Magnetoconvection Dynamics in a Stratified Layer.
I. 2D Simulations and Visualization

Steven R. Lantz and R. N. Sudan
Laboratory of Plasma Studies
Cornell University
Ithaca, NY 14853

Submitted to: *The Astrophysical Journal*
April, 1993
(Revised: October, 1993)

ABSTRACT

To gain insight into the problem of fluid convection below the solar photosphere, time-dependent magnetohydrodynamic convection is studied by numerical simulation of the magneto-anelastic equations, a model appropriate for low Mach numbers. Numerical solutions to the equations are generated on a two-dimensional Cartesian mesh by a finite-difference, predictor-corrector algorithm. The thermodynamic properties of the fluid are held constant at the rigid, stress-free top and bottom boundaries of the computational box, while lateral boundaries are treated as periodic. In most runs the background polytropic fluid configuration is held fixed at Rayleigh number R=5.44 times the critical value, Prandtl number P=1.8, and aspect ratio a=1, while the magnetic parameters are allowed to vary. The resulting dynamical behavior is shown to be strongly influenced by a horizontal magnetic field which is imposed at the bottom boundary. As the field strength increases from zero, an initially unsteady "single-roll" state, featuring complex time dependence, is replaced by a steady "traveling-wave" tilted state; then, an oscillatory or "sloshing" state; then, a steady two-roll state with no tilting; and finally, a stationary state. Because the magnetic field is matched onto a potential field at the top boundary, it can penetrate into the nonconducting region above. By varying the magnetic diffusivity, the concentrations of weak magnetic fields at the top of these flows can be shown to be explainable in terms of an advection-diffusion balance.

1. INTRODUCTION

Plasma convection in the presence of magnetic fields is a process of great importance to stellar astrophysics. Not only is convection the dominant mode of heat transport in convectively unstable layers, it is believed to be the energy source powering nearly all observed types of solar and stellar activity, including sunspots, the solar cycle, acoustic oscillations, and solar and stellar coronae and winds (*e.g.*, Foukal 1990; Moffatt 1978). However, theoretical progress in problems of plasma convection has long been hindered by the mathematical intractibility of the governing magnetohydrodynamic (MHD) equations—a situation which has been alleviated only relatively recently, by the availability of supercomputers that are powerful enough to construct approximate numerical solutions to these nonlinear partial differential equations (Weiss 1990). While any numerical approach is necessarily restricted by the small size of practicable computational grids, and correspondingly low values of relevant dimensionless numbers such as the Lundquist number, one can still hope that the lessons learned from simple, well-controlled numerical "experiments" might shed some light on more complex stellar counterparts.

Pioneering numerical studies of stellar magnetoconvection were done by Glatzmaier (1984, 1985a, b), in which he attempted to simulate global patterns of flow and magnetism in the solar interior. Given the severe limitations on the dynamic range of his model as compared to the real Sun, its successes were remarkable, particularly inasmuch as it displayed recognizable periods of dynamo activity. However, the "solar cycle" implied by the model did differ substantially from the real one. The possibility that his model's fidelity could be enhanced simply by increasing the resolution of the computer code must remain an open question, because even modest increases in resolution tend to require prohibitive increases in computer time. In any case, it is clear that in a global model of this type, there will always exist unresolved space and time scales; but, it is just as apparent that the *effects* of the latter must still be correctly represented, in order to get appropriate behavior from the former. This makes it worthwhile to pursue other, simpler models of MHD convection, especially, models at the level of cellular coherent structures, like the granules and supergranules that are observed on the Sun. These structures are, at least in principle, more accessible to the technique of large eddy simulation (LES), in which the unresolved scales are considered to be sufficiently featureless that they can be parametrized. In practice, such small-scale motions generally enter into one's model merely as an enhanced, or "artificial", diffusion. A more satisfactory and rigorous representation of these yet-unobserved smaller scales will depend on the outcome of theories and simulations of fully-developed MHD turbulence which are beyond the scope of this paper. Recent attempts to employ renormalization group techniques for subgrid modeling—Yakhot & Orszag (1986) for hydrodynamics, Fournier *et al.* (1982) for MHD, and Longcope & Sudan (1991) for low-β

reduced MHD—are only partially satisfactory.

Turning to the intermediate, cellular scale, one would wish in particular to examine how convection cells lend spatial and temporal structure to the magnetic fields they interact with, and vice versa. To this end, a number of studies have been made of plasma convection in an initially uniform magnetic field. The linear stability theory of this subject was thoroughly and elegantly treated by Chandrasekhar (1961), who showed (among many other things) that, due to the stabilizing nature of the Lorentz forces, the onset of convection can take the form of overstable oscillations. The subsequent, nonlinear development of convection can lead to "flux expulsion," a phenomenon in which weak magnetic fields are discharged from the center of a convection cell, in a time on the order of an eddy turnover period (see Weiss 1966; Moffatt 1978). This anticipated behavior was later confirmed in 2D simulations of a Boussinesq fluid (Weiss 1981; Galloway & Weiss 1981). Numerical investigations of 2D, fully compressible "magnetoconvection," in which magnetic and velocity fields are on a more equal dynamical footing, have uncovered a rich variety of time-dependent behavior, including both traveling-wave and standing-wave solutions (Hurlburt & Toomre 1988; Hurlburt *et al.* 1990; Weiss *et al.* 1990; Fox *et al.* 1991). These are characterized by milder undulation—with little expulsion—of the imposed magnetic field. Simulations like these can be analyzed from the standpoint of bifurcation theory, wherein the full model is approximated by a highly truncated Galerkin model. This has been used, for instance, to explain a transition to asymmetry that can sometimes occur (Nagata *et al.* 1990). In addition to the above results which pertain to magnetic fields that are initially vertical, analogous standing and traveling waves have been observed and theoretically identified in a uniform horizontal magnetic field, for 2D fluids in the Boussinesq (Arter 1983; Knobloch 1986) and compressible (Weiss 1991) cases, as well as for 2D compressible MHD convection in an oblique geometry (Matthews *et al.* 1992).

But numerical observations of convection in a *uniform* magnetic field are most likely pertinent only to the interiors of sunspots or active regions (or, perhaps, laboratory experiments). It is of real interest to examine a situation where magnetic fields are nonuniform and highly localized, as is more typical of the Sun—for instance, in the spatially intermittent magnetic network of the quiet Sun, which is associated with the boundaries of supergranule convection cells. Comparatively few simulations have been done of cellular convection possessing intermittent but dynamically significant magnetic fields, because of the larger range of spatial scales that must be resolved. Stein, Brandenburg, & Nordlund (1992) have made such an attempt, in 3D—and they did meet with some success in reproducing various features of the solar granulation, such as the clustering of quite narrow flux tubes into the downdraft regions with an inhibition of the velocity fluctuations, plus a reduction in the size of the convection cells as seen at the surface. But it is mainly evidence from observations, rather than simulations, that provides the impetus for the work to be described in subsequent sections of this paper; and so, it will be apropos next to review some pertinent observations of magnetic fields on the Sun.

Recent data on the solar magnetic network and its relation to the supergranules have been provided by the videomagnetogram (VMG) system at Big Bear Solar Observatory (BBSO) (Zirin 1988). It has been known for some time that on the quiet Sun, vertical magnetic fields emerge from the photosphere in isolated and highly concentrated “magnetic elements,” and that these regions of high magnetic flux congregate preferentially at the downflowing supergranule boundaries, as shown by forming a cross-correlation of a VMG with a Dopplergram of the horizontal velocity field (Wang & Zirin 1988; Wang 1988b). There is reason to believe on theoretical grounds that magnetic fields in the photosphere are even more concentrated than one is able to see: the typical quiet-Sun mean field strength of 5-10 gauss may be locally enhanced to as much as 1 or 2 kilogauss. (Magnetic elements need to be able to keep intact against the jostling of the surrounding fluid, so their internal magnetic pressure $B^2/8\pi$ should be on a par with local thermodynamic pressure; simply balancing these two pressures implies kilogauss-strength magnetic fields in the photosphere.) This prediction is further reinforced by calculations indicating that weaker vertical flux tubes will quickly undergo a collapse to a concentrated state due to a convective-type instability (Parker 1978; Spruit 1979; Spruit & Zweibel 1979). By the same token, though, magnetic fields of such strength would act to inhibit "normal" convection, implying that the filling factor of strong elements must be quite small. Thus, the argument goes, what one is really seeing in the BBSO data are concentrated "magnetic pores" whose great intrinsic strength is being washed out observationally by the tiny filling factor over the minimum resolved area (the latter is set by seeing limitations due to turbulent fluctuations in the Earth's atmosphere). There is little reason to disbelieve this logic: in fact, with the recent application of speckle interferometric techniques to the observations, there is now some confirmation of the existence of magnetic pores (Keller 1992). But despite the limitations, data like the BBSO data remain useful for present purposes, because their resolution turns out to be comparable to the resolution of the simulation code to be presented in the sections to follow—wherein smaller-scale structures are similarly neglected.

Another striking feature of the BBSO VMG data is that it shows the magnetic network to be a dynamic entity: a population of sorts, in which individual magnetic elements are born and die, even as they contribute to the formation of the greater pattern. About 1% of the magnetic elements are replaced every hour (Wang 1988a). How these elements might be connected to one another above and below the surface is unknown. The magnetic network is built up from a mixture of magnetic elements of both polarities, which are present in nearly equal numbers, and each one of these unipolar elements seems to execute a random walk of its own around—and sometimes through—

the shifting web of supergranule boundaries. Undoubtedly, the underlying magnetic topology is very complicated, but it suffices to state here that the network is a far-from-static phenomenon—it is a consequence of magnetoconvection dynamics.

It happens that it is not difficult to induce complicated dynamical behavior in nonlinear MHD simulations, even in fairly simple cases such as the 2D Boussinesq magnetoconvection work cited above. And, even without magnetic fields, Ginet and Sudan have demonstrated that dynamical chaos can and does occur in 2D simulations of anelastic convection (Ginet 1987; Ginet & Sudan 1987). This paper extends the previous work of Ginet and Sudan to include fully nonlinear MHD interactions. Magnetic flux is resupplied at the bottom boundary in order to replenish diffusive losses; such losses will occur due to the condition imposed at the top boundary, where the field is required to match onto a decaying, potential-field solution in the vacuum region above. The boundary conditions in particular distinguish the present work from that of others (Arter 1983, Weiss 1991). The goals of this study will be: first, to see the effect of the magnetic fields on fluid motions; and then, conversely, to observe the effects of the motion on resultant magnetic structures, particularly at the top boundary. In a second paper directly following this one (Lantz 1993, hereafter referred to as Paper II), it is demonstrated that a "low-order" or highly truncated Galerkin model works very well to explain qualitatively what is seen in these magneto-anelastic simulations. The reader should note that both this paper and Paper II are accompanied by videotape segments to aid in the visualization of the dynamics.

