3.9393

Table 1. Average cumulative cost over 500 paths for one hedging opportunity

Strike	95			100			105		
Method	1	2	3	1	2	3	1	2	3
	1.0886	1.9194	1.6959	1.8960	2.7052	2.5366	3.0545	3.5402	3.4721

Table 2. Average incremental risk over 500 paths for one hedging opportunity

The average cumulative cost for Methods 1 and 3 is much smaller than the cost for Method 2. This is especially the case for Method 1 for which the cumulative cost is almost two thirds the cost of Method 2.

Similar results are illustrated in Table 2. Since the incremental risk accumulates the absolute values of all the unplanned costs or profits of the strategy, it is a measure of the riskiness of the strategy. Using this risk measure, we infer that Method 1 is the least risky among the three methods. Method 3 is also less risky than Method 2.

Consider now that we have more hedging opportunities. Tables 3 and 4 show the average values over 500 paths of the cumulative cost and incremental risk, for different number of binomial tree periods per rebalancing time. We remark that the last column corresponds to the above case, that is, hedging only once, at time 0.

		Number of periods per rebalancing time							
Strike	Method	1	5	10	25	50	100	300	600
95	1	2.3943	1.7133	2.0579	2.1604	1.8537	1.6241	1.2702	1.0886
	2	2.3943	2.3848	2.3942	2.3892	2.2846	2.2268	2.1105	1.7124
	3	2.3943	2.3131	2.3368	2.2723	2.1758	2.0999	1.7392	1.3267
100	1	3.7578	3.0487	3.3202	3.5450	3.1924	2.9145	2.2254	1.8960
	2	3.7578	3.7356	3.7283	3.7443	3.6421	3.5174	3.4581	2.8826
	3	3.7578	3.6465	3.6640	3.6195	3.5083	3.4147	3.0779	2.3667
105	1	5.5266	5.0367	5.4258	5.4036	5.2196	4.8707	4.4567	3.0545
	2	5.5266	5.5233	5.5150	5.5389	5.4099	5.3005	5.1893	4.5074
	3	5.5266	5.4329	5.4507	5.4084	5.3125	5.2421	4.9151	3.9393

Table 3. Average value of the cumulative cost over 500 paths

		Number of periods per rebalancing time							
Strike	Method	1	5	10	25	50	100	300	600
95	1	0.0047	0.0151	0.0343	0.0856	0.1645	0.2920	0.7459	1.0886
	2	0.0047	0.0187	0.0394	0.0938	0.1956	0.3600	1.0762	1.9194
	3	0.0047	0.0196	0.0395	0.0961	0.1959	0.3554	1.0218	1.6959
100	1	0.0060	0.0242	0.0458	0.1154	0.2425	0.4457	1.2655	1.8960
	2	0.0060	0.0245	0.0474	0.1207	0.2518	0.4750	1.4841	2.7052
	3	0.0060	0.0256	0.0478	0.1241	0.2526	0.4713	1.4660	2.5366
105	1	0.0072	0.0341	0.0648	0.1411	0.3238	0.5920	1.9134	3.0545
	2	0.0072	0.0296	0.0612	0.1437	0.3042	0.5820	1.9053	3.5402
	3	0.0072	0.0311	0.0617	0.1480	0.3065	0.5745	1.9294	3.4721

Table 4. Average value of the incremental risk over 500 paths

Consistent with the single hedging opportunity case, the average cumulative cost for Method 1 is the smallest. Method 3 has an average cumulative cost close in value to Method 2, although smaller in most of the cases. As mentioned above, the differences tend to increase when rebalancing is infrequent. With respect to the average incremental risk, Method 1 is the best for the out-of-money and at-the-money put. The incremental risk for Method 3 is also smaller (or very close) to the risk for Method 2. In the case of the in-the-money put, Method 2 tends to be marginally better.

The expected total cost and incremental risk can be approximated even without simulation by analyzing the three methods in the binomial view of the world, where the stock price can only follow the paths given by the binomial tree. For each method we compute the expected cumulative cost and expected incremental risk over the entire binomial tree.

- Expected time T cumulative cost discounted to time 0:

$$E(C_M) = E(H - \sum_{k=0}^{M-1} \xi_k (X_{k+1} - X_k)).$$

This is the expected cost of the hedging strategy which can be easily computed once ξ_k has been computed. It is an indication of how much the hedging strategy costs on average.

