DETERMINING POINTS OF A CIRCULAR REGION REACHABLE BY JOINTS OF A ROBOT ARM

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Abstract

An "arm" is a sequence of links whose endpoints are connected consecutively by movable joints. The location of the first endpoint is fixed. This report gives a polynomial time algorithm for determining the regions that each joint can reach when the arm is restricted to a circular region of the plane.

1. Introduction

In an earlier report [1], we gave a polynomial time algorithm for determining whether the end of an arm can reach a given point from a given initial configuration when the arm is restricted to a circular region of the plane. In this report, we give a polynomial time algorithm for computing the boundaries of the <u>regions</u> each joint can reach. The presentation assumes some

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familarity with the earlier report.

An arm consists of a sequence of n links $L_1,\ldots,L_n$ that are hinged together consecutively at their endpoints. The links may rotate freely about their joints and are allowed to cross over one another. The endpoints are consecutively labeled $A_o,\ldots,A_n$, and the length of L_i is denoted by l_i. The location of A_o remains fixed in the plane.

Consider an embedding of the arm inside a circle C with center O, radius r and diameter d. Let S_j be the set of points to which A_j can be moved, and let $R_j = C \cap S_j$ be the points that A_j can reach on the circle C. In the earlier report [1], we showed that R_j consists of at most two arcs of C. A point p is on the boundary of S_j if, and only if, each neighborhood of p contains a point that belongs to S_j and a point inside or on C that does not belong to S_j. Normally the only points of R_j that are boundary points are the endponts of the arcs.

We will show that for each j, $0 \le j \le n$, the boundary of S_j can be covered by a finite number of circles. Furthermore, the number of circles needed for a given joint is bounded above by a constant that does not depend on the arm in any way. The entire process of computing the centers and radii of the circles needed for each joint in turn is p(n), where p(n) is a polynomial in the number of links. It is then straight forward to piece together the actual boundary of each region S_j once a set of covering circles has been found. One begins with the circle C and selects one of the possible boundary circles, tests whether the circle is in fact a boundary (using the algorithm described in [1]) and if so intersects it with the circle C and then proceeds with the next possible boundary circle.

The proof that the boundary of any S_j can be covered by a constant number of circles is quite technical and involves handling many special cases. We have organized the details into several sections whose contents we now briefly outline.

Section 2 considers the possibility that A_o is not the only "fixed" joint. By our definition of "arm", A_o is the only joint that is fastened to the plane. However, it may be that other joints are effectively fixed for geometric reasons. For example, A_o may be located on C, and the first link L_1 may have length equal to the diameter d of C so that the location of joint A_1 cannot change. Theorem 1 shows that there is a joint index j such that the location of A_i can change if, and only if, $j<i\le n$. Furthermore, this index can be found quickly. This result takes care of regions consisting of single points and allows us to assume without loss of generality that A_o is the only fixed joint.

Section 3 introduces "basic" circles and "elbows". Basic circles are natural candidates for inclusion in a cover of the boundary of a region S_j by circles. For example, C itself will be called a basic circle. Joints are called "elbows" if they lie off C but are neither straight nor folded. If there is an elbow between A_i and A_k, then the location of A_i can often be held fixed while A_k is moved to points in an open set containing the original location of A_k. Hence elbows are important because they can tell us that a joint is not on the boundary of its region. The lemmas given in Section 3 are concerned with the occurrence of elbows.

In Section 4, it is shown in Theorem 2 that as long as a joint A_m is not on a "basic" circle, then some joint A_j, $0 < j < m$, must be on C if A_m is on the boundary of its region. These joints on C can be thought of as dividing the part of the arm between A_o and A_m into "segments". Theorem 3 shows that the intermediate segments consist of straight lines of links while the initial and final segments may each have one joint that is folded.

In Section 5, the possibility that the final segment lies on a diagonal of C is handled as a special case. This special case motivates the definition of "supplementary" circles, which are then assumed to belong to the covering sets for the boundaries of the regions. The main result of this section is Theorem 5, which lists the possible configurations for the part of the arm between A_o and A_m when A_m is on the boundary of its region but is located at a point not covered by a basic or supplementary circle.

The main result of the entire paper, the fact that the number of covering circles needed is bounded by a constant independent of the arm, is given in Theorem 6 of Section 6. The proof consists of handling each of the configurations enumerated in Theorem 5. The machinery needed to do this is given in Lemmas 6-8. These Lemmas state that either certain inequalities in the lengths of links hold, or certain configurations cannot place A_m on the boundary of S_m. These inequalities restrict the possibilities for the last joint A_j on C before A_m. The possible locations for A_j then become centers for covering circles of radii determined by the final segment before A_m.

Before we begin the technical sections, we mention two notational matters. First, an expression such as "Joint A_j is <u>between</u> A_i and A_k" usually means that $i < j < k$, not that A_j lies between A_i and A_k on the line they determine in the plane. Another example of this usage is "Joint x is <u>beyond</u> or <u>past</u> joint y," meaning x has the higher index. It should be clear from the context when the words "between" and "beyond" are used in a geometric way. Second, if joints A_i and A_j are connected by a straight line made up of links

$L_{i+1},\ldots,L_j$, we will denote this line of links by $\lfloor A_iA_j \rfloor$ and the length $\mathbf{l}_{i+1}+\ldots+\mathbf{l}_j$ of this line by $|\lfloor A_iA_j \rfloor|$.

2. Determining the Immovable Joints

Given an arm with A_o fixed, it may be the case that certain other A_i are immovable. For example, if A_o is fixed on C and $\mathbf{l}_1 = d$, then certainly the location of A_1 cannot change. Another example of an immovable joint beyond A_o is shown in Figure 1.

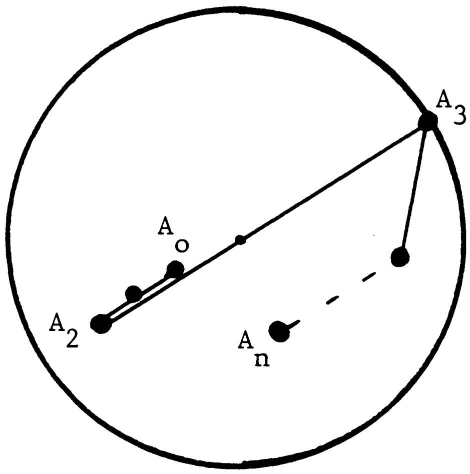


Figure 1. Joints A_o-A_3 cannot move.

In general, if A_o lies at a point distance d_o from C, $d_o \neq 0$ or r, and if $\mathbf{l}_i$ satisfies $\mathbf{l}_i = d - d_o + \mathbf{l}_1+\ldots+\mathbf{l}_{i-1}$, then joints $A_o,\ldots,A_i$ are effectively fixed. In Theorem 1, we prove that the immovable joints form a consecutive sequence from A_o up to some A_k, where k can be computed in time on the order of a polynomial in n. After proving this we will assume without loss of generality that k=0 so that A_1 and its successors are free to move.

<u>Theorem 1</u>: Suppose an arm is constrained to move inside a circle C and that x is the last fixed joint of the arm, i.e., the last joint whose set of reachable points contains only one element. Then the joints (if any) between A_o

and x must be fixed. Furthermore, x can be found in p(n) time, where p is a polynomial in the number n of links.

Proof: In [1], it was shown that an arm can always be moved to a certain "normal form" in which a straight line of links $[A_oA_i]$ stretches from A_o along a radius toward C, where $l_1+\ldots+l_i$ is equal to or less than the distance d_o from A_o to C but $l_1+\ldots+l_{i+1}$ is greater than d_o. (See Figure 2.)