In Section 2, the mathematical model underlying the simulations is presented, with attention to its physical basis and to the selected parametrization and boundary conditions. An overview of the results of 2D supercomputer simulations is given in Section 3, emphasizing the novel phenomena that were encountered and their "photospheric" magnetic signatures. Concluding remarks are offered in Section 4.

2. MAGNETO-ANELASTIC MODEL EQUATIONS

A useful mathematical model for many astrophysical situations is the "anelastic approximation" of Gough (1969) which was proposed earlier by Ogura and collaborators (Ogura & Charney 1960; Ogura & Phillips 1962). A recent and very clear derivation of these equations has been provided by Gilman & Glatzmaier (1981). Magnetic fields were introduced into the approximation by Ginet & Sudan (1982) and by Glatzmaier (1984). However, the anelastic approximation really has much deeper roots, going back to the standard one-dimensional model of astrophysical convection known as "mixing length theory" (MLT). MLT anticipates one of the fundamental assumptions of the anelastic approximation: it says that a stellar convection zone will tend to a state so close to adiabatic and hydrostatic equilibrium that it should be possible to regard convective motion as a perturbation superimposed upon a non-convecting "reference state" (Schwarzschild 1958; see also Clayton 1968). Accordingly, MLT yields the proper set of scalings to utilize in a perturbation expansion about a zero-order (O(1)) static equilibrium, because it describes scaling relationships among the first-order ($O(\varepsilon)$) variables for motions driven by buoyancy forces.

2.1 *Relation to Mixing Length Theory*

The small parameter ε of the anelastic approximation, which indicates the scale of perturbations relative to the background, has a direct relation to a superadiabatic gradient determined by mixing length theory. The MLT superadiabatic parameter, denoted $\Delta\nabla$, is a nondimensional number defined by

$$\Delta\nabla \equiv \left| \frac{d\ln T}{d\ln p} - \frac{\delta_p r_*}{c_p} \right| \tag{1}$$

wherein T is temperature, p is pressure, $\delta_p \equiv -(\partial \ln\rho \,/\, \partial \ln T\,)_p$ is the dimensionless coefficient of thermal expansion (equal to 1 for an ideal gas), $r_* \equiv R_* \mu_*^{-1}$ is the universal gas constant divided by the average mass per particle, and c_p is the specific heat at constant pressure. It is a principal finding of MLT that $\Delta\nabla$ is a very small quantity for deep convection in stars—for example, it is 10^{-5} or so at the presumed base of the supergranular cells in the Sun. This number is determined by equating the known surface solar energy flux, $L_\odot/(4\pi R_\odot^2)$, to a mixing-length estimate of the convective heat flux F_{conv} due to a buoyancy-driven vertical velocity v_z in a (constant) gravitational field g,

$$L_\odot/(4\pi R_\odot^2) = F_{conv} \approx \rho_0 T_0 c_p v_z \cdot(\Delta\nabla) \ , \tag{2}$$

$$where \quad v_z \approx [\delta_p g H\cdot(\Delta\nabla)]^{1/2} \ , \tag{3}$$

then solving for the order of magnitude of $\Delta\nabla$. Here, the mixing length has been set equal to a pressure scale height H; estimates of the other quantities—such as the mass density ρ_0—are based on an adiabatic and hydrostatic background stratification, which can then be justified *a posteriori*, once $\Delta\nabla$ is proved to be small.

It is precisely the smallness of this $\Delta\nabla$ parameter that is exploited in the anelastic approximation. One begins by assuming the temperature (for instance) can be expressed as

$$T(\mathbf{x},t) = T_0(z) + \varepsilon T_1(\mathbf{x},t) + \varepsilon^2 T_2(\mathbf{x},t) + \ldots \tag{4}$$

where the numbered subscript on an expansion variable on the right hand side refers to its order in ε. Notice that the lowest order depends on height z only. First-order fluctuations thus scale as $\varepsilon \sim T_1/T_0$. But equation (2) suggests that $\varepsilon \sim \Delta\nabla$, making $\Delta\nabla$ the appropriate measure of the relative smallness of the fluctuations. It is easy to show that $(\Delta\nabla)^{1/2}$ is also related to the Mach number, as follows. The speed of sound c_s at any given depth is

$$c_s \equiv \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2} \approx \left(\frac{d p_0/dz}{d\rho_0/dz}\right)^{1/2} = \left(\frac{-\rho_0 g}{d\rho_0/dz}\right)^{1/2} \sim (gH)^{1/2} , \tag{5}$$

where the same assumptions that led to equations (2) and (3) have been exploited—the adiabatic assumption permitted equating derivatives at constant specific entropy s to derivatives in z, and hydrostatic balance was also invoked. Then, by equation (3), one verifies that the Mach number $\mathrm{M} \equiv v_z/c_s \sim (\Delta\nabla)^{1/2}$. This is an *overall* scaling for components of the velocity $\mathbf{v}$, making it clear that the flow must be occurring at *low* Mach number, and this is what enables one consistently to filter out the sound waves in the anelastic approximation, via an assumed ordering of $\partial/\partial t \sim \mathbf{v}\cdot\boldsymbol{\nabla} \sim \mathrm{O}(\varepsilon^{1/2})$. An analogous reasoning process is implicit in the more familiar incompressible hydrodynamic equations

(Zank & Matthaeus 1991). Again, in any purported solution to the anelastic equations, this low-Mach-number assumption must be borne out *a posteriori*, or else the results are invalid physically.

Magnetic fields may be added to the model in a straightforward way provided they are not too strong (Ginet & Sudan 1982; Glatzmaier 1984). To avoid upsetting the original O(1) hydrostatic equilibrium, it is clear that the magnetic field must scale with velocity, as $O(\varepsilon^{1/2})$—or to put it another way, the Alfvén-speed-to-sound-speed ratio v_A/c_s must be small. This translates to a condition on the plasma beta parameter:

$$\beta \equiv \frac{8\pi p}{B^2} = \frac{4\pi\rho}{B^2} \cdot \frac{2p}{\rho} \sim \frac{c_s^2}{v_A^2} \sim \varepsilon^{-1} \quad . \tag{6}$$

But this high-β condition does not relegate the magnetic field **B** to a merely passive role—convection is after all driven by the fluctuations, whereas β pertains to the equilibrium, which here serves as nothing more than a backdrop to the dynamics. In fact, the magnetic pressure $B^2/8\pi$ is generally of the same order as the fluctuating fluid pressure p_1; and it is therefore significant dynamically, as will be seen later.

One must next decide how to treat the dissipation terms in the proposed magneto-anelastic perturbation expansion. Inside stars like the Sun, all *physical* diffusion coefficients, whether of viscosity, conductivity, or resistivity, are extremely small, in the sense that their associated Reynolds or Péclet numbers are huge (primarily because the convective structures of interest are enormous), indicating that the nonlinear process of advection dominates over any linear processes of dissipation. Such a high degree of fluid turbulence poses problems for direct numerical simulation, because limitations on computer memory and processor speed greatly restrict the dynamic range of wavenumbers and frequencies that can be resolved. The only feasible option is to assume that dissipative terms are standing in for subgrid-level turbulence. In that case, it is reasonable to scale an arbitrary diffusion coefficient, say $K \sim v_z H \sim O(\varepsilon^{1/2})$, where K is an eddy diffusivity based on the large-scale velocity. This puts advection and diffusion back on a more equal, and more manageable, footing—although this comes at the expense of the usual laboratory-based interpretation of the physics. Particular care must be taken in the interpretation of the conduction term $\nabla\cdot(\kappa\nabla T)$ in the equation of heat transfer; the subgrid model for this term is really $\nabla\cdot(\rho_0 T_0 K \nabla s) \approx \nabla\cdot(\kappa\nabla T_1)$ where $\kappa \equiv \rho_0 c_p K$ is the "turbulent eddy" thermal conductivity, representing random mixing of entropy at the small scales (Lantz 1992). Thus, κ stands for that portion of the *convective* heat transfer that is not being resolved numerically: it is not at all the thermal conductivity in the usual sense. So it is clear that here, a calculation to find the "onset of convection" will signify not an initial instability to bulk motion as it overcomes the dominance of pure conduction, but rather the initiation of motion at the largest, numerically resolved scales, over and above a milieu of unresolved, small-scale turbulence.

2.2 *The Magneto-Anelastic Equations in Two Dimensions*

In the anelastic approximation, the perturbation expansion one must perform is necessarily more subtle than simply writing out all the variables in powers of ε, because, as the mixing-length analysis indicates, the velocity must scale to lowest order as $\varepsilon^{1/2}$. To cope with the fact that different variables can have different *relative* scalings, a two-step approach—which might best be dubbed a "scaled-variable expansion"—is generally implemented. The complete derivation of the magneto-anelastic equations will not be reproduced here, since it has been adequately treated in other sources (Gilman & Glatzmaier 1981; Ginet 1987; Lantz 1992). With their dimensional units restored, the equations that form the starting point of this study are as follows:

Momentum Equation:

$$\mathrm{O}(1), \qquad \frac{dp_0}{dz} = -g\rho_0 \quad (\textit{hydrostatic balance}) \ , \tag{7}$$

$$\mathrm{O}(\varepsilon), \qquad \rho_0\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p_1 - g\rho_1\hat{\mathbf{z}} + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} + \nabla\cdot\boldsymbol{\sigma}' \tag{8}$$

$$\textit{with viscous stress tensor}, \qquad \sigma'_{ij} \equiv \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}(\nabla\cdot\mathbf{v})\delta_{ij}\right) \ ; \tag{9}$$