- Expected incremental risk per rebalancing time, discounted to time 0

$$E(\frac{1}{M}\sum_{k=0}^{M-1} E(|C_{k+1}-C_k|\ |\mathcal{F}_k)) = \frac{1}{M}\sum_{k=0}^{M-1} E(|C_{k+1}-C_k|).$$

This is a measure of the average unplanned intermediate costs or income of the strategy.

The expected cost for all three methods is presented in Table 5.

		Number of periods per rebalancing time							
Strike	Method	1	5	10	25	50	100	300	600
95	1	2.3977	1.7969	2.0085	2.1282	1.7763	1.6797	1.1971	0.9682
	2	2.3977	2.3912	2.3832	2.3593	2.3203	2.2455	1.9929	1.7353
	3	2.3977	2.3284	2.3283	2.2460	2.1976	2.1381	1.6401	1.2611
100	1	3.7499	3.1634	3.3994	3.5006	3.1313	2.9887	2.1038	1.6570
	2	3.7499	3.7422	3.7325	3.7035	3.6557	3.5626	3.2321	2.8703
	3	3.7499	3.6674	3.6695	3.5739	3.5236	3.4766	2.8802	2.2359
105	1	5.5191	5.1031	5.2926	5.3356	5.0523	4.9430	4.2796	2.6471
	2	5.5191	5.5103	5.4994	5.4667	5.4122	5.3045	4.9042	4.4337
	3	5.5191	5.4274	5.4322	5.3294	5.2836	5.2607	4.6578	3.7352

Table 5. Expected cumulative cost $E(C_M)$

We observe that when we rebalance the portfolio every period, all the methods yield the same expected cost since we are able to exactly replicate the options on all the paths of the binomial tree. When the number of periods per rebalancing time is larger than 1, Method 1 gives a smaller expected cost than Method 2. Method 3 is also better than Method 2 but the differences are less significant. We remark also that the expected cost for piecewise linear local risk-minimization is much smaller when we rebalance infrequently.

Let us analyze now the expected incremental risk.

		Number of periods per rebalancing time							
Strike	Method	1	5	10	25	50	100	300	600
95	1	0.0000	0.0137	0.0298	0.0800	0.1515	0.2982	0.7046	0.9682
	2	0.0000	0.0188	0.0369	0.0921	0.1841	0.3672	1.0339	1.8108
	3	0.0000	0.0180	0.0354	0.0919	0.1829	0.3622	0.9792	1.5635
100	1	0.0000	0.0217	0.0429	0.1075	0.2296	0.4555	1.2355	1.6570
	2	0.0000	0.0241	0.0473	0.1188	0.2389	0.4817	1.4197	2.6152
	3	0.0000	0.0231	0.0455	0.1189	0.2381	0.4775	1.4054	2.3824
105	1	0.0000	0.0299	0.0550	0.1332	0.3014	0.5979	1.8135	2.6471
	2	0.0000	0.0287	0.0563	0.1423	0.2878	0.5856	1.7967	3.4558
	3	0.0000	0.0275	0.0543	0.1426	0.2867	0.5793	1.8207	3.2905

Table 6. Expected incremental risk per rebalancing time

When we rebalance the portfolio every period, we replicate the options exactly on all the paths of the binomial tree, therefore the risk is zero for all the methods. When we rebalance the portfolio less frequently, Method 1 yields the smallest expected incremental risk in most of the cases and Method 3 is an intermediate method. We notice again that the performance of the piecewise linear local risk-minimization depends on the moneyness of the put options, the best results being obtained for out-of-money and at-the-money put options. As shown in Table 6, the expected incremental risk per rebalancing time increases as we rebalance less frequently.

As we can see from Tables 5 and 6, the values of the expected total cost and expected incremental risk follow closely the values of the average cumulative cost and average incremental risk.

The above numerical results show that the three methods perform differently with respect to the total cost and incremental risk. Next we illustrate that their hedging styles are also different. Consider the particular case of hedging the out-of-money put option with 12 hedging opportunities. We can compute the discounted cumulative cost C_k of the portfolio at each hedging time t_k along a simulated path. Figure 2 shows the typical evolution of the discounted cumulative cost along such a path.