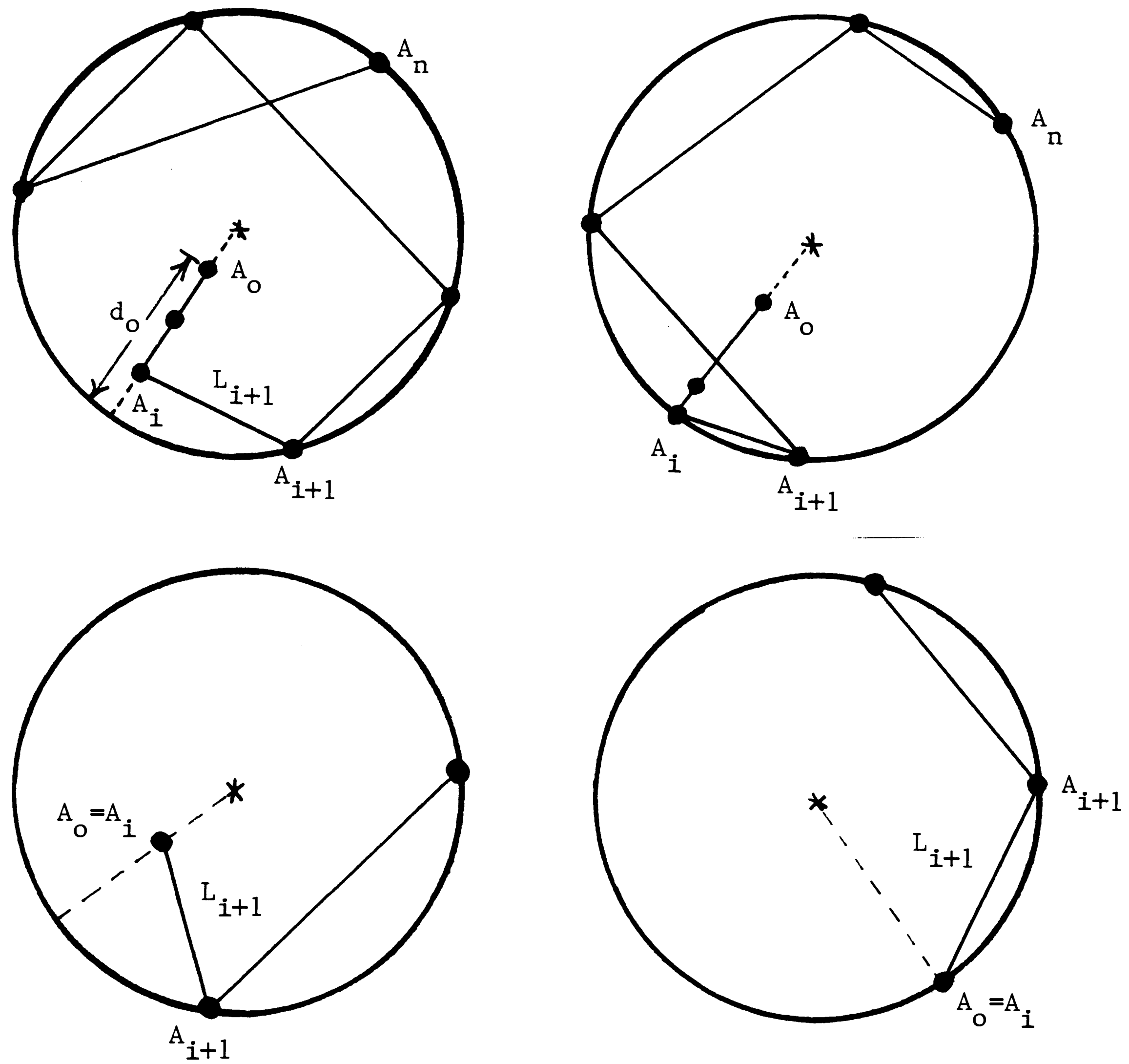


Figure 2. Arms in normal form A_{i+1} is the first joint for which $l_1+\ldots+l_{i+1}$ exceeds d_o, the distance between A_o and C.

The joint A_i may or may not lie on C, depending on whether $l_1+\ldots+l_i \le d_o$ holds with equality. However, in this normal form all joints beyond A_i lie on C.

Note that if $l_1 > d_0$, perhaps because A_0 is on C, then $A_i = A_0$ and the line of links $\lfloor A_0A_i \rfloor$ is empty. Also note that L_{i+1} may lie on the diagonal through A_0 even though it is too long to fit on the radius. (See Figure 1.) Obviously, the location of any immovable joint must be consistent with this normal form. This fact is useful in determining which joints are immovable.

If A_0 is positioned at 0, the center of C, or if $l_1 + \ldots + l_n \leq d_0$ so that the arm at best can just reach C, it is obvious that the last immovable joint A_k is A_0 itself. This is because the entire arm can be rotated about A_0. We assume from now on that A_0 is not at 0 and that $l_1 + \ldots + l_n > d_0$. We also assume that the arm has been moved to normal form.

We begin by showing that if A_0 is not the only immovable joint, that is, if $A_k \neq A_0$, then A_{i+1} and all its predecessors must be fixed, where again, A_i is the last joint such that L_1 through L_i form a straight segment that lies on the radius through A_0 when the arm is in normal form. The cases $A_i = A_0$ and $A_i \neq A_0$ will be treated separately.

If $A_k \neq A_0$ but $A_i = A_0$ so that A_0 itself is the last joint on the radius, then either L_1 must cross over 0 in reaching from A_0 to A_1 on C or L_2 must be as long as the diameter d of C. Otherwise, all joints beyond A_0 would be able to move, contradicting the assumption that $A_k \neq A_0$. If L_1 crosses over 0 or if $l_2 = d$, then A_1 is fixed.

If $A_k \neq A_0$ and $A_i \neq A_0$ so that there is at least one link on the radius, L_{i+1} must cross 0 in reaching from A_i to A_{i+1} on C. Otherwise, all joints beyond A_0 would be able to move. The fact that L_{i+1} crosses 0 implies that A_{i+1} is fixed on C, as it cannot move closer to A_0. Also, all joints between A_0 and A_{i+1} are fixed because they can move neither farther from A_0 nor closer to

A_{i+1}. This completes the proof that if $A_k \neq A_o$, A_{i+1} and all its predecessors are fixed.

Now note that if A_j is _any_ immovable joint lying on C, _all_ joints beyond A_j can move unless $l_{j+1}=d$ or $l_{j+2}=d$. Also if either of these conditions is satisfied, then A_{j+1} is immovable. This implies that each successor of A_{i+1} up to A_k must be immovable.

It is easy to see that A_k can be found in polynomial time. □

3. Some Preliminary Lemmas

We now begin the determination of the boundaries of S_j. In light of Theorem 1, let us adopt the notation that A_o _is the last fixed joint of the arm_.

Given a joint A_m beyond A_o, we want to build up a small collection of circles whose union covers the boundary of the set S_m of points A_m can reach. In order to find such a collection quickly and easily, _we do not insist that each circle in the collection contain a boundary point_.

There are four circles, not necessarily distinct, that it is natural to place in the collection immediately -- namely, the two circles centered at O whose radii correspond to the minimum and maximum distances that A_m can move off C, and two circles centered at A_o whose radii are obvious bounds for the minimum and maximum distances that A_m can move from A_o. These circles, which we call _basic circles_, are discussed in more detail below.

In [1] it was shown that the minimum and maximum distances that a joint can move from C can be computed in p(n) steps, where p is a polynomial in the number of links. Hence the first two basic circles can be found quickly.

(Note that one of these is often C itself and that the other may consist of the point 0 considered as a circle of zero radius.) Summing the lengths of the links preceeding A_m gives an upper bound for the maximum distance A_m can move from A_o. If A_m is preceded by a link L_j that is so long that

$$l_j - \sum_{i=1, i \neq j}^{m} l_i > 0,$$

then this difference gives a positive lower bound for the minimum distance between A_o and A_m; otherwise, 0 is a bound. The two remaining basic circles are defined to be the circles centered at A_o with these radii, which are easy to compute.

Before continuing to build up a collection of circles covering the boundary points of S_m, we first need to observe some facts about joints. These are developed in Lemmas 1 through 3 below.

Consider a joint A_j that does not lie on the circle C. If L_j and L_{j+1} form a 0°(=360°) v 180° angle, A_j is said to be a <u>fold</u> or a <u>straight joint</u>, respectively. If A_j does not lie on the circle and is open to any other angle, it is called an <u>elbow</u>. (It is important to note that the definition of an elbow requires that the joint not be on C.) The next lemma gives a simple but fundamental observation about elbows.

<u>Lemma 1</u>: Suppose that no joint strictly between A_i and A_k lies on circle C but that some joint A_j between them is an elbow (A_i and A_k may or may not lie on C.) Then the location of A_i can be held fixed while A_k is moved to all those points in some open ball centered at A_k that do not violate the minimum and maximum distances that A_k can be located from the circle. (See Figures 3a and b.)