Entropy Equation:

$$\mathrm{O}(\varepsilon^{1/2}), \qquad s_0 = \textit{const.} \quad (\textit{adiabatic stratification}) \ , \tag{10}$$

$$\mathrm{O}(\varepsilon^{3/2}),\quad \rho_0 T_0 \left(\frac{\partial s_1}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla}) s_1 \right) \;=\; \boldsymbol{\nabla}\cdot(\kappa \boldsymbol{\nabla} T_1) + \frac{\eta}{4\pi} \left| \boldsymbol{\nabla}\times\mathbf{B} \right|^2 + (\boldsymbol{\sigma}'\cdot\boldsymbol{\nabla})\cdot\mathbf{v}\;; \quad (11)$$

Continuity Equation:

$$\mathrm{O}(\varepsilon^{1/2}),\qquad \boldsymbol{\nabla}\cdot(\rho_0 \mathbf{v}) \;=\; 0\;; \qquad (12)$$

MHD Ohm's Law:

$$\mathrm{O}(\varepsilon),\qquad \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) \;=\; \boldsymbol{\nabla}\times(\eta \boldsymbol{\nabla}\times\mathbf{B})\;, \qquad (13)$$

$$\textit{with initial condition,}\qquad \boldsymbol{\nabla}\cdot\mathbf{B} \;=\; 0\;; \qquad (14)$$

Equation of State:

$$\mathrm{O}(1),\qquad p_0 \;=\; r_* \rho_0 T_0\;, \qquad (15)$$

$$\mathrm{O}(\varepsilon),\qquad \frac{\rho_1}{\rho_0} \;=\; \alpha_T \frac{p_1}{p_0} - \delta_p \frac{T_1}{T_0}\;; \qquad (16)$$

First Law of Thermodynamics:

$$\mathrm{O}(1),\qquad T_0 ds_0 \;=\; 0 \;=\; c_p\, dT_0 - \delta_p \frac{dp_0}{\rho_0} \qquad \Rightarrow \qquad \frac{dT_0}{dz} \;=\; -\frac{\delta_p g}{c_p}\;, \qquad (17)$$

$$\mathrm{O}(\varepsilon),\qquad T_0 s_1 \;=\; c_p T_1 - \delta_p \frac{p_1}{\rho_0} \qquad \Rightarrow \qquad s_1 \;=\; c_p \frac{T_1}{T_0} - r_* \delta_p \frac{p_1}{p_0}\;, \qquad (18)$$

where the generalized thermodynamic coefficients are $c_p = c_p(p_0, T_0)$ and $r_* = R_* \cdot \mu_*^{-1}(p_0, T_0)$; and the nondimensional coefficients of isothermal compressibility and isobaric expansion are defined to be

$$\alpha_T \equiv \left(\frac{\partial \ln\rho}{\partial \ln p}\right)_{T,0} = 1 + \left(\frac{\partial \ln\mu_*}{\partial \ln p}\right)_{T,0}, \quad \delta_p \equiv -\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_{p,0} = 1 - \left(\frac{\partial \ln\mu_*}{\partial \ln T}\right)_{p,0}. \tag{19}$$

Note that $\alpha_T = \delta_p = 1$, and $c_p = 5r_*/2 = const.$, for a 3D ideal gas, which is what will be assumed throughout this paper. Again, the lowest-order variables depend on height z only.

Use of the magneto-anelastic approximation turns out to confer a number of advantages computationally. First, sound waves have been eliminated by virtue of the weakening of full compressibility implicit in equation (12); this permits one to take larger time steps. Moreover, the degree of nonlinearity in the hydrodynamics has been reduced, as $\rho_0(z)$ becomes merely a prescribed function. Thus, after one applies hydrostatic balance (eqn. (7)) and the adiabatic condition (eqn. (17)), the z-dependence of all zero-order variables is specified once the values of any two of them have been specified at a reference level z_r. This reduction in the number of variables that must be time-advanced goes still further if one restricts the equations to two dimensions. In planar geometry, the divergence-free condition on the momentum, equation (12) again, allows one to define a momentum stream function $\varphi(x,z)$: $\rho_0(z)\mathbf{v}(x,z) = \hat{\mathbf{y}}\times\nabla\varphi(x,z)$. For magnetic fields lying in the (x,z) plane, one can similarly use $\nabla\cdot\mathbf{B}=0$ to define an analogous magnetic flux function $\psi(x,z)$: $\mathbf{B}(x,z) = \hat{\mathbf{y}}\times\nabla\psi(x,z)$. A closed set of 2D equations is then found by taking the curl of equation (8), to obtain a rule for the time advance of

$\varpi \equiv \nabla^2\varphi$, and also the divergence of equation (8), to obtain an elliptic equation for p_1 which must hold at each instant of time t. Finally, one additional elliptic equation must be solved at each time t in order to extract φ from ϖ. (There is no analogous equation involving $\nabla^2\psi$ because equation (13) permits one to time-advance ψ directly.) In two dimensions, then, the magneto-anelastic equations are reduced to time evolution equations for the three quantities ϖ, ψ, and s_1, plus elliptic equations that hold at each time step for the quantities φ and p_1.

Recorded below are the 2D equations upon which the simulation results of this paper are based. The dynamic viscosity μ, the thermal diffusivity κ, and the magnetic diffusivity η have all been taken as constants. (Notice that these three quantities differ in their physical units; only η has the dimensions of a true diffusion coefficient.) The equations are otherwise perfectly general, and in particular, they are not dependent on the form of any equation of state.

Momentum Vorticity Equation:

$$\frac{\partial\varpi}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\varpi}{\rho_0}\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\varpi}{\rho_0}\frac{\partial\varphi}{\partial x}\right) =$$

$$\frac{\partial}{\partial x}\left[\frac{1}{2\rho_0 h}\left(\frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial z}\right)\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\frac{1}{\rho_0 h}\left(\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial z}\right)\right] + g\,\frac{\partial\rho_1}{\partial x} +$$

$$\frac{1}{4\pi}\left[\frac{\partial}{\partial x}\left(\nabla_\perp^2\psi\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\nabla_\perp^2\psi\frac{\partial\psi}{\partial x}\right)\right] + \mu\,\nabla_\perp^2\left(\frac{\varpi}{\rho_0} + \frac{1}{\rho_0 h}\frac{\partial\varphi}{\partial z}\right), \qquad (20)$$

$$\textit{where} \qquad \varpi \equiv \hat{\mathbf{y}}\cdot\nabla\times(\rho_0\mathbf{v}) \ , \quad \rho_0\mathbf{v} \equiv \hat{\mathbf{y}}\times\nabla\varphi \ , \tag{21}$$

$$\textit{implying} \qquad \varpi = \nabla_\perp^2\varphi \ , \quad \textit{with} \quad \nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \ ; \tag{22}$$

$$\textit{and} \qquad \frac{1}{h} \equiv -\frac{1}{\rho_0}\frac{d\rho_0}{dz} = \left(\frac{\alpha_T}{r_*} - \frac{\delta_p^2}{c_p}\right)\frac{g}{T_0} \ . \tag{23}$$

Entropy Equation:

$$T_0\left[\rho_0\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}\left(s\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(s\frac{\partial\varphi}{\partial x}\right)\right] = \kappa\nabla_\perp^2 T_1 + \eta\left|\nabla_\perp^2\psi\right|^2 +$$

$$\frac{\mu}{\rho_0^2}\left[\left(-\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} + \frac{1}{h}\frac{\partial\varphi}{\partial z}\right)^2 + \left(2\frac{\partial^2\varphi}{\partial x\partial z} + \frac{1}{h}\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{3}\left(\frac{1}{h}\frac{\partial\varphi}{\partial x}\right)^2\right] \ . \tag{24}$$

Pressure Equation:

$$\nabla_\perp^2 p_1 + \frac{\partial}{\partial z}\left(\frac{p_1}{h}\right) = \frac{2}{\rho_0}\left[\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial z^2} - \left(\frac{\partial^2\varphi}{\partial x\partial z}\right)^2\right] + \frac{2}{\rho_0 h}\left[\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial z}\right] -$$

$$\frac{\partial^2\varphi}{\partial x^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho_0 h}\right) + \frac{\partial}{\partial z}\left(\frac{\delta_p g}{c_p}s_1\right) + \frac{1}{4\pi}\nabla\cdot(\nabla_\perp^2\psi\nabla\psi) + \frac{4\mu}{3}\nabla_\perp^2\left(\frac{1}{\rho_0 h}\frac{\partial\varphi}{\partial x}\right) \ . \tag{25}$$

Flux Function Equation:

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\psi}{\rho_0}\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\psi}{\rho_0}\frac{\partial\varphi}{\partial x}\right) = -\frac{1}{\rho_0 h}\psi\frac{\partial\varphi}{\partial x} + \eta\nabla_\perp^2\psi \ , \tag{26}$$

$$\textit{with} \qquad \mathbf{B} \equiv \hat{\mathbf{y}}\times\nabla\psi \ . \tag{27}$$

To the extent possible, the above equations have been written in "conservative form," so that most terms appear as a derivative of one of the space variables; this is a desirable property for the finite-difference (or more precisely, finite-volume) discretization scheme that was employed.

In most finite-difference methods to solve a given set of partial differential equations, it is the widely-known Courant-Friedrichs-Lewy condition that restricts the maximum allowable time step. By using the magneto-anelastic approximation, this condition is reduced in its severity: the upper limit that would normally apply—in the case of the full MHD equations in the high-plasma-β limit considered here, it would be the sound transit time across a grid spacing—is increased by several orders of magnitude (by approximately a factor of $M^{-1}\cdot Re^{1/2}$ where Re is the Reynolds number), to be the viscous diffusion time across this smallest resolved distance. Again, this is perhaps the most significant computational advantage of the magneto-anelastic model: one does not have to resolve acoustic waves, which are presumably unimportant in the low-Mach-number energetics of the system. Recent simulations have given some evidence that acoustic modes are capable of transporting a significant heat flux, depending on how the boundary conditions are imposed (Hossain & Mullan 1991, 1993); however, this is still a matter of some controversy.

2.3 *Definition of Nondimensional Parameters*

The task of mapping out the behavior of the magneto-anelastic equations is difficult, because the parameter space of the magneto-anelastic equations is defined by eight (or more) coordinates. For the problem at hand, the physical, dimensional constants that are required to specify the system—the "cgs" parameters—can be assigned to four different categories and enumerated as follows: *reference* parameters, $\{\rho_r, T_r, (\nabla s)_r, B_r\}$; *physics* parameters, $\{g, r_*, c_p\}$; *diffusion* parameters, $\{\kappa, \mu, \eta\}$; and *geometry* parameters, the depth and width of the domain, $\{d, L\}$. (Two dimensionless constants, δ_p and α_T, are ignored because they are naturally equal to unity anyway, for the ideal gas which will be assumed.) Four out of the twelve parameters are needed to define the physical mass, length, time, and temperature scales, leaving a total of eight free parameters. These are usually rearranged into dimensionless combinations. There is of course no unique way to write down such combinations; in fact, one can choose among an infinite number of alternatives, depending on how one happens to group together the various factors. Table 1, which constitutes the main parameter set for this paper, is just one of the possibilities.