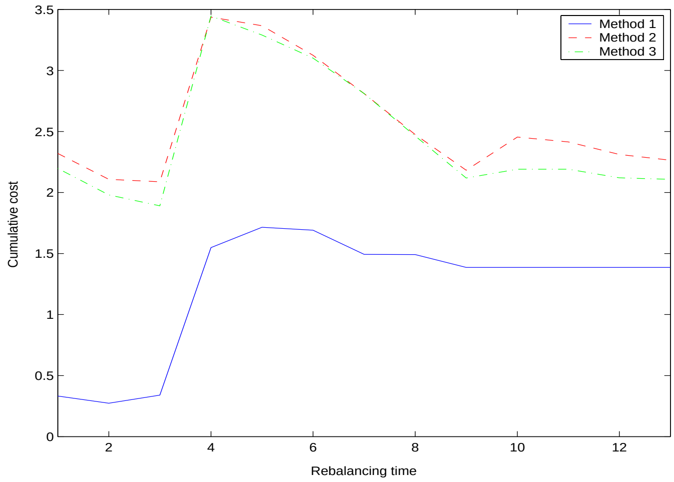

Figure 2: Cumulative costs along a simulated path for the stock price

We notice that the plots of the cumulative costs for the mean-self-financing Methods 2 and 3 are almost identical. Method 1 starts with a much smaller initial investment in the hedging portfolio and gradually increases it along the path in order to match the option payoff. The fact that the initial cost for setting the portfolio tends to be much smaller for Method 1 than for the other two methods is illustrated in Table 7. The initial portfolio cost is computed by $C_0 = \xi_0 X_0 + \eta_0$, where ξ_0, η_0 are the holdings in the binomial tree at time 0.

		Number of periods per rebalancing time							
Strike	Method	1	5	10	25	50	100	300	600
95	1	2.3977	0.1530	0.6673	1.3139	0.3328	0.3679	0.0000	0.0000
	2	2.3977	2.3912	2.3832	2.3593	2.3203	2.2455	1.9929	1.7353
	3	2.3977	2.3284	2.3283	2.2460	2.1976	2.1381	1.6401	1.2611
100	1	3.7499	0.5544	1.4451	2.3361	0.8783	0.7925	0.0000	0.0000
	2	3.7499	3.7422	3.7325	3.7035	3.6557	3.5626	3.2321	2.8703
	3	3.7499	3.6674	3.6695	3.5739	3.5236	3.4766	2.8802	2.2359
105	1	5.5191	1.5201	2.8201	4.0033	2.2123	2.4485	2.6349	0.0000
	2	5.5191	5.5103	5.4994	5.4667	5.4122	5.3045	4.9042	4.4337
	3	5.5191	5.4274	5.4322	5.3294	5.2836	5.2607	4.6578	3.7352

Table 7. Initial cost of the hedging portfolio

Next we analyze the distributions of the cumulative cost for the three methods with respect to the hedging of the out-of-money put option with 12 rebalancing opportunities. From Table 4 we see that the average cumulative costs for the three methods are 1.8537, 2.2846 and 2.1758, respectively. Figure 3 shows the histograms for the cumulative costs over the 500 simulated paths.