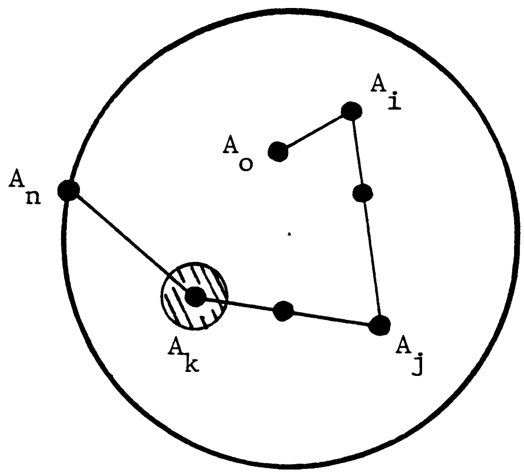


Figure 3a. The elbow at A_j enables A_k to reach the points in the shaded area while the location of A_i remains fixed.

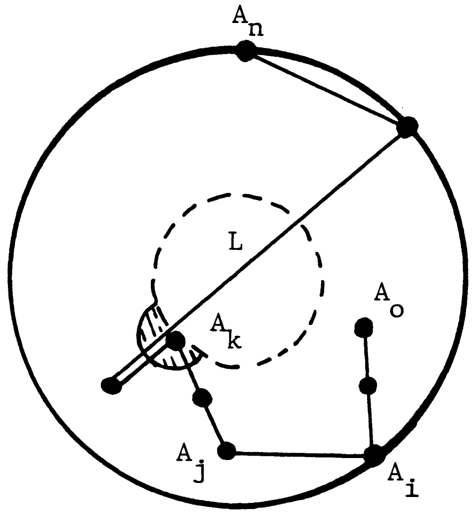


Figure 3b. Link L is so long that A_k cannot reach any points inside the dashed circle.

<u>Proof</u>: Note that the distance between A_i and A_k can be both increased and decreased by adjusting the angle at A_j. Simultaneously, the entire configuration of links between A_i and A_k can be rotated about A_i, provided that the links beyond A_k do not prevent this motion. But [1] showed that the links beyond a given joint never constrain its motion along any path that stays within the minimum and maximum distances that the joint can be located off C. $\sqcup$

Another basic observation is that a fold can sometimes be turned into an elbow.

Lemma 2: Suppose that u and v are two joints of an arm enclosed in circle C and that all joints between u and v are straight with one exception, x, which is a folded joint not lying on C. If the lines of links $\lfloor xu \rfloor$ and $\lfloor xv \rfloor$ from x to u and x to v are not equal in length and if the longer contains at least two links, then an elbow can be created at x without changing the locations of u and v. If the lines $\lfloor xu \rfloor$ and $\lfloor xv \rfloor$ (possibly of equal length) *each* contain at least *two* links, then again, x can be turned into an elbow without moving u and v. (See Figure 4.)

Proof: If $|\lfloor xu \rfloor| = |\lfloor xv \rfloor|$, the second statement is obvious. If $|\lfloor xu \rfloor| \neq |\lfloor xv \rfloor|$, let the line $\lfloor xu \rfloor$ be the longer one, and let y be a joint between x and u. With the locations of u and v fixed, the line $\lfloor yu \rfloor$ can be rotated about u, forcing y to move *away* from v. This can be done because y can be moved away from v by opening x, and at the same time, the configuration between v and y can be rotated about v to keep y on a circle of radius $|\lfloor yu \rfloor|$ centered at u. □

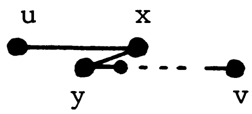
a) $|[xy\rfloor| \le |\lfloor xv \rfloor|$ b) $|\lfloor xy \rfloor| > |\lfloor xv \rfloor|$

Figure 4. Creating an elbow at x.

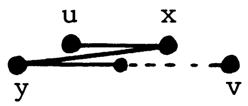
A final basic observation is that an elbow can be created from two folds that are joined by a straight line of links unless the line consists of a single "long" link.

<u>Lemma 3</u>: Let u and v be joints of an arm embedded in a circle C. Suppose all joints strictly between u and v are straight with two exceptions, which are folds. Then the locations of u and v can be held fixed while the arm is moved to create an elbow between u and v unless the folds are joined by a single link that is at least as long as the sum of the lengths of all the other links between u and v.

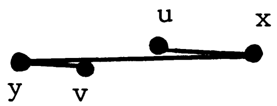
<u>Proof</u>: Let x and y be the folds, and let u, x, y, v be the order of the joints in the arm. First, suppose that $|\lfloor ux \rfloor| \geq |\lfloor xy \rfloor|$. (See Figure 5a.)



a) $|\lfloor ux \rfloor| \geq |\lfloor xy \rfloor|$



b) $|\lfloor ux \rfloor| < |\lfloor xy \rfloor|$ and $|\lfloor xy \rfloor| < |\lfloor ux \rfloor| + |\lfloor yv \rfloor|$



c) $|\lfloor xy \rfloor| \geq |\lfloor ux \rfloor| + |\lfloor yv \rfloor|$

Figure 5. A Pair of folds. An elbow can be created between u and v except in c) when $\lfloor xy \rfloor$ is a single link.

Then the locations of u and v can be kept fixed, and also all angles except those at u, x, y, and v can be kept fixed, while $\lfloor yv \rfloor$ is rotated about v. This is because y will be moving <u>away</u> from u, which can be accomplished by opening the folded joint x into an elbow. The configuration between u and y can be rotated about u to give y the proper angular displacement with respect to u and v.

Now suppose that $|\lfloor ux \rfloor| < |\lfloor xy \rfloor|$ and that $|\lfloor xy \rfloor| < |\lfloor ux \rfloor| + |\lfloor yv \rfloor|$. (See Figure 5b.) Then while the locations of u and v are held fixed, $\lfloor ux \rfloor$ can be rotated about u, which moves x <u>away</u> from v, by unfolding y into an elbow and rotating the configuration between v and x about v.

The only remaining possibility is that $|\lfloor xy \rfloor| \geq |\lfloor ux \rfloor| + |\lfloor yv \rfloor|$, as shown in Figure 5c. Then an elbow can be created unless x and y are joined by a single link. $\sqcup$

<u>Corollary</u>: Let u and v be joints of an arm embedded in a circle. Suppose all joints strictly between u and v are straight joints or folds. If there are three or more folds between u and v, then the locations of u and v can be held fixed while the arm is moved to create an elbow between u and v.

<u>Proof</u>: Let e, f, and g be the first three folds past u between u and v. Let h be the next fold past g if one exists; otherwise, let h be v. If an elbow cannot be created between u and v, then Lemma 3 applied to u, e, f and g shows that $|\lfloor ef \rfloor| > |\lfloor fg \rfloor|$. But Lemma 3 applied to e, f, g and h shows that $|\lfloor fg \rfloor| > |\lfloor ef \rfloor|$, a contradiction. $\sqcup$

4. Segments

Now we are ready to continue studying the boundary of S_m, $m>0$. In this section, we show that when a configuration of the arm places A_m on the boundary of S_m but not on a basic circle, the part of the arm between A_o and A_m divides into an initial segment reaching from A_o to C, possibly some intermediate segments between joints on C, and a final segment reaching from a joint on C to A_m. The intermediate segments consist of straight lines of links, but the initial and final segments may each contain a joint that is not straight. The next theorem shows that there must be a joint between A_o and A_m that lies on C.

<u>Theorem 2</u>: If the arm has been moved so that A_m lies on the boundary of S_m but does not lie on a basic circle, then some joint strictly between A_o and A_m must lie on circle C.