It is important to realize that in the magneto-anelastic equations, *all* of the "cgs parameters" listed above are really depth-dependent functions, at least in principle. But typical dimensionless combinations of these quantities, like the Rayleigh number, are simply scalars in the usual definitions. To define these scalar dimensionless parameters uniquely, one must specify some level z_r at which the cgs parameters are to be evaluated. The values of the temperature, density, and so on at this level will additionally serve to normalize their corresponding functions of depth. In this paper, the assumption of an ideal gas, plus the choice of certain dissipation coefficients and the gravitational acceleration to be constants, means that only a few cgs parameters—those categorized as "reference" parameters above—need to be specified at the level z_r. Such depth-indexed quantities are marked, both above and in Table 1, with an "r" subscript. Of course, in order for the whole problem to be well-defined, the full depth dependence of the reference quantities must be determined, and this task is covered in Section 2.4.

Regardless of how one chooses to label the eight axes of parameter space, it is clear that to explore more than a tiny subspace of this multi-dimensional volume via large-scale simulation would be a practical impossibility. Commonly, one limits one's sampling by fixing most of the parameters and varying only one or two of them. Since the main concern here is for magnetic effects, a number of the purely fluid parameters—namely, ε, a, G, S, and the product $R \cdot P$—will never be altered. The quantities to be varied are the ratio R/P (the Grashof number, corresponding to the dynamic viscosity μ), the magnetic Prandtl number M (corresponding to the magnetic diffusivity η), and the Chandrasekhar number Q (corresponding to the horizontal magnetic field at the bottom boundary, B_{x0}). A perfectly acceptable alternative is to express μ, η, and B_{x0} in their natural units. Generally the nondimensional parametrization is the one that is preferred, but physical units will frequently be cited to aid in interpreting the results in the context of solar physics. In addition, there are two alternative dimensionless parameters that will prove to be especially useful. The Reynolds number $\mathrm{Re} \equiv v_{rms} d(\mu/\rho_r)^{-1}$ is a measure of fluid turbulence, while the magnetic Reynolds number, $\mathrm{Rm} \equiv v_{rms} d\eta^{-1}$ is a similar measure of magnetic turbulence. Because these numbers depend on v_{rms}, the root mean square of the fluid velocity, which varies during the course of a run, these particular dimensionless numbers are not the optimal nondimensional forms for μ and η. But they are helpful in characterizing the *type* of fluid flow one expects to see, in the video, *e.g.*

For the runs covered in this paper, many of the parameters will be kept fixed. Their values at the midplane, in cgs units, are as follows: ρ_r=3.01×10^{-6}; T_r=3.69×10^{4}; $(\nabla s)_r$=2.99×10^{-3}; g=2.74×10^{4}; r_*=8.248×10^{7} (atomic hydrogen); c_p=2.062×10^{8}; κ=3.848×10^{14}; d=L=4.297×10^{8}. The key to translating from the remaining physical (μ, η, B_r) to nondimensional (P or R, M, Q) units is then: P=5.37×10$^{-7}\mu$, implying R=6.56×10$^{10}\mu^{-1}$ at the midplane, because R=3.52×10$^{4}P^{-1}$ there (see Sec. 3); M=1.61×10$^{-12}\eta$, also at the midplane; and Q=1.47×10$^{16}B_{x0}^2\mu^{-1}\eta^{-1}$, bottom. Notice that all the reference values are taken at the midplane, except B_r=B_{x0} in the definition of Q, where B_{x0} is the horizontal magnetic field imposed at the bottom. The form of a parameter deemed most appropriate to understanding the physics of a particular run will be the one stated.

For further details on the nondimensionalization and parametrization of the magneto-anelastic equations, see Section 3 and also Lantz (1992).

2.4 *Background Stratification and Boundary Conditions*

To specify completely the problem at hand, the scalar parameters presented in Table 1, together with equations (20) through (27), must now be augmented with explicit functions for the depth dependence of the lowest-order thermodynamic functions, plus all requisite boundary conditions. The first of these situations can easily be rectified. Since the background quantities $\rho_0(z)$, $T_0(z)$, etc. are assumed to be related by an ideal gas law with $r_* = const.$ and $c_p = 5r_*/2$, one can prove that the adiabatic condition, as it is given in equation (17), will require the zero-order stratification to obey a polytropic law: $(p_0/p_r) = (\rho_0/\rho_r)^\gamma$, where γ is equal to 5/3. This can immediately be put together with hydrostatic balance (eqn. (7)) to arrive at

$$\frac{p_0}{p_r} = \left(\frac{z_t - z}{z_t - z_r}\right)^{m+1}, \quad \frac{\rho_0}{\rho_r} = \left(\frac{z_t - z}{z_t - z_r}\right)^{m}, \quad with \quad m = \frac{1}{\gamma - 1} = \frac{3}{2} \tag{28}$$

where z_t marks the "top" of the fluid, at which level $p_0 = \rho_0 = 0$. Working back from equation (28), one can show that both T_0 and the temperature scale height are linear in $(z_t - z)$. Since the S parameter is the ratio of the zone depth to the temperature scale height, it is S that determines the location of z_t. Likewise, G determines the value of γ, and thus specifies the polytropic index m.

To drive this configuration unstable, an excess is imposed in the constant value of the temperature gradient, beyond its adiabatic value from equation (17). In effect, the polytropic index is shifted from m=1.5 to m=1.49; this small shift is reflected in the smallness of the expansion parameter ε as defined in Table 1. As the entropy gradient departs from zero, initial perturbations can grow, and their subsequent evolution is governed in part by their initial and boundary conditions. There are five different variables to be discussed in this context: ϖ, s_1, ψ, φ, and p_1. Initial data must be provided for the first three of these functions of (x, z, t), while the initial states of φ and p_1 can be computed from their instantaneous elliptic equations. Initial conditions in this paper were taken from a prescribed set of small, random fluctuations; the same set was used for all runs. But many runs were continuations of the final state of others, and more will be said on this in Section 3.1.

For the boundary conditions, it is appropriate to let simplicity guide many of the choices, especially in the absence of clear-cut solar models. Computationally, it is most convenient to have rigid walls situated at the top and bottom, where the temperature and pressure are fixed, and this common practice is also followed here. For all variables, a periodic condition is prescribed in the horizontal direction. Additionally, the velocity field is made to be horizontal-stress-free, at the top and bottom, which gives boundary conditions on ϖ in terms of φ; specifically, the condition that prevails at both top and bottom is

$$v_z = 0 \ , \quad \frac{\partial v_x}{\partial z} = 0 \quad \Rightarrow \quad \varphi = const. \ , \quad \varpi = -\frac{1}{\rho_0 h}\frac{\partial \varphi}{\partial z} \quad . \tag{29}$$

Taking the constant value of φ to be identical (equal to 0, *e.g.*) at both boundaries ensures that the frame of reference is not moving laterally.

The pressure boundary conditions are linked to global conservation of mass (Ginet 1987). The pressure integration is split into two parts, conceptually: (i) an integral equation for the horizontally-averaged pressure $\langle p_1 \rangle$, which comes from applying the mass-conserving and periodic conditions to a direct x-integration of equation (8); and (ii) an elliptic equation for the remaining, fluctuating pressure, $p_1' = p_1 - \langle p_1 \rangle$. This procedure has been recorded at length by Lantz (1992). The end result is that $\langle p_1 \rangle$ must be allowed to vary in a well-defined manner at the boundaries in order to conserve the total mass integral within the simulated region, while p_1' and T_1 are set to be zero at the boundaries. All this, when taken together, constitutes a boundary condition on s_1 through the various thermodynamic relations. While there appear to be "extra" derivatives in some equations—*i.e.*, beyond second order—no further conditions are needed in these cases, because the equations in question (ϖ and p_1') do not require time-stepping at the boundaries, where the possible higher-order conditions would enter.

The boundary conditions on ψ warrant special attention due to their importance. Making the top and bottom walls perfect electrical conductors would be the simplest choice, but this would disallow the transport of magnetic flux across the boundaries—a process which is necessary for flux emergence to occur, and which may be significant for the solar cycle (Golub *et al*. 1981). Since the nature of emerging flux is of interest to observers, boundary conditions were adopted at the top which permit field lines to penetrate out into a perfectly insulating, or vacuum, region. Requiring that this vacuum field die away at large distances constrains the allowed form of the flux function ψ that can appear at the top boundary. In particular, ψ must match smoothly onto a solution of the Laplace equation $\nabla_\perp^2 \psi = 0$. Simple continuity of **B** leads to a condition on $\{\hat{\psi}(k_x)\}$, which is the set of Fourier components of ψ (recall ψ is periodic in x):

$$\frac{\partial \hat{\psi}(k_x)}{\partial z} = -|k_x|\,\hat{\psi}(k_x) \ , \quad top \tag{30}$$

Numerically, fast Fourier transform techniques can be employed to enforce equation (30) at the top. This condition, unlike a perfectly conducting wall, permits field lines to "escape to infinity," leading to the ultimate decay of the initial magnetic field, if the bottom boundary is a perfect conductor ($B_z = -\partial\psi/\partial x = 0$, at bottom). Thus, magnetic flux must be continually replenished at the bottom boundary in some way—for example, by fixing the electrostatic field **E** (so that $E_y = \partial\psi/\partial t =$ *const.*) or by fixing the horizontal magnetic field ($B_x = \partial\psi/\partial z = const.$) there. Each of these techniques is examined in more detail in Section 3, where the full synopsis of numerical methods is given.