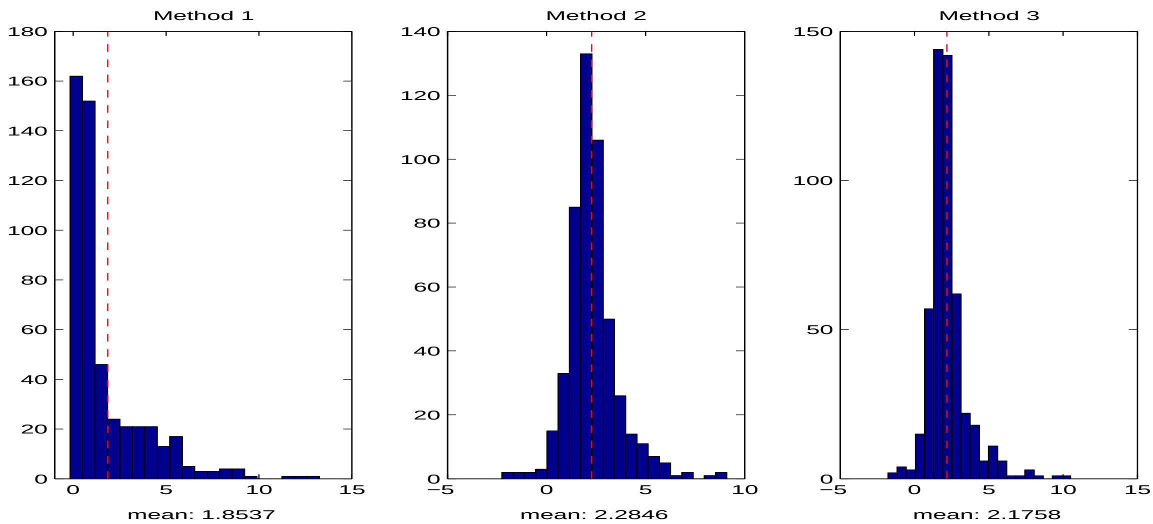

Figure 3: Histogram of cumulative costs

We can see that Method 1 is more asymmetric about its mean compared to Methods 2 and 3. Almost 70% of the time the cumulative costs for Method 1 are less than their mean and 55% are less than half of the mean. Since the average cumulative cost is very close to the expected total cost, it means that almost 70% of the time the cost for rebalancing the portfolio will be less than what we expect. In the case of Method 3, only 60% of the time the cumulative costs are less than their mean, while in the case of Method 2 the median is approximately equal to the mean. However, Figure 3 also shows that Method 1 has a very small probability of having larger costs than the costs for the other two methods.

The skewness of the cumulative costs is another indication of the asymmetry of the data around the mean. In our case, the skewness for Method 1 is 2.1639, while the skewness for Method 2 is 1.1562 and the one for Method 3 is 1.1538.

The histograms for the incremental risks over the same paths are presented in Figure 4. The average incremental risks for the three methods are very close in value. However, the median for Method 1 is 0.0480, while the medians for the other two methods are 0.0623 for Method 2 and 0.0604 for Method 3. On the other hand, as in the case of the cumulative cost, we can also notice that the incremental risk for Method 1 is more widely spread. The skewness for the three methods are 1.0774, 0.7988, and 0.8221 respectively.

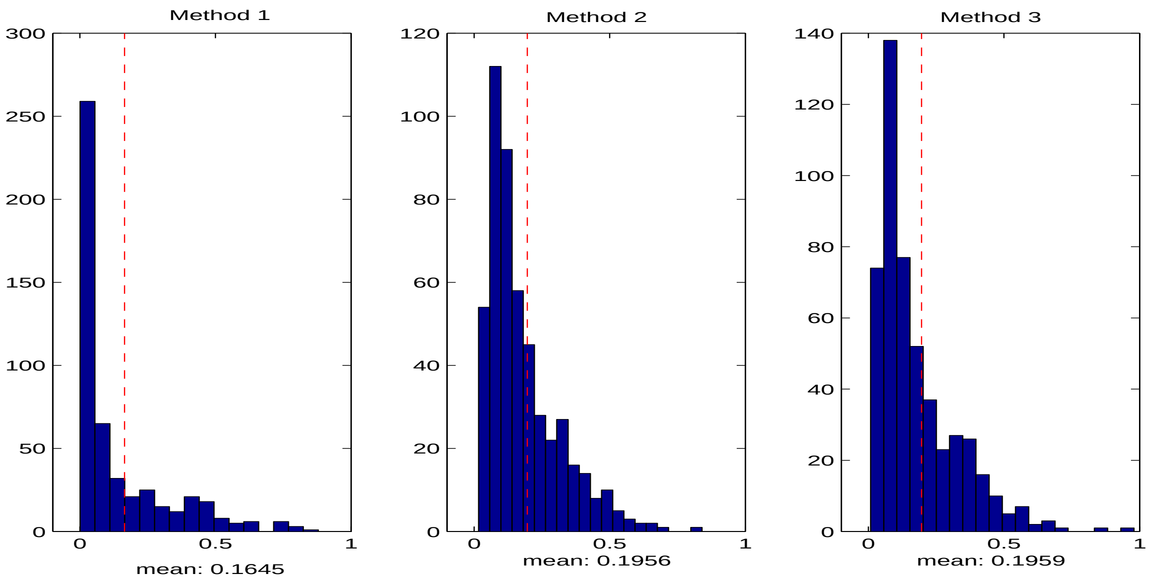

Figure 4: Histogram of incremental risks

The distributions of the cumulative cost and incremental risk become more and more asymmetrical as we hedge less frequently or the put option is more deeply out-of-money. Tables 8 and 9 show the skewness of the distributions of the cumulative cost and incremental risk for at-the-money and out-of-money puts with very few hedging opportunities.