<u>Proof</u>: Suppose that there is no joint A_i, $0<i<m$, that lies on C. Since A_m is on the boundary of S_m but not at its minimum or maximum distance from C, Lemma 1 implies that there can be no elbows between A_o and A_m, and the corollary to Lemma 3 implies that there can be no more than two folds. There cannot be exactly two folds between A_o and A_m, since it follows from Lemma 3 that A_m would be as close as possible to A_o, contradicting the assumption that A_m is not on a basic circle. There cannot be exactly one fold between A_o and A_m as the longer straight line of links would be either a single link, putting A_m on a basic circle, or a multiple-link line, which according to Lemma 2 allows the formation of an elbow. On the other hand, if all the joints between A_o and A_m were straight, A_m would be on a basic circle. □

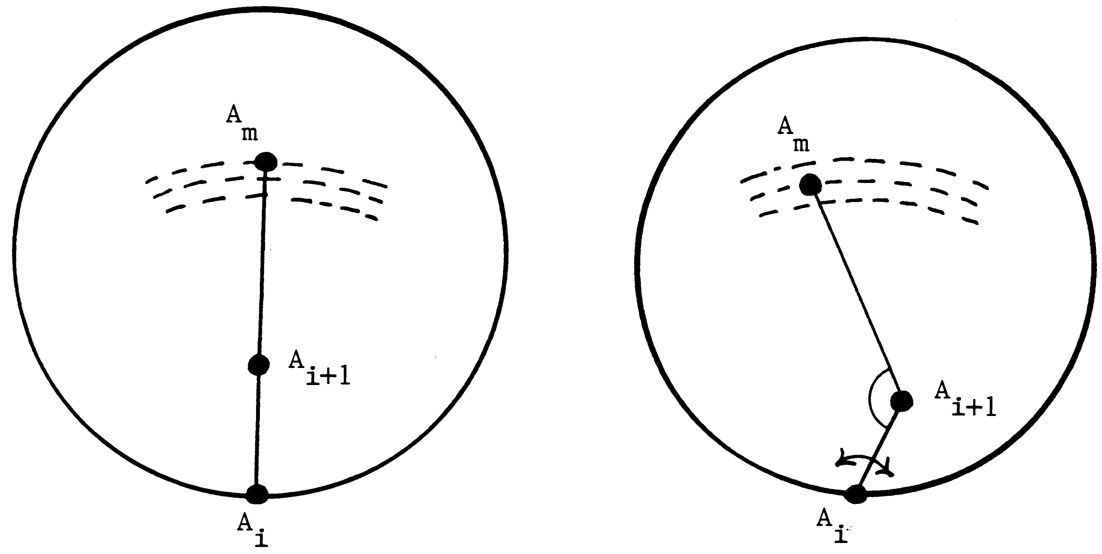
Suppose that the arm has been moved so that A_m is on the boundary of S_m but not on a basic circle. Then A_m is not on C but, according to Theorem 2, we can find some last joint between A_o and A_m that is on C. We will say that the links between this last joint and A_m form the <u>final segment of the configuration before</u> A_m, or simply, the <u>final segment</u>. Similarly, we will say that the links between A_o and the first joint <u>beyond</u> A_o on C form the <u>initial segment</u> of the configuration. (Here, A_o may or may not be on C.)

It is clear from Lemmas 1, 2, and 3 that <u>the final segment is made up of either a straight line of links from a joint on C to</u> A_m <u>or a single link from a joint on C to a fold that is followed by a straight line of links to</u> A_m. In either case, the final segment lies along a line. The next lemma will help us to show that if A_m is on the boundary of S_m but not on a basic circle, then the configuration of the arm has no elbows anywhere before A_m and only two joints that might be folds. Also, it will enable us to treat the possibility that the final segment lies on a diagonal of C as a special case.

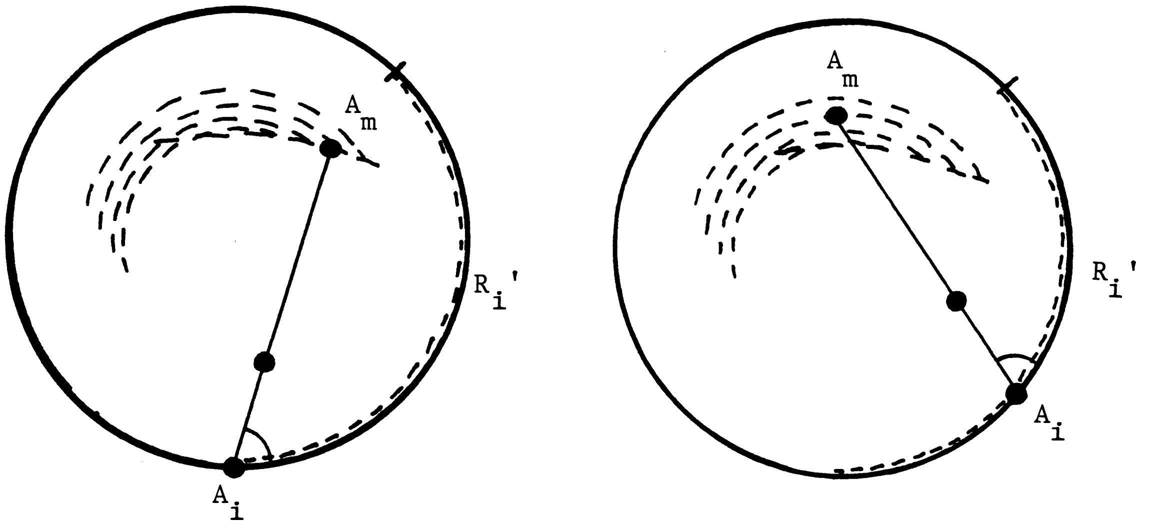
<u>Lemma 4</u>: Suppose that an arm has been moved to a position that places A_m on the boundary of S_m but not on a basic circle. If the final segment before A_m begins at A_i on C and lies on a diagonal of C, then R_i', the arc of $S_i \cap C$ to which A_i belongs, consists of a single point.

<u>Proof</u>: Since A_m does not lie on a basic circle, A_m cannot be located at 0. For the same reason, if A_m lies on the opposite side of 0 from A_i, the final segment must be a multiple-link straight line. But then A_m cannot be on the boundary of S_m unless R_i' is a single point. To see this, note that the distance between A_m and C can be decreased by rotating the final segment about A_i and that it can be increased by bending a joint in the final segment. (See Figure 6a.) Also, $[A_iA_m]$ can be rotated about A_i, and A_i can be repositioned along R_i' to make A_m sweep out arcs of circles. (See Figure 6b.) Taken together, these circular arcs cover an open ball centered at the original position of A_m unless R_i' is a single point. (It may not be possible to slide A_i along R_i' without removing A_i from C. Nevertheless, A_i can be repositioned anywhere along R_i', and the orientation of $[A_iA_m]$ can be restored to achieve the same effect. See [1].)

Suppose A_m lies between A_i and 0 and that the final segment is a single link. If R_i' is not a single point, then A_m can be moved along arcs of circles centered at 0 and having radii at least $r-|[A_iA_m]|$. (See Figure 7a.) This is because $[A_iA_m]$ can be rotated about A_i to move A_m farther from 0, and then A_i can be repositioned along R_i' with $[A_iA_m]$ in its new angular position. To contradict the assumption that A_m is on the boundary of S_m, it remains only to



a) A_m can be moved to points on the dashed arcs by bending A_{i+1} and then rotating the configuration between A_i and A_m back and forth about A_i while keeping the location of A_i fixed.



b) $\lfloor A_iA_m \rfloor$ can be rotated about A_i while the location of A_i remains the same. Then if A_i is not at the ccw endpoint of R_i', A_i can be repositioned ccw of its initial position while $\lfloor A_iA_m \rfloor$ is held at some fixed angle. In this way, A_m can reach the points on the dashed arcs.

Figure 6. A final segment that is a multiple-link straight line on the diagonal with 0 between A_i and A_m, where R_i' extends counterclockwise (and possibly clockwise) from the initial location of A_i.

show that A_m can reach the neighbors of its initial location that lie outside the circle of radius $|\lfloor A_iA_m \rfloor|$ centered at the initial location p_i of A_i.