3. RESULTS OF SIMULATIONS

A computer program has been devised to calculate approximate numerical solutions to the 2D magneto-anelastic equations outlined in the previous section. This computer code falls into the technical class of a finite-difference predictor-corrector method for solving a set of partial differential equations; it is accurate to second order in both space and time. The method is a direct adaptation of the anelastic algorithm constructed by Ginet which is described in detail in his Ph.D. thesis (Ginet 1987). Conceptually, the magneto-anelastic equations are time-advanced on a rectangular, two-dimensional grid, on which space and time derivatives are approximated by their representative centered differences, for optimum accuracy and stability. (Note that centering in time is accomplished by means of the corrector steps, *i.e.*, it becomes possible to compute the time-centered differences once the predictor step has projected ahead the variables' values at each grid point.) In the case of the elliptic equations, a fast Fourier transform is first performed in the periodic, horizontal direction, resulting in a set of simultaneous finite-difference equations in z, which can be solved as tridiagonal matrix inversions. The precise finite-difference formulas that have been used are not recorded here, as they have already been documented in sufficient detail previously.

Good visual representation of the simulation data is extremely important for achieving an understanding of the dynamical behavior of the system in the parameter regime here under study. Therefore, in addition to studying the figures provided in this section, the reader is advised to view the accompanying videotape as a way of more fully appreciating the complex time dependence inherent in the dynamics. A guide to the video is provided at the end of this section.

3.1 *Initial Data and Magnetic Field Kinematics*

The previous results obtained by Ginet & Sudan (1987) showed a remarkable variety of flow patterns, including quasiperiodic and non-repeating behavior for selected parameter values. In order to explore further their results in this paper, many of the fluid parameters will be kept fixed at the same values that Ginet and Sudan adopted, as follows: $\varepsilon = 6.23\times10^{-3}$; $a = 1.0$; $G = 0.4$; $S = 7.0$; and the product $R{\cdot}P = 3.52\times10^{4}$. These are midplane values; at the top, $\varepsilon = 2.80\times10^{-2}$, while $R{\cdot}P = 1790$. To avoid the arbitrariness of specifying a reference level, the Rayleigh number will henceforward be expressed as a multiple of R_c, its critical value at the onset of convection when the other parameters are fixed as above. This takes care of the depth-dependence entirely, so one can state unambiguously that $R{\cdot}P = 9.8R_c$ for the Ginet & Sudan runs.

The salient characteristic of nearly all their runs, as they varied R and P with the product $R{\cdot}P$ fixed but without any magnetic fields, was the predominance of a single large convection roll stretching across the full width of the box. The chosen aspect ratio of 1.0 was favorable for the formation of such a state, as opposed to a more symmetric state with two counterrotating rolls which would be typical of convection just above the onset at R_c (and also at larger aspect ratio). The shear flow generated by this periodic array of corotating vortices tended to “stabilize” the convection, in the sense that heat transport was greatly inhibited whenever the shear flow was a maximum. This had the counterintuitive consequence of *decreasing* the net heat transport with growing Rayleigh number (and concomitant shrinking of the Prandtl number) in the range $4R_c \lesssim R \lesssim 100R_c$.

To determine what effect magnetic fields might have on this kind of behavior, it was desirable at the outset to mimic conditions from the prior results as closely as possible. Thus, initial data for the first group of runs were created by introducing magnetic fields suddenly into an already-evolved fluid solution (here and in Section 3.2). Alternatively, much stronger magnetic fields could be “switched on” underneath the bottom boundary, below a given set of random, small initial velocity perturbations (Sections 3.3 and 3.4). In the first technique, the magnetic field, when weak, serves merely as a tracer of the flow (especially for low magnetic diffusivity), so reviewing some of these first results will give a useful introduction to the unmagnetized flow dynamics.

In the kinematic regime where $B^2/8\pi << p_1$, one expects the fluid to behave as if there were no magnetic fields at all. Figure 1 illustrates this passive character of magnetic field evolution in the kinematic regime. No new magnetic flux is being supplied to this “run-down” calculation, so the initial field must decay away over time. In this and many of the other figures in this section, two kinds of contour plots are displayed: contours of the "momentum stream function" φ, which are streamlines of the flow; and contours of the magnetic flux function ψ, which are the magnetic field lines. Because the presence of magnetic fields is dynamically unimportant in these runs, the fluid motions are identical to the results of Ginet (1987) for the matching parameters.

The contrast between high and low magnetic diffusivity is in accord with expectations. With lower diffusivity, the field follows more closely along streamlines (Figure 1c), and is more prone to intervals of turbulence (Figure 1f). Here and in Section 3.2 only, the initial horizontal field was placed in the bottom 10 cells of the domain, and the bottom boundary was made to be a perfect conductor. In most of the runs to be covered hereafter, η will be kept at the smaller of the two values, the smallest that could be resolved on a 32×32 grid: 5×10^{10} $\mathrm{cm}^2/\mathrm{sec}$, making Rm ≈ 500. (Henceforth the grid will be 64×64, just to make certain.) This choice of η is motivated by observations of the Sun, where magnetic fields are generally seen to be clumped into tight bundles, implying a low diffusivity. Overall, the number of grid points is ultimately constrained by the large number of time steps that must be taken to observe long-period motions. Later, one will see that some types of oscillations have periods up to 20 times the eddy overturning time. These may require 10^5 integration time steps or more to exhibit one full cycle on a 64×64 grid.

For the runs at $P = 0.5$, $R = 19.6R_C$, the fluid dynamics is clearly dominated by the presence of the one large roll. The single sign of vorticity implies that there is a net vertical shear in the horizontal velocity: and at the top, there is indeed overall motion to the right, while at the bottom there is leftward movement. A clue to what is happening is provided by the contour plots of density perturbations in Figure 2. As the shear first tilts over, and then tears apart, the density contours, it is as if two rows of convection cells have now been formed, one on top of the other. In Paper II it will be shown how this sub-pattern of convection is able to "bootstrap" the shear that created it in the first place: it turns out that the shear is reinforced nonlinearly as the sub-pattern interacts with the original cellular pattern via the Reynolds stress $\mathbf{v}\cdot\nabla\mathbf{v}$. The nonlinear tilting instability seen in these first few runs is a pervasive feature of all the simulations, including those done by Ginet & Sudan (1987). That this is a genuine, symmetry-breaking bifurcation, and not some kind of asymmetric flaw in the code, has been tested by mirror-reversing the initial conditions while making the appropriate sign changes in pseudoscalar fields; as expected, the subsequent evolution is identical, although mirror-reflected.

3.2 *Nonlinear Dynamics of Magnetic Fields: Tension vs. Buoyancy*

In the dynamical or nonlinear regime, magnetic fields are not passively following the fluid motions, but have a substantial back-reaction upon the plasma, significantly altering the observed pattern of convection. Strong magnetic fields can drastically alter the fluid flow, either by exerting drag via magnetic tension, or by driving convection themselves through magnetic buoyancy (Parker 1955). Experience has shown that it is difficult to observe both of these effects concurrently in a 2D code due to the limitations of the geometry. When the field lines lie in the simulation plane, as they do here, tension tends to dominate; whereas when fields are perpendicular to the plane (*e.g.*, Cattaneo & Hughes 1988) there is no magnetic tension, and magnetic buoyancy will prevail. In 2D, parallel and perpendicular fields cannot be combined, because Lorentz forces in the (forbidden) third direction would arise. This is a serious, but not crippling, limitation.

Figure 3 shows results from two integrations of the magneto-anelastic equations, each of which was started from a quiescent state and then run for some time, before it suddenly had a magnetic field inserted near the perfectly-conducting bottom. For the run at the top of Figure 3, the peak value of the inserted field is too weak to affect the flow; these frames should be compared with the ones directly beneath them, which have the same parameters and initial conditions except for a 100 gauss (peak) strength of the inserted field. There is a clear tendency in the lower frames for the field to deflect the flow so that it follows along with, rather than crosses, field lines. Magnetic buoyancy is unimportant here: the tension in the field keeps the flux localized near the bottom until it becomes too weak to be buoyant. If the field is made even stronger, say a kilogauss (kG), the field is even less affected by the convection, and it acts as an effective wall, until it is sufficiently weakened by diffusion.

A further limitation of a 2D finite-difference code having these boundary conditions is that the magnetic field will always decay due to resistivity. Even purely numerical resistivity (the lowest possible) would lead to the virtual disappearance of appreciable field strength after just a few eddy turnover periods. In Figure 4, an initial 100 gauss flux sheet, initially placed roughly two-thirds of the way to the upper boundary, has been advected to near the top of the zone—except for a piece of it, which has gotten caught in the downdraft and pulled into the main eddy. The stretching and squeezing of the field at the top causes the protrusion of field lines into the vacuum region. Although this eruption event is dramatic in appearance, the field decays so swiftly due to the advection and subsequent reconnection of field lines that the event has little ultimate influence on the convection. Such activity seems to be independent of the initial field strength, implying that magnetic buoyancy is once again irrelevant here. But the eruption is still interesting as a phenomenon caused by passive advection. The full event is portrayed in an animated form in the first video segment.

3.3 *Flux Feeding at the Bottom Using a Fixed Electric Field*

To combat the tendency for strong fields to decay away before they could have much effect, it was clearly necessary to resupply magnetic flux during the simulation, and the most natural way to do it was to provide new flux at the bottom boundary. This is consistent with current theories of solar convection, which maintain that the strongest magnetic fields will be generated in a stable layer at the base of the convection zone (*e.g.*, Schüssler 1987). Some experimentation with initial and boundary conditions was needed to determine exactly how to provide a magnetic field in a stable manner, because in many runs very steep gradients in the magnetic field would form near the bottom, leading to drastic numerical instability. Two techniques proved most successful in achieving a steady state with high magnetic fields; in one, a fixed electrostatic field (proportional to $\partial\psi/\partial t$) is maintained at the bottom boundary, while in the other, the horizontal component of the magnetic field, or B_{x0}, is held constant there. As a solar analog, one could imagine a long, magnetically buoyant flux tube arising from deep within the Sun, gradually pushing itself into a region of smaller-scale convection cells closer to the surface.

The first technique (a "flux-feeding" condition) is best illustrated by the second and third video segments, but a verbal description should suffice here. By fixing $\partial\psi/\partial t$, the entry rate of magnetic field lines into the system is held constant (ψ labels field lines). Resupplying flux at a moderate rate to a $P = 0.5$ quasiperiodic state results in a quasiperiodic cycle: magnetic fields are scooped up from the bottom and made to circle around the eddy, and reconnect; then, as the growth of the shear mode reduces the vertical transport, leftover flux relaxes back down to the bottom, to be built up for the next scooping episode. By contrast, fixing $\partial\psi/\partial t$ for the $P = 2.25$ ($R = 4.36R_c$) state results in two steady, counterrotating rolls with steady magnetic behavior, in which each magnetic field line is pulled along by upwelling fluid, traveling around the tops of the eddies on either side and reconnecting with neighboring field lines in the downdraft. In both cases, the convection produces a narrow region of flux emergence at the top, centered on the downdraft.