		Number of periods per rebalancing time		
Strike	Method	100	300	600
100	1	1.9501	2.2083	3.6606
	2	1.0964	1.5402	2.3062
	3	1.4971	2.0253	3.4866
95	1	3.3159	3.1150	3.6968
	2	2.1452	1.9755	3.0489
	3	2.9816	2.8925	3.7503
90	1	3.2092	3.9750	6.3692
	2	1.9215	3.3258	5.8251
	3	2.5941	3.9460	6.4418

Table 8. Skewness of the cumulative cost

		Number of periods per rebalancing time		
Strike	Method	100	300	600
100	1	1.6664	1.8297	3.6606
	2	1.3280	1.9650	3.2254
	3	1.5124	2.1018	4.0890
95	1	3.0099	2.7516	3.6968
	2	2.1937	2.5577	4.0712
	3	2.3636	2.8494	4.2680
90	1	3.1017	3.6850	6.3692
	2	2.5482	3.4817	6.9718
	3	2.8981	3.6814	6.9192

Table 9. Skewness of the incremental risk

We note that the skewness increases as the number of hedging opportunities decreases and has the largest value when the put option is deep out-of-money.

5. Discrete hedging put-call parity

In Section 4 we have only analyzed the case of put options. We will see that hedging call options is closely related to the hedging of put options. Suppose that we have computed the optimal holdings ξ^{put}, η^{put} in the portfolio for hedging a put option with maturity T, discounted strike price K and M hedging opportunities at $0 = t_0 < t_1 < ... < t_{M-1} < t_M := T$. We can derive a relation between these holdings and the corresponding optimal holdings ξ^{cal}, η^{call} for the call option on the same underlying asset and with the same maturity, strike price and hedging opportunities. Namely, we have the following property:

$$\begin{cases} \xi_k^{call} = \xi_k^{put} + 1 \\ \eta_k^{call} = \eta_k^{put} - K \end{cases}, \tag{10}$$

for all $0 \leq k \leq M-1$. Thus if we know the optimal holdings for the put option, we can compute the optimal holdings for the call, directly, without solving any optimization problems.

Moreover, the discounted values of the portfolios for hedging the put and the call options, V_k^{put} and V_k^{cal}, satisfy the following put-call parity relation for all $0 \leq k \leq M$:

$$V_k^{call} - V_k^{put} = X_k - K. \tag{11}$$

This is certainly true at time T, since:

$$V_M^{call} - V_M^{put} = H^{call} - H^{put} = X_M - K,$$

where $H^{call} = (X_M - K)^+$ and $H^{put} = (K - X_M)^+$.

For any $0 \leq k \leq M-1$, relation (11) follows immediately from (10). Indeed,

$$V_k^{call} - V_k^{put} = (\xi_k^{call} - \xi_k^{put})X_k + \eta_k^{call} - \eta_k^{put} = X_k - K.$$

Similarly, we can deduce the relation between the cumulative costs for the call and put options:

$$C_M^{call} = C_M^{put} + X_0 - K.$$

Moreover, the incremental costs for the call and put options are equal, that is, for all $k = 0, 1, \ldots, M-1$:

$$C_{k+1}^{call} - C_k^{call} = C_{k+1}^{put} - C_k^{put}.$$

Using the above relations, we can easily translate the numerical results for hedging put options, presented in Section 4, to the hedging of call options. Piecewise linear local risk-minimization often leads to smaller expected total hedging cost and risk, with the best results obtained in the case of in-the-money and at-the-money call options.

Let us now prove relation (10). The proof given below is valid for all the three methods.

Recall that we have $\xi_M = 0,\ \eta_M = H$, where H is the discounted payoff. For $k \le M-1$, in order to find the number of shares ξ_k and the number of units of bond η_k at time t_k, we have to solve an optimization problem of the form:

$$\min\ E(f(C_{k+1} - C_k)|\mathcal{F}_k) = \min_{\xi_k,\eta_k} E(f((\xi_{k+1} - \xi_k)X_{k+1} + (\eta_{k+1} - \eta_k))|\mathcal{F}_k). \tag{12}$$

where $f(x) = x^2$ for Method 2 and $f(x) = |x|$ for Methods 1 and 3. In the case of Method 3 we also have the constraint:

$$E(C_{k+1} - C_k|\mathcal{F}_k) = 0 \ \Leftrightarrow\ \eta_k = E((\xi_{k+1} - \xi_k)X_{k+1} + \eta_{k+1}|\mathcal{F}_k). \tag{13}$$

Let us first show that relation (10) holds at time t_{M-1}.