To prove that A_m can reach its neighbors outside the circle centered at

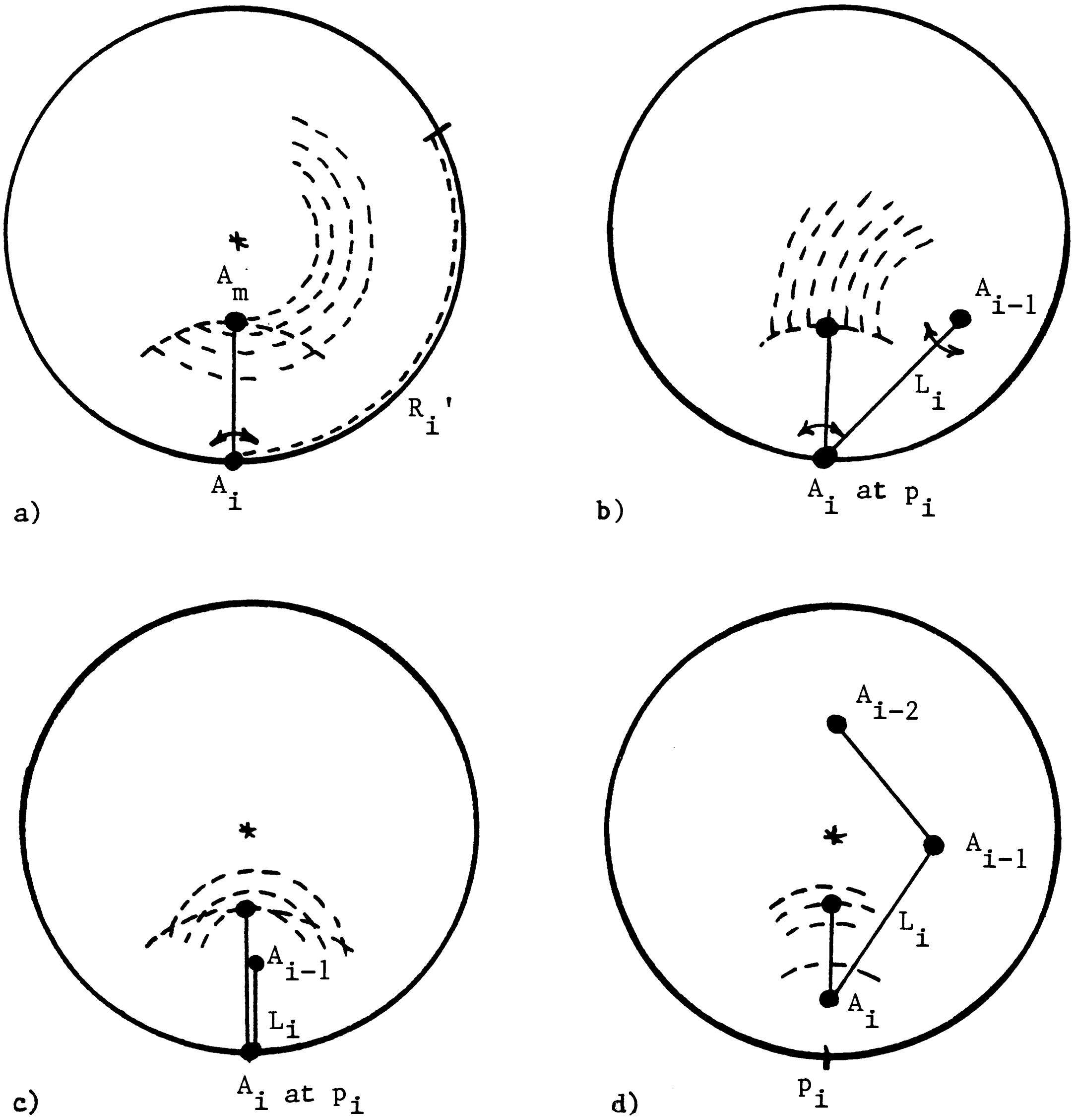


Figure 7. A_m lies between A_i and 0, and the final segment is a single link.

p_i, we now consider the possibilities for the configuration of L_{i-1} and L_i. If L_i does not lie on the diagonal, or if L_i does lie on the diagonal but has length no greater than $|\lfloor A_iA_m \rfloor|$, then $\lfloor A_iA_m \rfloor$ can be rotated about A_i while the configuration between A_{i-1} and A_m is rotated about A_{i-1}. In this way, A_m can move along arcs of circles centered at A_{i-1}, as shown in Figures 7b and 7c.

Now we can assume that L_i lies on the diagonal and that $\mathbf{l}_i > |\lfloor A_iA_m \rfloor|$. Since we are assuming that A_m is not on a basic circle, $A_{i-1} \neq A_o$ and $\mathbf{l}_i \neq d$;

otherwise, A_m would be as close as possible to A_o or to 0, respectively. If A_{i-1} is an elbow, then the configuration between A_{i-2} and A_m can be rotated about A_{i-2} while $[A_iA_m]$ is rotated about A_i. This situation is similar to the one shown in Figure 7b. If A_{i-1} is a straight joint, this joint can be bent to move A_i closer to 0. If $[A_iA_m]$ is simultaneously rotated about A_i, A_m reaches its neighbors outside the circle centered at p_i, as shown in Figure 7d.

Now we can assume that A_{i-1} is a fold and lies at the end of a straight line of links $[A_xA_{i-1}]$ that begins at some predecessor A_x of A_{i-1}. By Lemmas 1 and 3, A_m is not on the boundary of S_m if A_x is a fold. If A_x is an elbow, we are again in a situation similar to the one shown in Figure 7b. We can assume that A_x does not lie on C (on top of A_i) because in that situation, we could keep the location of A_i and A_x fixed and rotate L_i and $[A_xA_{i-1}]$ off the diagonal, a situation that has already been discussed. This completes the proof of the theorem for the situation in which A_m lies between A_i and 0 and the final segment is a single link.

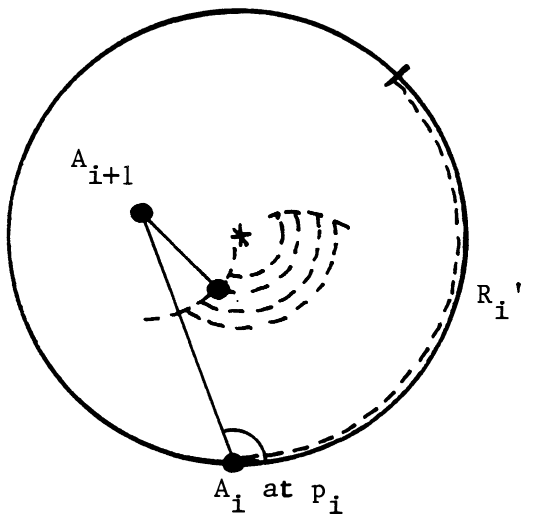


Figure 8. The final segment has a fold at A_{i+1} and places A_m between p_i and 0.

If A_m lies between A_i and 0 and the final segment is a straight line of links containing more than one link, then A_m cannot be on the boundary of S_m. This is because the arguement that we just presented for a single link shows how to reach points in a neighborhood outside the circle centered at p_i of radius $|[A_iA_m]|$ and the argument given earlier when A_m reached beyond 0 shows how to reach points closer to p_i.

Finally, suppose that A_m lies between A_i and 0 and that the final segment has a fold at A_{i+1}. If R_i' is not a single point, then A_m can be moved to reach the neighbors of its initial location. To see this, rotate the final segment about p_i to an off-diagonal position. Now rotate $\lfloor A_{i+1}A_m\rfloor$ about A_{i+1} to increase and decrease the distance between A_m and 0 while simultaneously repositioning A_i along R_i'. (See Figure 8.)

In every case, we have found that R_i' must be a single point. $\sqcup$

Now we can describe the general form of a configuration that places A_m on the boundary of S_m but not on a basic circle.

<u>Theorem 3</u>: Suppose that an arm has been moved to a configuration that places A_m on the boundary of S_m but not on a basic circle. Let A_i be the first joint beyond A_o on C, and let A_j be the last joint before A_m on C. Then all joints between A_o and A_m that do not lie on C are straight, with the possible exceptions of A_{i-1} and A_{j+1}. If A_{i-1} and A_{j+1} are not straight, then they must be folds.

<u>Proof</u>: As noted previously, Lemmas 1, 2 and 3 imply that the final segment is as described in the statement of the theorem. If an elbow appears before A_j, then it leads to a joint on C that by Lemma 1 is not at an endpoint of the arc(s) it can reach on C. This means that A_j is also at an interior

point of its arc. By Lemma 4, the final segment cannot lie on a diagonal of C. But the final segment can be rotated about A_j to both increase and decrease the distance between A_m and C. Then since A_j is at an interior point of its arc, A_j can be repositioned along C so that A_m sweeps out arcs of circles that cover a neighborhood of its original position. (See Figure 9.) This contradicts the assumption that A_m is on the boundary of S_m. Hence there are no elbows before A_m.