3.4 *Fixed Horizontal Magnetic Field at the Bottom Boundary*

It was the second flux-regeneration technique—the fixed-B_{x0} condition—that was used for most of the runs. The combination of fixing B_{x0}, and starting from a quiescent initial state with extremely weak but nonzero perturbations in both the magnetic and velocity fields, was found to be the best at producing statistically steady-state behavior with strong magnetic fields. It then became of interest to explore how a convecting state might change as B_{x0} was ramped up. Magnetic fields generally have a stabilizing effect, so it was of particular interest to see how B_{x0} would influence the previously witnessed transition between two-roll, steady flow and single-roll, unsteady, shear-dominated flow that occurs in the vicinity of $P = 2.0$, $R = 4.9R_c$ (Ginet & Sudan 1987). A series of runs was therefore made just to the unstable side of the transition, at $P = 1.8$ ($R = 5.44R_c$). Figure 5 shows that in the absence of magnetic fields, there is a nontrivial periodicity as $t \rightarrow \infty$ in the total (*i.e.*, area-integrated) vorticity and in the Nusselt number, a dimensionless measure of heat transport. The temporal pattern of these global quantities is consistent with an oscillation that has undergone a period doubling *en route* to chaos—and chaos is precisely what is observed at slightly higher R (Ginet & Sudan 1987; Ginet 1987). (Note that any *local* quantities would display an additional periodicity, due to the horizontal drift of the spatial pattern.)

Several transitions were indeed found to occur as the field strength at the bottom was increased. At low strength, the magnetic fields merely caused the oscillations to become irregular, although the same basic pattern of alternation between a shear-dominated and a roll-dominated state persisted, as illustrated in Figure 6. But, beginning at $B_{x0} \approx 280$ gauss and subsequently, two new and unanticipated states of magnetoconvection were identified. In the first of these, shown in Figure 7, the periodically-varying single roll has been replaced by two steady counterrotating rolls having unequal magnitudes—a "lopsided" pattern. This is a traveling wave: the pattern is time-stationary in a frame that moves with a steady horizontal velocity, as one can see by examining the sequence presented in the figure. Conservation of momentum is not violated by this because there is a background density gradient, so it is possible for the net *velocity* across a vertical plane to be nonzero, even though the net momentum is zero. Global quantities like the Nusselt number are time-stationary. The physical parameters corresponding to this type of behavior are: $B_{x0} = 0.5$ kG (with similar behavior occurring for $B_{x0} = 0.28, 0.30, 0.35$ kG); length scale, 4300 km; and velocity, 0.3 km/sec.

Next, in Figure 8, one can observe that for $B_{x0} = 0.62$ kG the uniform translation has been converted to a horizontal oscillation, because the field has become strong enough to reverse the direction of flow at the bottom periodically. The motion is shown near its leftmost turning point. A large, counter-clockwise rotating eddy is blocked by the region of growing magnetic fields, and as it comes to a stop, the other eddy grows up and eventually forces the system to roll back the opposite way. After one full cycle there is no net drift in the pattern, so this can be categorized as a pulsating wave. Similar behavior was also seen to occur for $B_{x0} = 0.68$ kG. Clearly this stage serves as a prelude to the steady two-roll state, being a compromise of sorts between lopsided and symmetric two-roll convection. One might expect the amplitude of this "sloshing" oscillation to decrease with increasing B_{x0}. What is surprising about the oscillation is that its period is very much longer—by a factor of 20 to 30—than the eddy turnover period (about 2×10^4 seconds), as evidenced by Figure 9. Animations of both this state and the lopsided state are included in the video that accompanies Paper II.

Next, as B_{x0} rises up to 0.75 kG, and on through 1.0 kG, the shear mode is suppressed altogether, and the expected symmetric two-roll state is attained, as shown in Figure 10. Figure 10 also shows that a flux sheet can be strong enough to be buoyant but still be unable to rise, because it is held down by down-flowing fluid and by its own tension—an effect that may or may not exist in three dimensions, where further instabilities may well come into play. As the magnetic parameter is increased in this regime, the magnetic field occupies more and more of the domain close to the bottom. This affords less and less room for the convection to take place. Finally, prior to B_{x0} = 1.3 kG, convection is stopped completely, and very strong horizontal fields are able to diffuse uniformly throughout the zone. This final state is to be understood as a limitation of the 2D model, when given an arbitrarily large source of magnetic energy.

The magnetic fields modify the thermal efficiency of the fluid layer in an interesting way. Global heat transport is often expressed in terms of the Nusselt number Nu, defined as the ratio of total heat flux due to both convection and conduction to the heat flux due to conduction as if it could act alone. This number is conveniently calculated by dividing the average temperature gradient at one boundary by the (unstable) conductive temperature gradient. Figure 11 demonstrates that at first, magnetic fields actually improve the Nusselt number—presumably because the fields are able partially to stabilize the shear and tilting that causes the complicated time-dependent behavior. But as B_{x0} grows stronger, the convection slows down and ultimately stops, leaving only the turbulent eddy conduction to carry the heat.

In the transitions at $P = 1.8$, one has seen that a strong magnetic field at the bottom influences the fluid flow globally, and therefore, observing characteristics of the velocity pattern can reveal something about the fluid's deeper magnetic character. Nearer to the surface, it appears that magnetic fields are advected by the flow passively. Thus, the *surface* magnetic fields are a signature or tracer of the underlying flow field; and as such, they must have something to say regarding the *deep* magnetic fields. To illustrate the point, Figures 12 and 13 show two of the unique signatures. Figure 12 corresponds to the "lopsided" state at $B_{x0} = 620$ kG, and one notices several distinctive features here: B_z is predominantly unipolar (in the sense that B_z is peaked more sharply in one sign than the other, even though the net flux at the top is zero); is concentrated in the downdraft; and is 5 to 10 gauss in peak strength. It thus reflects the skewed nature of the underlying flow field. For the two-roll state, in Figure 13 ($B_{x0} = 750$ kG), B_z is of slightly higher magnitude but is now balanced and bipolar in nature—as it must be, from the underlying symmetry in the velocity field. For the intermediate, oscillatory states, B_z is of course time-varying, but similar in amplitude and character to the lopsided configuration.

Intuitively, one might expect that the magnetic profile at the top is best described by a passive advection model of some kind, since fields there are always extremely weak compared to the fields at the bottom. In the Appendix, it is shown that the surface fields can be analyzed in terms of a simple advection-diffusion balance, the approach one uses naturally to estimate the thickness of any boundary layer. A key feature of the passive advection picture is that the width of the flux emergence region—the distance between the two peaks of the bipolar magnetic field—is proportional to the square root of the magnetic diffusivity η. This hypothesis was tested by varying η for the two-roll state at B_{x0} = 750 G. Figure 14 shows that the separation between the opposite-polarity peaks is extremely well fit by a power law that goes as $(\mathrm{Rm})_{top}^{-1/2}$, corresponding to $\eta^{1/2}$.

More precisely, from arguments presented in the Appendix based on boundary layer theory, the measured separation Δ between peaks of B_z is theoretically equal to $\sigma\delta$, where $\delta \equiv (\eta\,|\,\partial v_x/\partial x\,|_0^{-1})^{1/2}$, and σ is an O(1) constant, $2.0 < \sigma \lesssim 2.6$, roughly. Given that the x-velocity varies sinusoidally with a period L (a good approximation for all the pertinent simulation runs), one can write Δ/L in terms of a magnetic Reynolds number as

$$\frac{\Delta}{L} = \frac{\sigma}{\sqrt{2\pi V_{max} L \eta^{-1}}} = \frac{\sigma}{\sqrt{4\pi(\mathrm{Rm})_{top}}} \tag{31}$$

Taking $\sigma = 2.0$ as a lower bound, this relation has been drawn as a solid line in Figure 14. One can see that the estimate it provides is too low by about 60%—although the data is generally fit very well by a $-1/2$ power law (dotted line). Since the condition $(\mathrm{Rm})^{1/2} >> 1$ is not particularly well satisfied, some disagreement should not be too surprising.

3.5 *Guide to the Video Visualization*

The video gives a fuller description than can be provided verbally (or even pictorially) of the magnetic phenomena which have been seen in this series of runs of the SUMO simulation code. The format of the frames of the video is much the same as in the preceding figures, except that the two kinds of plots are overlaid instead of being placed side-by-side: φ contours are the boundaries of colored regions (negative φ corresponding to shades of red, positive φ in shades of green); while ψ contours are solid black lines. The use of colors allows images of ψ to be superimposed on those of φ with minimal confusion. Note that in some segments of the video, the periodic property is exploited to place two identical copies of the simulation region side-by-side. Over this, the extension of the magnetic field above the top boundary (satisfying $\nabla^2\psi = 0$) is depicted against a blue background. Further explanation is provided in the video titles and in the corresponding figure captions found within Section 3.

4. SUMMARY AND CONCLUSIONS

A variety of magnetically-influenced dynamical behaviors can be seen in 2D simulations of anelastic convection. For the most part, the magnetic field provides an extra restoring force that helps to stabilize tilting and shearing in the velocity field—but such stabilization is neither immediate nor total. As the parameter representing imposed magnetic field strength is increased, convection that was originally highly unsteady is observed to undergo a sequence of bifurcations into progressively more regular patterns. These were found to be, in order of increasing magnetic field, a “lopsided” traveling wave, a “sloshing” or direction-reversing wave, two steady counterrotating rolls, and a nonconvecting state. With magnetic boundary conditions that feed in flux at the bottom and electrically insulate the top, the global dynamics are seen to be controlled by the bottom boundary region, while at the top, the magnetic fields have been shown to act passively, providing a possible signature of the underlying dynamics.

There are a number of implications for solar physics that might be drawn from these results. First, the simulations show that imposing a moderately strong field at just one boundary of a convective roll can dramatically alter its long-term dynamical behavior and its ability to transport heat. Thus, it is hazardous to neglect such magnetic fields when discussing the fluid properties of the Sun’s convection zone—and this would be true even if the magnetic field were confined to an extremely narrow boundary layer at the bottom. Qualitative changes in the fluid's behavior at low Rayleigh numbers due to magnetic fields could persist into the turbulent regime, in 3D as well as in 2D.