For all $(\xi_{M-1}^{put}, \eta_{M-1}^{put})$, let

$$\begin{cases} \xi_{M-1}^{call} = \xi_{M-1}^{put} + 1 \\ \eta_{M-1}^{call} = \eta_{M-1}^{put} - K. \end{cases} \tag{14}$$

Then

$$\begin{aligned} E(f(C_M^{call} - &C_{M-1}^{call})|\mathcal{F}_{M-1}) \\ &= E(f(H^{call} - \xi_{M-1}^{call}X_M - \eta_{M-1}^{call})|\mathcal{F}_{M-1}) \\ &= E(f(H^{call} - X_M + K - \xi_{M-1}^{put}X_M - \eta_{M-1}^{put})|\mathcal{F}_{M-1}) \\ &= E(f(H^{put} - \xi_{M-1}^{put}X_M - \eta_{M-1}^{put})|\mathcal{F}_{M-1}) \\ &= E(f(C_M^{put} - C_{M-1}^{put})|\mathcal{F}_{M-1}). \end{aligned} \tag{15}$$

Conversely, for all $(\xi_{M-1}^{call}, \eta_{M-1}^{call})$, consider $(\xi_{M-1}^{put}, \eta_{M-1}^{put})$, defined by (14). We have again that relations (15) hold. It follows that:

$$\min\ E(f(C_M^{call} - C_{M-1}^{call})|\mathcal{F}_{M-1}) = \min\ E(f(C_M^{put} - C_{M-1}^{put})|\mathcal{F}_{M-1})$$

and $(\xi_{M-1}^{put}, \eta_{M-1}^{put})$ is optimal if and only if $(\xi_{M-1}^{call},\ \eta_{M-1}^{call})$ is optimal.

We remark also that $(\xi_{M-1}^{put}, \eta_{M-1}^{put})$ satisfies the constraint (13) if and only if $(\xi_{M-1}^{call}, \eta_{M-1}^{call})$ satisfies it. We conclude that relation (10) holds at time t_{M-1} for all the three methods.

Suppose now that we have proved relation (10) at time t_{k+1} and we want to argue it holds at time t_k. The proof follows exactly in the same way as above by keeping in mind that the optimal holdings at time t_{k+1}, that is $(\xi_{k+1}^{call}, \eta_{k+1}^{call})$ and $(\xi_{k+1}^{put}, \eta_{k+1}^{put})$ satisfy (10).

Therefore, (10) hold for all $0 \le k \le M-1$.

6. Conclusions

In an incomplete market, the optimal hedging strategy depends on the criteria for measuring the risk. The traditional strategies found in literature are based on quadratic risk-minimization. The numerical results presented in the paper illustrate that piecewise linear risk-minimization typically leads to hedging strategies with smaller expected total hedging cost and risk. These new strategies have quite different, and often preferable, properties compared to traditional strategies. The distributions of the total hedging cost and risk show that hedging strategies obtained by piecewise linear risk-minimization have a larger probability of small cost and risk, though they also have a very small probability of larger cost and risk. We remark that the performance of the optimal hedging strategies for piecewise linear local risk-minimization depends on the moneyness of the option, namely, the best results are obtained for out-of-money put options and in-the-money call options.

Although there is no analytic solution to the piecewise linear local risk-minimization problem, an optimal strategy can be easily computed. In order to compute the hedging strategy for an option with maturity T, we generate a binomial tree on the time horizon $[0, T]$ and compute the portfolio holdings in each state of this tree at hedging times. However, in practice, each time we hedge we may have new information on the volatility and drift of the stock. Therefore, a new binomial tree can be computed at any rebalancing time on the remaining time horizon.

Our future research will address the theoretical questions related to the uniqueness and convergence of the optimal trading strategy computed by piecewise linear local risk-minimization. In addition, we plan to study the piecewise linear total risk-minimization criterion. We have seen that the total risk measure $E(|C_M - C_0|)$ involves $M + 1$ variables (the number of shares of stock in the hedging portfolio at each time $t_0, ..., t_{M-1}$ and its initial cost C_0). On the other hand, the local risk measure $E(|C_{k+1} - C_k| \; |\mathcal{F}_k)$ refers to only two variables (the number of shares of stock and the number of units of the bond at time t_k). Using, for example, a binomial tree for the stock price, the local risk-minimization can be expressed as a collection of two-variable piecewise linear minimization problems on subtrees and it is very easy to implement a solver for this case. However, it is much more computationally expensive to solve the total risk-minimization problem. Since $C_M - C_0$ depends on the entire path of the stock price, a direct approach to the total risk-minimization problem min $E(|C_M - C_0|)$ would have to consider all the paths in the binomial tree, and therefore exponentially many variables. We are investigating the implementation of an efficient algorithm to compute the optimal trading strategies using this criterion.