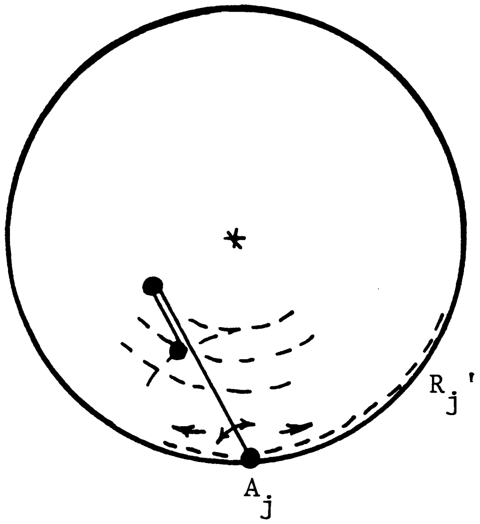


Figure 9. A_j is at an interior point of R_j', and the final segment is off the diagonal, so A_m does not lie on the boundary of S_m.

For the same reason that there are no elbows before A_m, there can be no folds connected by straight lines of equal length to superimposed joints on C. Lemma 3 and its corollary show that there cannot be two or more folds between joints on the circle or between A_o and A_i. Finally, Lemma 2 shows that if there is a fold between A_o and A_i, it must be A_{i-1}. ⊔

5. Configurations with A_m on the Boundary of S_m

The next theorem will help us to treat the situation in which the final segment lies on a diagonal of C as a special case. After that case has been handled, we will enumerate the configurations having A_m on the boundary of S_m.

Theorem 4: Suppose that an arm has been moved to a position that places A_m on the boundary of S_m but not on a basic circle and that the final segment before A_m lies on a diagonal of C. Then the initial segment consists of a single link, L_1, which is connected directly to the final segment.

Proof: By Theorem 2, there is at least one joint between A_o and A_m that lies on C. Let A_i and A_j be the first and last such joints. According to Theorem 3, A_{i-1} and A_{j+1} are the only joints off C that may not be straight, in which case they must be folds. Also, the arc of R_i containing A_i must be a single point because by Lemma 4, the arc of R_j containing A_j is a single point. This has several consequences. First, if the initial segment does not lie on a diagonal, it must consist of a single link. Second, A_o cannot be at 0. Third, if the initial segment lies on a diagonal of C, it cannot be a multiple-link straight line longer than r, the radius of C, nor can it have A_o positioned between A_i and 0 and a fold at A_{i-1}.

Assume that the initial segment lies on the diagonal. If 0 lies between A_i and A_o, then the initial segment cannot have a fold at A_{i-1} or be a single link because A_i would be fixed, contradicting the assumption that A_o is the last fixed joint. Since we have already ruled out a multiple-link straight line longer than r, we conclude that 0 cannot lie between A_i and A_o. Thus if the initial segment lies on the diagonal, A_o must lie between A_i and 0. In this situation we have already ruled out the possibility that A_o is at 0 and the possibility that the initial segment has a fold at A_{i-1}. Hence if the initial segment lies on the diagonal, it must be a straight line of links of length less than r.

Whether the initial segment is a line of links lying on a radius of C or a single off-diagonal link, A_j (the last joint before A_m on C) must be A_i. Otherwise, the first intermediate segment would have to be a diagonal chord in order to prevent its second endpoint from being able to move on C. In fact, this diagonal chord would have to be the single link L_{i+1}, but then A_{i+1} would be a fixed joint.

To prove the theorem, we now need only to rule out the possibility that the initial segment is a multiple-link straight line lying on a radius. Suppose, for the purpose of contradition, that this is the case. If the final segment had a fold at $A_{j+1} = A_{i+1}$, then Lemma 3 could be applied even though A_j lies on C, because A_j cannot move beyond C even when C is removed. But Lemma 3 would imply that A_m lies on a basic circle. It is obvious that the final segment could not be a multiple-link straight line. If the final segment consisted of a single link $L_{j+1} = L_m$, A_m would lie on a basic circle if L_{j+1} reached as far as A_o, and on the other hand, L_{j+1} could be rotated about A_m to form an elbow if L_{j+1} did not reach to A_o. This rules out all possibilities for the final segment. Hence the initial segment consists of one link if it lies on a diagonal. $\sqcup$

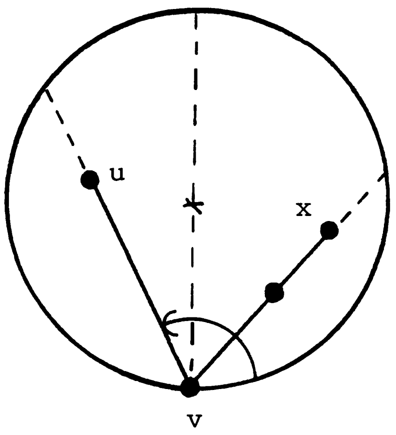
It follows from Theorem 4 that if the final segment lies on a diagonal of C, then A_m lies on one of at most four "supplementary" circles that we are about to describe. Note that in this situation, there are at most two possible locations for A_1, corresponding to the two possible orientations for L_1 (see [1] for the definition of orientation). Then, for a fixed position of A_1, A_m lies on a circle centered at A_1 of radius either $\sum_{j=2}^{m} l_j$ or, when positive, $l_2 - \sum_{j=3}^{m} l_j$. This defines at most four circles, which we call <u>supplementary</u>.

From now on, we assume that A_m is neither on a basic circle nor on a supplementary circle so that we need only concern ourselves with situations in which *the final segment before A_m does not lie on a diagonal of C.*

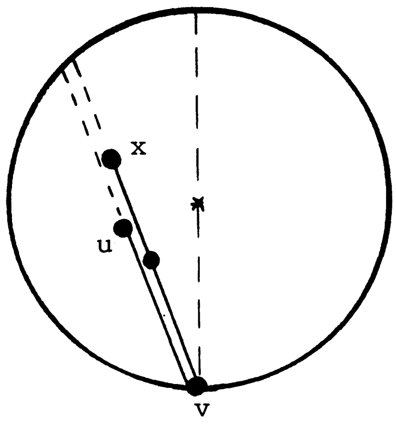
In the next lemma, we list several configurations that can be moved to form elbows without ever changing the location of their endponts. Thus by Theorem 3, these configurations cannot occur between A_o and A_m if A_m is on the boundary of S_m but not on a basic circle. Then we will use these forbidden configurations in enumerating the possibilities for the whole configuration of the arm from A_o to A_m.

In what follows, we will use the expression xy to denote the infinite line determined by points or joints x and y. As before, $\lfloor xy \rfloor$ will denote a straight line of links connecting joints x and y.

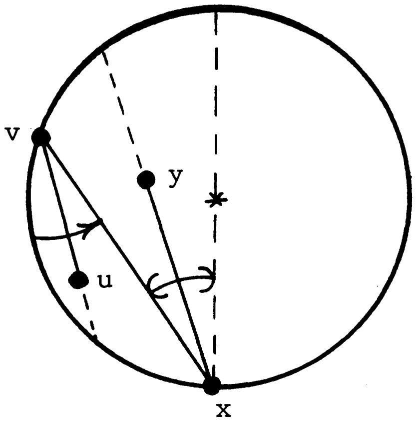
Lemma 5: In each of the configurations shown in Figure 10, the locations of the two endpoints can be kept fixed while the configuration is moved to form an elbow.



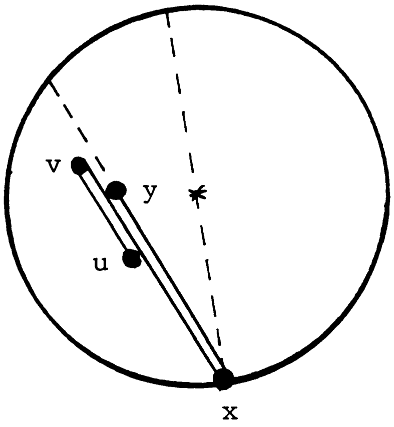
Ia. $\lfloor uv \rfloor$ lies off the diagonal.