The connection to solar *observations* is necessarily more tentative, due to the simulations' obvious limitations in resolution, boundary conditions, and geometry. Nevertheless, some intriguing possibilities do emerge. One is that it may be feasible to estimate a "turbulent eddy" diffusivity for the upper convection zone, simply by measuring the widths of emerging flux regions. Such regions are currently suspected to be composed of even smaller-scale magnetic pores, whose width is determined by a local pressure balance; however, it may be that in the *aggregate*, the size of a given clump of these pores (near a supergranule boundary, say) is determined by an advection-diffusion balance of the kind suggested by simulations. The simulations have reproduced the essential feature of concentrating magnetic flux at the downdrafts. One estimates a photospheric η by using equation (31) plus the available data on supergranulation and the magnetic network (Wang 1988a, b). Taking V_{max} = 0.38 km/sec, $L = 3.37\times10^4$ km from Dopplergram data and Δ = 3250 km from the autocorrelation of vertical magnetic fields, with $\sigma = 2.0$ (a Gaussian), one arrives at η = 190 km^2/sec. Wang (1988a) has separately measured this effective diffusivity at the photosphere, by cross-correlating magnetograms taken at different lag times. He obtains a value for η of 150 to 200 km^2/sec, which is in rough agreement with the advection-diffusion value.

One important difference between the simulations and Wang's observational data is that in the latter, the emerging flux elements are mostly unipolar. But simulations could offer an explanation for such an apparent predominance of "unipolar" flux elements: they might really be bipolar in nature, but possessing a highly skewed distribution due to translational motion. Reentrant flux of the "missing" polarity would be too spread-out to be noticeable by comparison (cf. Figure 12). Turning to solar data, again, one can look at the degree of tilting of a typical magnetic element as being a good indicator of the proximity of strong fields of opposite polarity; the higher the tilt, the closer together the poles. Surface fields on the Sun are commonly seen to be tilted over by 6 to 35 degrees, with an average of 10° or so (Topka *et al*. 1992). This compares more favorably with the 20° angle seen in "skewed" simulations than it does with the 30° and more in symmetric cases—just as expected. Better evidence for the existence of fundamentally bipolar but skewed fields would be provided by a negative dip in the autocorrelation curve of the vertical magnetic field. This could occur at a distance extending as far as $L/2$, where L is the supergranule size of around 30,000 km. Here, unfortunately, the conjecture is not borne out very well by observations (Wang 1988a), although this could be due to an overall imbalance of positive and negative flux in the region that was under study.

In all these comparisons, the validity of the conclusions depends on whether the assumptions behind the model are justified—whether features too small to be resolved by telescopes can truly be represented by an effective turbulent diffusivity, whether the boundary conditions at the "photosphere" have been chosen appropriately, *etc*., all of which are open to question. Still, with this awareness of the restrictions of the model, the proposed extrapolations of this numerical study to solar physics are not unreasonable. Perhaps it is more significant, however, that the numerical data arising from the model are able to provide new insights into basic magnetoconvection dynamics, which are interesting in their own right: these will form the subject matter of Paper II.

ACKNOWLEDGMENTS

The authors give their thanks to Philippe Similon, for suggestions regarding passive scalar advection; to Dana Longcope and Bruce Land, for help with the graphics software; and to Greg Ginet, for assistance in developing the simulation code. Special thanks go to Nigel Weiss for many helpful comments. This research was supported by the National Science Foundation under grant no. ATM-8600308, and by the Cornell Theory Center, which receives major funding from the NSF and IBM Corporation, with additional funding from New York State Science and Technology Foundation and members of the Corporate Research Institute. An early version of this paper was submitted to the 1990 IBM Supercomputing Competition.

APPENDIX: 2D ANALYSIS OF MAGNETIC FLUX ADVECTION NEAR A STAGNATION POINT

The structure of 2D magnetic boundary layers has been elucidated by Perkins & Zweibel (1987), who drew upon earlier work by Childress (1979). The approach here is more akin to Bender & Orszag (1978), but generalized to 2D. Assume that close to the stagnation point at the top of the downdraft, the velocity can be represented by the linear terms in its Taylor series. Recalling that $\mathbf{\nabla}\cdot\mathbf{v} = v_z h^{-1} = 0$ at the stress-free upper boundary, the steady state situation for ψ is described locally by

$$\left|\frac{\partial v_x}{\partial x}\right|_0 \left(-x\frac{\partial\psi}{\partial x} + z\frac{\partial\psi}{\partial z}\right) = \eta\nabla_\perp^2\psi \ . \qquad \text{(A1)}$$

At high Rm, solutions to this "inner equation" can be asymptotically matched onto solutions of the $\nabla_\perp^2 \to 0$ "outer equation," in such a way that the boundary conditions are satisfied. But, rather than perform this detailed matching to obtain a global approximation to the answer, it will suffice to examine the character of just one class of solutions to equation (A1). Attempting a separation of variables—even though technically it is impossible for ψ to obey its top boundary condition *globally* with a solution of this type—one can make the substitution $\psi(x,z) = F(x)G(z) + \overline{\psi}$ (where $\overline{\psi}$ is an adjustable constant) to find

$$\frac{d^2F}{dx^2} + x\frac{dF}{dx} + (\nu+1)F = 0 \ , \qquad \frac{d^2G}{dz^2} - z\frac{dG}{dz} - (\nu+1)G = 0 \qquad \text{(A2)}$$

The equations have been nondimensionalized by $\delta \equiv (\eta \,|\, \partial v_x / \partial x \,|_0^{-1})^{1/2}$ which is the natural unit of length in the boundary region. Equations (A2) are reminiscent of the harmonic oscillator eigenvalue problem of quantum mechanics except that here the separation constant (ν+1) can take on any value. The solutions in x and z are therefore products of a Gaussian with linear combinations of parabolic cylinder functions,

$$\left.\begin{aligned} F(x) &= e^{-x^2/4}[a_+ D_\nu(x) + a_- D_\nu(-x)] \\ G(z) &= e^{+z^2/4}[b_+ D_{-\nu-1}(z) + b_- D_{-\nu-1}(-z)] \end{aligned}\right\} . \tag{A3}$$

In the case of symmetric two-roll convection, F must be an even function of x, implying $a_+ = a_-$; whereas in the z-direction, $b_+ = 0$ is the only way to prevent G from diverging as $z \to -\infty$. (Note that $z > 0$ is outside the domain, so divergence as $z \to +\infty$ can be ignored.) The leading behavior of G is $z^{-\nu-1}$ for $z \to -\infty$, which restricts ν to $\nu > -1$. In addition, $\nu < 0$ is required to prevent multiple inflection points (or, extrema in B_z) from occurring at distant x.

The resulting function $F(x)$ is smooth, with a single maximum: a Gaussian with lengthened tails. The peak-to-peak separation of the resulting extrema in B_z—equal to the distance between the zeros of the second derivative of F—ranges from a minimum of 2.0 at $\nu = 0$, to a limit of about 2.6 as $\nu \to -1$. (It is undefined at $\nu = -1$, which generates the function $\psi(x,z) = const$.) By comparison, the peak-to-peak distance in simulations was around 3.5 in these same units—but note that the scaling of $\delta \propto \eta^{1/2}$ comes out exactly as expected. Checking against Figure 13, it is evident that the vertical magnetic field could easily be fitted by the derivative of a Gaussian with long tails, while the power spectrum confirms there is a slightly-faster-than-exponential falloff at high wavenumbers. So, within factors of order unity, it appears that the passive advection model does correctly account for the structure of magnetic fields near the top boundary, and in particular that magnetic Lorentz forces are relatively unimportant there, at least for the cases considered in Section 3.

When the reflectional symmetry of the convection is broken by tilting, the even-parity solutions can be modified by taking $a_+ \neq a_-$ to arrive at configurations like those shown in Figure 12. The boundary condition $\int B_z \, dx = 0$ must in this case be maintained by a more spread-out and asymmetric reentry of vertical flux all across the top surface, away from the immediate vicinity of the downdraft.

TABLE 1: LIST OF DIMENSIONLESS PARAMETERS

Parameter	Definition	Name
R	$\dfrac{gd^4\rho_r^2(\nabla s)_r}{\kappa\mu}$	Rayleigh number
P	$\dfrac{\mu c_p}{\kappa}$	Prandtl number
S	$\dfrac{gd}{c_p T_r}$	Stratification parameter
G	$\dfrac{r}{c_p}^*$	Gas parameter
Q	$\dfrac{d^2 B_r^2}{4\pi\mu\eta}$	Chandrasekhar number
M	$\dfrac{\eta c_p \rho_r}{\kappa}$	Magnetic Prandtl number
a	$\dfrac{L}{d}$	Aspect ratio (*width*/*height*)
ε	$\dfrac{d(\nabla s)_r}{c_p}$	(*expansion parameter*)

REFERENCES

Arter, W. 1983, *Geophys. Astrophys. Fluid Dyn.*, **25**, 259

Bender, C. M., & Orszag, S. A. 1978, *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill), p. 455

Cattaneo, F., & Hughes, D. W. 1988, *J. Fluid Mech.*, **196**, 323

Chandrasekhar, S. 1961, *Hydrodynamic and Hydromagnetic Stability* (New York: Dover)

Childress, S. 1979, *Phys. Earth Planet. Int.*, **20**, 172

Clayton, D. D. 1968, *Principles of Stellar Evolution and Nucleosynthesis* (Chicago: Univ. of Chicago Press)

Foukal, P. V. 1990, *Solar Astrophysics* (New York: Wiley)

Fournier, J. D., Sulem, P. L., & Pouquet, A. 1982, *J. Phys. A: Math. Gen.* **15**, 1393

Fox, P. A., Theobald, M. L., & Sofia, S. 1991, *Ap. J.*, **383**, 860

Galloway, D. J., & Weiss, N. O. 1981, *Ap. J.*, **243**, 945

Gilman, P. A., & Glatzmaier, G. A. 1981, *Ap. J. Suppl.*, **45**, 335

Ginet, G. P. 1987, Ph. D. thesis, Cornell Univ.