Appendix

Let us analyze in detail the implementation of the local risk-minimizing problems (2), (5) and (7), when using an event tree to describe the stock price. Consider the filtration $(\mathcal{F}_k)_{k=\overline{0,M}}$, given by $\mathcal{F}_k = \sigma(X_j | j \leq k)$, the σ-field generated by the variables $X_0, \ldots, X_k$. Suppose the stock price is modelled using a binomial tree with N periods, but hedging can only take place on $M < N$ time dates $0 = i_0 < i_1 < ... < i_{M-1} < N := i_M$. Thus, for all $0 \leq k \leq M$, at time i_k there are $n_k = i_k + 1$ possible states for the stock price and given state j at time i_k, the stock price can only move to $\alpha_k = i_{k+1} - i_k + 1$ possible states at time i_{k+1}:

$$
\begin{tikzcd}[row sep=small]
S_k^j \arrow[r, "p_1"] \arrow[ddr, "p_j"] \arrow[ddddr, "p_{\alpha_k}"'] & S_{k+1}^j = u^{\alpha_k-1} S_k^j \\
& \vdots \\
& S_{k+1}^{j+i-1} = u^j d^{\alpha_k-1-j} S_k^j \\
& \vdots \\
& S_{k+1}^{j+\alpha_k-1} = d^{\alpha_k-1} S_k^j,
\end{tikzcd}
$$

where $p_j = \binom{\alpha_k-1}{j} p^{\alpha_k-1-j}(1-p)^j$ for all $0 \le j \le \alpha_k - 1$.

Recall that the discounted stock price is then given by:

$$X_k^j = \frac{S_k^j}{B_k}, \ \forall 0 \le k \le M, \ \forall 0 \le j \le n_k.$$

Suppose now that at time N, the discounted payoff of the option in state j is given by H_j.

Then, the piecewise linear risk minimization problem (5) becomes: starting from $V_M = H$, for all the states j at time t_k, $k = M-1, \ldots, 0$, minimize

$$\min E(|C_{k+1} - C_k| \ |X_k = X_k^j). \tag{16}$$

Therefore:

- For each $1 \le j \le n_M$ define $\xi_M^j = 0, \ \eta_M^j = H_j$.
- For each $k = M-1, ..., 0$ and for each $1 \le j \le n_k$, find the number of shares ξ_k^j and the number of units of the bond η_k^j at time i_k if state j occurs by solving the minimization problem:

$$\min_{\xi_k^j, \eta_k^j} \sum_{l=0}^{\alpha_k-1} p_l \ |X_{k+1}^{j+l}(\xi_{k+1}^{j+l} - \xi_k^j) + \ (\eta_{k+1}^{j+l} - \eta_k^j)|. \tag{17}$$

If we want to express (17) in a more compact way, for each $k = M-1, ..., 0$ and for each $1 \le j \le n_k$, denote by:

$$A = \begin{bmatrix} p_1 & p_1 X_{k+1}^j \\ \vdots & \vdots \\ p_{\alpha_k} & p_{\alpha_k} X_{k+1}^{j+\alpha_k-1} \end{bmatrix}, \ b = \begin{bmatrix} p_1(X_{k+1}^j \xi_{k+1}^j + \eta_{k+1}^j) \\ \vdots \\ p_{\alpha_k}(X_{k+1}^{j+\alpha_k-1} \xi_{k+1}^{j+\alpha_k-1} + \ \eta_{k+1}^{j+\alpha_k-1}) \end{bmatrix}, \ x = \begin{bmatrix} \eta_k^j \\ \xi_k^j \end{bmatrix}.$$

Then (17) becomes:

$$\min_{x \in \mathbb{R}^2} ||Ax - b||_1. \tag{18}$$

As we mentioned before, when the stock price follows a binomial model, the piecewise linear risk-minimization (5) reduces to solving L^1-minimization problems of the form (18). In order to solve these two-dimensional L^1-optimization problems, we have implemented an L^1-solver similar to the solver for the L^1-norm fit of a straight line described by Karst (1958) and Sadovski (1974).

The constrained piecewise linear risk-minimization method is implemented in a similar way. Using a binomial model for the stock price, problem (7) becomes: starting from $V_M = H$, for all the states j at time t_k, $k = M-1, \ldots, 0$, minimize

$$\begin{cases} \min\ E(|C_{k+1} - C_k|\ |X_k = X_k^j) \\ \\ \text{subject to } E(C_{k+1} - C_k | X_k = X_k^j) = 0. \end{cases} \tag{19}$$

Problem (8) is therefore given by:

- For each $1 \leq j \leq n_M$ define $\xi_M^j = 0,\ \eta_M^j = H_j$.
- For each $k = M-1, ..., 0$ and for each $1 \leq j \leq n_k$, find the number of shares ξ_k^j at time i_k if state j occurs by solving the minimization problem:

$$\min_{\xi_k^j} \sum_{l=0}^{\alpha_k - 1} p_l | X_{k+1}^{j+l}\ \xi_{k+1}^{j+l} - \sum_{s=0}^{\alpha_k - 1} p_s (X_{k+1}^{j+s} \xi_{k+1}^{j+s}\ + \eta_{k+1}^{j+s}) - \xi_k^j (X_{k+1}^{j+l} - \sum_{s=0}^{\alpha_k - 1} p_s\ X_{k+1}^{j+s})| \tag{20}$$

then define: $\eta_k^j = \sum_{l=0}^{\alpha_k - 1} p_l (\xi_{k+1}^{j+l}\ X_{k+1}^{j+l} + \eta_{k+1}^{j+l} - \xi_k^j X_{k+1}^{j+l})$.

We can also express this in a more compact way similar to problem (17). For each $k = M-1, ..., 0$ and for each $1 \leq j \leq n_k$, consider the matrices A and b defined before. Denote by $A^{(1)},\ A^{(2)}$ the columns of the matrix A. Consider also the vector $e = [1, ..., 1]^T$. Then (19) can be expressed as:

$$\begin{cases} \min_{x \in \mathbb{R}^2}\ ||Ax - b||_1 \\ \\ \text{subject to: } [1\ ,\ e^T A^{(2)}]\ x =\ e^T b, \end{cases} \tag{21}$$

with $x = \begin{bmatrix} \eta_k^j \\ \xi_k^j \end{bmatrix}$.

The above problem can be transformed in a one dimensional minimization problem by substituting η_k^j:

$$\min_{x \in \mathbb{R}} ||(A^{(2)} -\ (e^T A^{(2)}) A^{(1)}) x - b + (e^T b) A^{(1)}||_1$$

$$\begin{cases} \xi_k^j = x \\ \eta_k^j = e^T b - (e^T A^{(2)})\ x. \end{cases} \tag{22}$$

Therefore, the constrained piecewise linear risk-minimization problem (7) reduces to solving one-dimensional L^1-minimization problems of the form (22) for which a solver is very easily implemented.

Finally, in the framework of the binomial model for the stock price, the quadratic local risk-minimization problem (2) becomes: starting from $V_M = H$, for all the states j at time t_k, $k = M-1, \ldots, 0$, minimize

$$\min\ E((C_{k+1} - C_k)^2\ |X_k = X_k^j). \tag{23}$$

The explicit hedging strategy solving this problem is given by:

- For each $1 \leq j \leq n_M$ define $\xi_M^j = 0,\ \eta_M^j = H_j$.
- For each $k = M-1, ..., 0$ and for each $1 \leq j \leq n_k$ define

$$\begin{cases} \xi_k = \frac{\mathrm{Cov}(\xi_{k+1}X_{k+1}+\eta_{k+1}, X_{k+1}|X_k=X_k^j)}{\mathrm{Var}(X_{k+1}|X_k=X_k^j)} \\ \eta_k = E((\xi_{k+1}-\xi_k)X_{k+1}+\eta_{k+1}|X_k = X_k^j), \end{cases} \tag{24}$$

where

$$\mathrm{Cov}(\xi_{k+1}X_{k+1}+\eta_{k+1}, X_{k+1}|X_k = X_k^j) = \sum_{l=0}^{\alpha_k-1} p_l(\xi_{k+1}^{j+l}X_{k+1}^{j+l} + \eta_{k+1}^{j+l})X_{k+1}^{j+l} - (\sum_{l=0}^{\alpha_k-1} p_l(\xi_{k+1}^{j+l}X_{k+1}^{j+l} + \eta_{k+1}^{j+l}))(\sum_{l=0}^{\alpha_k-1} p_l X_{k+1}^{j+l}),$$

$$\mathrm{Var}(X_{k+1}|X_k = X_k^j) = \sum_{l=0}^{\alpha_k-1} p_l(X_{k+1}^{j+l})^2 - (\sum_{l=0}^{\alpha_k-1} p_l X_{k+1}^{j+l})^2$$

and

$$E((\xi_{k+1}-\xi_k)X_{k+1}+\eta_{k+1}|X_k = X_k^j) = \sum_{l=0}^{\alpha_k-1} p_l(\xi_{k+1}^{j+l}X_{k+1}^{j+l} + \eta_{k+1}^{j+l} - \xi_k^j X_{k+1}^{j+l}).$$

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