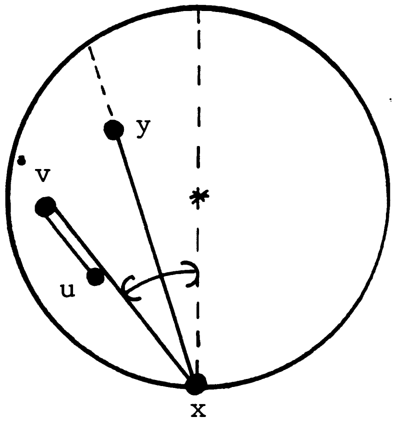
Ib. $\lfloor uv \rfloor$ lies off the diagonal, v is folded, and $|\lfloor xv \rfloor| > |\lfloor uv \rfloor|$ if $\lfloor uv \rfloor$ is a single link.



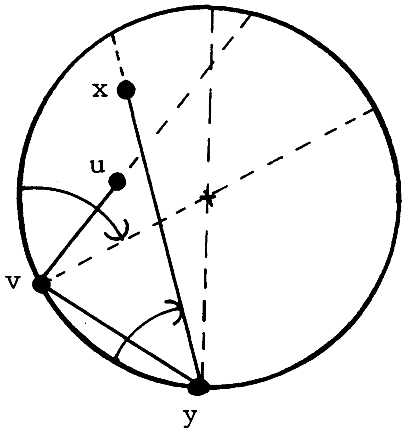
IIa. $\lfloor vx \rfloor$ lies off the diagonal.



IIb. $\lfloor vx \rfloor$ lies off the diagonal, v and x are folded, and y lies beyond u.



IIc. $\lfloor vx \rfloor$ lies off the diagonal, and v is folded.



III. $\lfloor xy \rfloor$ lies off the diagonal.

Figure 10. Configurations that give rise to elbows. Arrows indicate the angular range for a line of links. A sharp tip indicates that the endpoint of the arc belongs to the range, and a round tip indicates that it does not. No order is implied by the letters at joints: u could come before or after v. There may be additional joints between the ones that appear in the figure. A dashed extension of a link indicates that its endpoint may lie on C.

<u>Proof</u>: In Ia and Ib, the locations of u and x can be held fixed while $\lfloor uv \rfloor$ is rotated about u. This requires that v move closer to x, which can be accomplished by creating an elbow between v and x. In Ib, if $\lfloor uv \rfloor$ contains more than one link, v can be moved straight toward u and x by creating an elbow between x and v and between u and v. Hence $\lfloor xv \rfloor$ need not be longer than

$\lfloor uv \rfloor$ in this case.

In IIa, b, and c, the locations of u and y can be held fixed while $\lfloor xy \rfloor$ is rotated about y. This requires that x move away from u, which can be accomplished by opening the joint at v while simultaneously rotating $\lfloor uv \rfloor$ about u.

In III, the locations of u and x can be held fixed while $\lfloor xy \rfloor$ is rotated about x. If u and 0 lie on opposite sides of xy (the infinite line determined by x and y), then y must move away from u. This can be accomplished by opening the joint at v while simultaneously rotating $\lfloor uv \rfloor$ about u. If u and 0 lie on the same side of xy, then y must move toward u. This can be accomplished by closing the joint at v while simultaneously rotating $\lfloor uv \rfloor$ about u. If u lies on xy between x and y, at x, or on the opposite side of x from y, then y must either move away from u, remain at the same distance from u, or move closer to u, respectively. The important point is that the required rotation of $\lfloor uv \rfloor$ about u moves v closer to 0. $\sqcup$

Lemma 5 gives configurations that cannot appear between A_o and A_m in any configuration of the arm that places A_m on the boundary of S_m but not on a basic circle. We now use this information to enumerate the possibilities for the configuration of the arm from A_o to A_m. Then from this list we will be able to determine the additional circles that are needed to cover the boundary of S_m. It is important to note that in Figure 10, u can come before or after x and y. For example, u can correspond to A_m as well as to A_o.

According to the notation we have been using, the expression $\lfloor xy \rfloor$ denotes a straight line of links between x and y whereas the expression wx denotes an infinite line (not necessarily containing any links) through w and x. Suppose the line wx passes through C. We will say that $\lfloor xy \rfloor$ lies <u>under</u> wx if $\lfloor xy \rfloor$ and 0 are on opposite sides of wx and that $\lfloor xy \rfloor$ lies <u>above</u> wx if 0 and $\lfloor xy \rfloor$ lie on the same side of wx. (See Figure 11.)

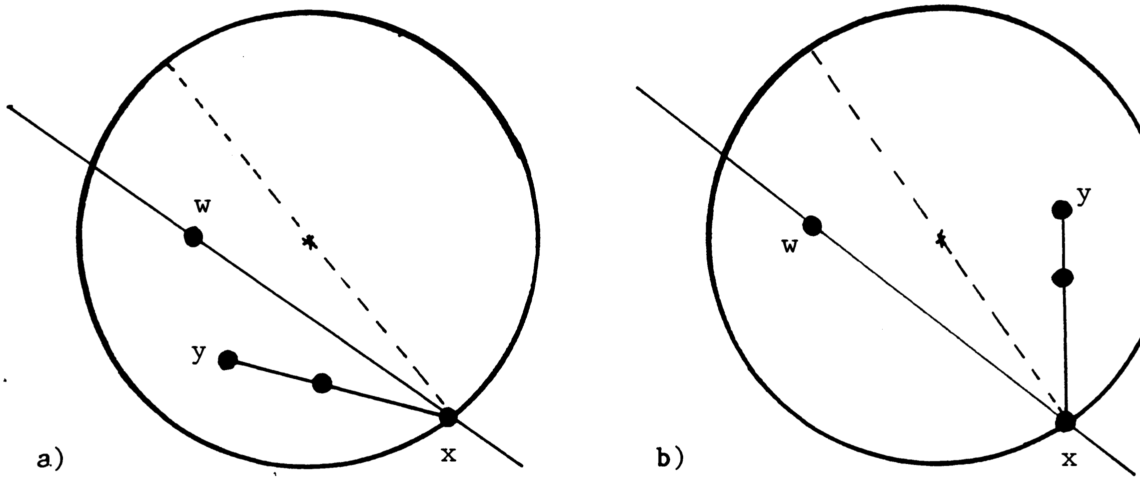
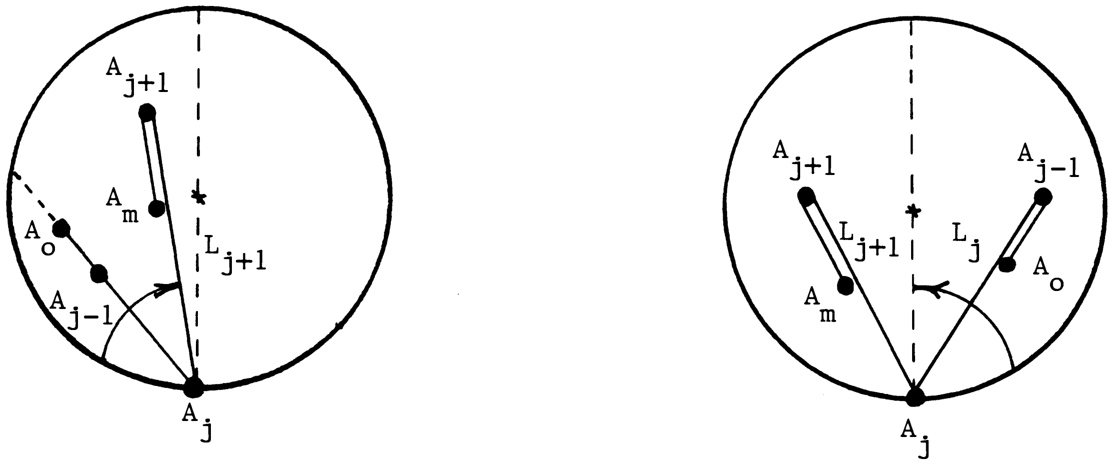


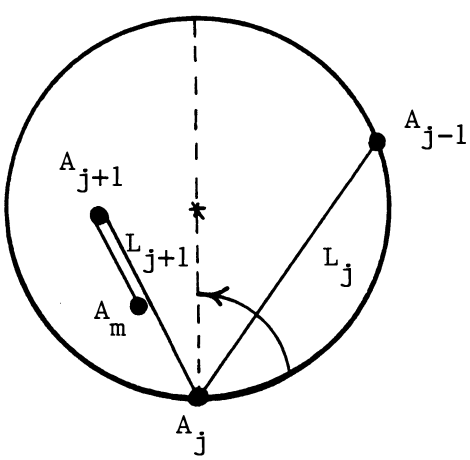
Figure 11. $\lfloor xy \rfloor$ lies "under" line wx in a) and "above" wx in b).