Ginet, G. P., & Sudan, R. N. 1982, *Bull. Am. Phys. Soc.*, **27**, 1024

———. 1987, *Phys. Fluids*, **30**, 1667

Glatzmaier, G. A. 1984, *J. Comput. Phys.*, **55**, 461

———. 1985a, *Ap. J.*, **291**, 300

———. 1985b, *Geophys. Astrophys. Fluid Dyn.*, **31**, 137

Golub, L., Rosner, R., Vaiana, G. S., & Weiss, N. O. 1981, *Ap. J.*, **243**, 309

Gough, D. O. 1969, *J. Atmos. Sci.*, **26**, 448

Hossain, M., & Mullan, D. J. 1991, *Ap. J.*, **380**, 631

———. 1993, *Ap. J.*, **416**, 733

Hurlburt, N. E., & Toomre, J. 1988, *Ap. J.*, **327**, 920

Hurlburt, N. E., Proctor, M. R. E., Weiss, N. O., & Brownjohn, D. P. 1989, *J. Fluid Mech.*, **207**, 587

Keller, C. U. 1992, *Nature*, **359**, 307

Knobloch, E. 1986, *Geophys. Astrophys. Fluid Dyn.*, **36**, 161

Lantz, S. R. 1992, Ph. D. thesis, Cornell Univ.

Lantz, S. R. 1993, *Ap. J.*, this volume, Paper II

Longcope, D. W., & Sudan, R. N. 1991, *Phys. Fluids B*, **3**, 1945

Matthews, P. C., Hurlburt, N. E., Proctor, M. R. E., & Brownjohn, D. P. 1992, *J. Fluid Mech.*, **240**, 559

Moffatt, H. K. 1978, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge Univ. Press)

Nagata, M., Proctor, M. R. E. & Weiss, N. O. 1990, *Geophys. Astrophys. Fluid Dyn.*, **51**, 211

Ogura, Y., & Charney, J. G. 1960, in *Proc. Intl. Symp. Num. Weather Prediction*, 431

Ogura, Y., & Phillips, N. A. 1962, *J. Atmos. Sci.*, **19**, 173

Parker, E. N. 1955, *Ap. J.*, **121**, 491

———. 1978, *Ap. J.*, **221**, 368

Perkins, F. W., & Zweibel, E. G. 1987, *Phys. Fluids*, **30**, 1079

Schüssler, M. 1987, in *The Internal Solar Angular Velocity*, ed. B. R.

Durney & S. Sofia (Dordrecht: D. Reidel), p. 303

Schwarzschild, M. 1958, *Structure and Evolution of the Stars* (New York: Dover)

Spruit, H. C. 1979, *Sol. Phys.*, **61**, 363

Spruit, H. C., & Zweibel, E. G. 1979, *Sol. Phys.*, **62**, 15

Stein R. F., Brandenburg, A., & Nordlund, Å. 1992, in *Cool Stars, Stellar Systems, and the Sun*, ed. M. S. Giampapa & J. A. Bookbinder (San Francisco: Astronomical Society of the Pacific), 148

Topka, K. P., Tarbell, T. D., & Title, A. M. 1992, *Ap. J.*, **396**, 351

Wang, H. 1988a, *Sol. Phys.*, **116**, 1

———. 1988b, *Sol. Phys.*, **117**, 343

Wang, H., & Zirin, H. 1988, *Sol. Phys.*, **115**, 205

Weiss, N. O. 1966, *Proc. Roy. Soc. A*, **256**, 310

———. 1981, *J. Fluid Mech.*, **108**, 273

———. 1990, *Computer Phys. Rep.*, **12**, 233

———. 1991, *Geophys. Astrophys. Fluid Dyn.*, **62**, 229

Weiss, N. O., Brownjohn, D. P., Hurlburt, N. E., & Proctor, M. R. E. 1990, *MNRAS*, **245**, 434

Yakhot, V., & Orszag, S. A. 1986, *J. Sci. Comp.*, **1**, 3

Zank, G. P., & Matthaeus, W. H. 1991, *Phys. Fluids A*, **3**, 69

Zirin, H. 1988, *Astrophysics of the Sun* (Cambridge: Cambridge Univ. Press)

FIGURE LEGENDS

FIG. 1.—Contrast of magnetic diffusivities at $P = 0.5$ ($R = 19.6R_c$), with a perfectly conducting bottom boundary condition, in the kinematic regime. Here, an initial weak B_x is decaying away in time. Shown at left are: (a) the streamlines of $\rho_0\mathbf{v}$; (b) magnetic field lines for Rm = 46 ($\eta = 5\times10^{11}$ cm^2/sec, $M = 0.82$); (c) magnetic field lines for Rm = 460 ($\eta = 5\times10^{10}$ cm^2/sec, $M = 8.2\times10^{-2}$). These same fields are then shown later in time, on the right: (d) the convection flow field, which, like (a), is the same for both runs; (e) magnetic field lines, still laminar at high η; and (f) the field lines at lower η, showing a tendency to turbulence.

FIG. 2.—Density contours at $P = 0.5$ ($R = 19.6R_c$), in the kinematic regime. Density perturbations, (a), stand out better after subtracting off their horizontal average, (b). The presence of net shear in the horizontal flow leads to stretching—then breaking, (c) and (d)—of the perturbed density contours. The tilted density pattern drives a tilted convection flow, which then reinforces the shear nonlinearly.

FIG. 3.—Contrast of dynamic and kinematic regimes at identical times; same parameters as in Figure 2. (a) streamlines of $\rho_0\mathbf{v}$, (b) field lines, for very weak, initially horizontal fields; (c) streamlines of $\rho_0\mathbf{v}$, (d) field lines, starting from a peak field strength of 100 gauss. The stronger field is better able to hold itself together against the flow, partly deflecting

the fluid to follow along field lines. Here and elsewhere, solid (broken) contour lines denote positive (negative) values of φ, the momentum streamfunction.

FIG. 4.—Eruption of a flux sheet through the surface. Same parameters as Figure 2, but in this case, the 100-gauss sheet of magnetic field was located nearer to the top, about two-thirds of the way up. Magnetic field lines (solid) have here been overlaid on φ contours (broken).

FIG. 5.—The temporal behavior of some global properties of the flow at $P = 1.8$ ($R = 5.44R_c$), no magnetic fields. Period-doubled oscillations are evident in: (a) the area-integrated momentum vorticity, $\varpi = \hat{\mathbf{y}}\cdot\nabla\times(\rho_0\mathbf{v})$, (b) the integral of the absolute value of ϖ, and (c) the Nusselt number based on the average of the temperature gradient at the top boundary.

FIG. 6.—Showing the two phases of irregularly modulated, oscillatory magneto-anelastic convection at $B_{x0} = 0.18$ kG, $M = 8.2\times10^{-2}$; other parameters as in Figure 5. (a) streamlines of $\rho_0\mathbf{v}$, (b) field lines, in the shear-dominant phase; (c) streamlines of $\rho_0\mathbf{v}$, (d) field lines, in the roll-dominant phase.

FIG. 7.—A sequence in time for a run having the same parameters as in Figure 5 but with $M = 8.2\times10^{-2}$ and $B_{x0} = 0.5$ kG at bottom. In the absence of magnetic fields, oscillatory convection would result, but in

their presence the shearing tendency is partially stabilized, resulting in a lopsided pattern that propagates slowly rightward. Frames (a), (c), (e), and (g) show streamlines of the momentum; frames (b), (d), (f), and (h) show magnetic field lines.

FIG. 8.—A series of snapshots near the turning point of an oscillatory state: (a), (c), (e), and (g), streamlines of $\rho_0\mathbf{v}$, and (b), (d), (f), and (h), magnetic field lines, for the same parameters as in Figure 5 except B_{x0} = 0.62 kG. The bigger (solid-line) left-rolling eddy runs against a region of strong magnetic fields, and as it shrinks, the other (dashed-line) eddy grows until it is big enough to push everything back to the right.

FIG. 9.—Comparing the time histories of oscillations in the sloshing and shear/roll modes. The instantaneous Nusselt number across the top boundary is shown (a) for B_{x0} = 0.62 kG (cf. Figure 8), and then (b) for B_{x0} = 0.18 kG (cf. Figure 6). Only one-half of a period is displayed in (a)—the signature in Nu repeats, with the opposite sign of vorticity, in the other half-cycle. The length of the half-period seen in (a) is clearly much greater than the characteristic time scale evident in (b); the latter arises from the B_{x0} = 0 oscillatory behavior (see Figure 5).

FIG. 10.—A two-roll steady-state configuration for B_{x0} = 0.75 kG; other parameters as in Figure 5. Shown are (a) streamlines of $\rho_0\mathbf{v}$; (b) field lines; (c) and (d), the horizontal averages of magnetic pressure and the

perturbed density, respectively. The latter shows a dip near the bottom due to the effect of magnetic buoyancy.

FIG. 11.—The Nusselt number, a dimensionless measure of the total heat transport, as a function of B_{x0}, the horizontal magnetic field imposed at the bottom boundary. When B_{x0} is very high, convection is suppressed, and heat transport is purely conductive (Nu = 1.0). The decrease of Nu to 1.0 as B_{x0} is increased is not monotonic, however, because flows with reduced net shear seem to be able to carry heat more efficiently than the time-varying single-roll state at $B_{x0} = 0$, resulting in a maximum in Nu for intermediate values of B_{x0}.

FIG. 12.—Magnetic profiles and power spectrum at the top for the run of Figure 7. The flux emergence is skewed, emerging with one sign at the downdraft, and reentering in a spread-out fashion with opposite sign. The power spectrum exhibits a quasi-Gaussian character.

FIG. 13.—Magnetic profiles and power spectrum at the top for the run of Figure 10. Notice that flux emergence is bipolar with a peak strength of 10 gauss, 75 times less than the field at the bottom; that all the flux is emerging at the downdraft; and that field profiles are approximately Gaussian (or Gaussian-derivative) and centered on the downdraft.

FIG. 14.—The distance between the peaks of the bipolar emergence

region (normalized to L), as a function of magnetic Reynolds number $(\mathrm{Rm})_{top}$; other parameters as in Figure 10. The solid line is a $-1/2$ power law suggested by a simple model of passive advection near the surface (see Appendix). The error bars indicate the uncertainty in the leading constant. The dashed line is a $-1/2$ power law fit to the data for comparison.

AUTHORS' PRESENT ADDRESSES

Steven R. Lantz
Cornell Theory Center
Cornell University
Ithaca, NY 14853

R. N. Sudan
Laboratory of Plasma Studies
Cornell University
Ithaca, NY 14853