<u>Theorem 5</u>: Suppose an arm has been moved to a configuration that places A_m on the boundary of S_m, but not on a basic or supplementary circle. Then if the final segment of the arm before A_m contains more than one link, it contains one of the configurations shown in Figure 12.

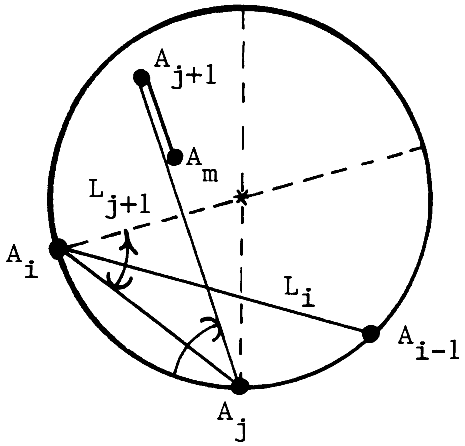


Configuration 1: L_{j+1} lies off the diagonal, and A_{j+1} is folded. If $\lfloor A_o A_j \rfloor$ lies on L_{j+1}, then A_o does not reach A_m.

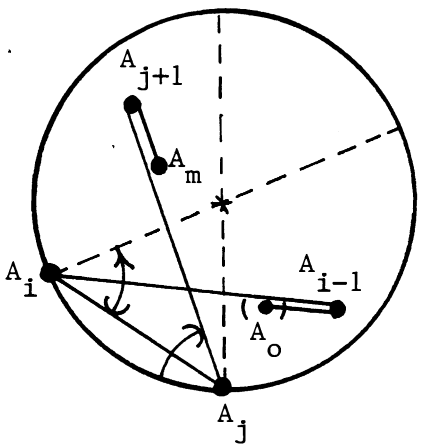
Configuration 2: L_{j+1} lies off the diagonal



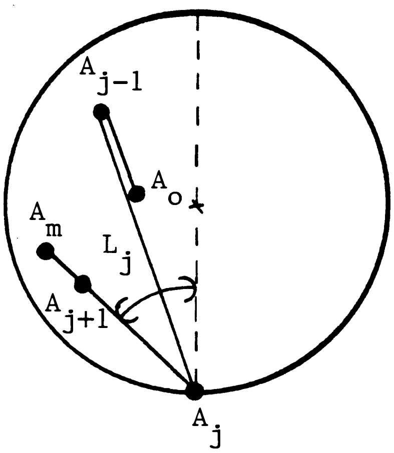
Configuration 3



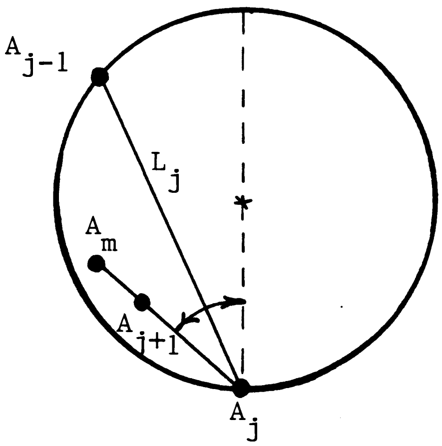
Configuration 4



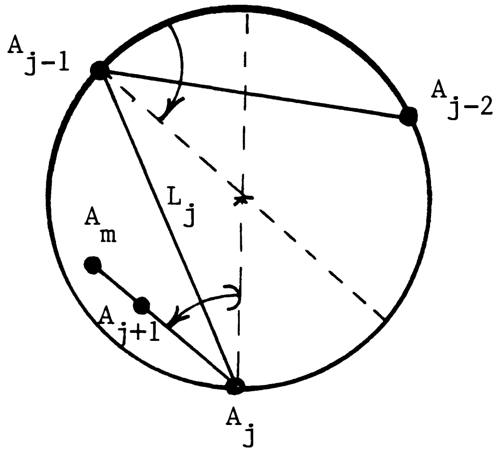
Configuration 5: Either A_{i-1} is A_o or A_{i-1} folds back to A_o.



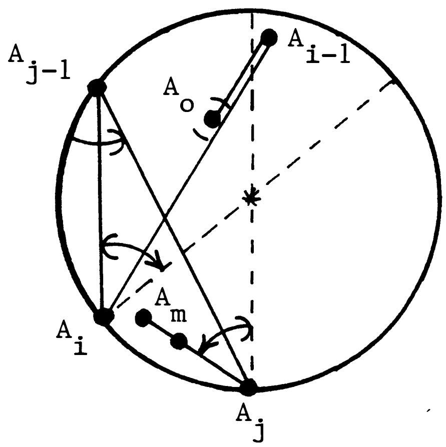
Configuration 6



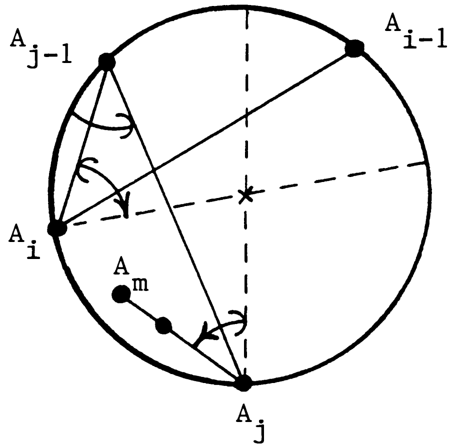
Configuration 7: Either L_j or L_{j-1} is the last link before A_m with length = $\max(l_1,\dots,l_m)$.



Configuration 8



Configuration 9: Either A_{i-1} is A_o or A_{i-1} is a fold leading to A_o.



Configuration 10

Figure 12. The final segment contains more than one link. Notation is the same as for Figure 10.

__Proof__: Since A_m does not lie on a basic circle, A_m does not lie on C. Also, Theorem 2 implies that there is some joint strictly between A_o and A_m that lies on C. Let A_j be the last such joint. Since A_m does not lie on a supplementary circle, $A_j \neq A_1$, so $1<j<m$. This also means, according to Theorem 4, that the final segment before A_m cannot lie on a diagonal of C.

We will break the proof into two parts. First we will assume that the final segment before A_m has a fold at A_{j+1}, and second we will assume that the final segment is a straight line $\lfloor A_j A_m \rfloor$ consisting of more than one link. (Note that the hypothesis of the theorem rules out the possibility of the final segment consisting of a single link.)

__Proof for the case of a fold at__ A_{j+1} __in the final segment__:

First suppose that A_j is the __only__ joint strictly between A_o and A_m that lies on C. Configuration 1 of Figure 12 covers the situation in which A_o is connected to A_j by a straight line $\lfloor A_o A_j \rfloor$, where by assumption $j>1$. This is because configuration Ia of Figure 10 cannot be a configuration in the arm, so

$\lfloor A_oA_j \rfloor$ must lie either on or beneath the infinite line A_jA_{j+1}. If $\lfloor A_oA_j \rfloor$ lies on A_jA_{j+1}, then by IIb, A_m does not lie between A_o and A_j. Also, A_o does not coincide with A_m since A_m does not lie on a basic circle. Configuration 2 of Figure 12 covers the situation in which A_{j-1} is a fold. This is because configurations IIb and IIc are forbidden, so L_j must lie on the diagonal or on the opposite side of the diagonal from L_{j+1}. (Note that either A_o or A_m could correspond to u in configurations IIb and IIc.)

Next, suppose that A_j is not the only joint between A_o and A_m on C, so that for some i, $0<i<j$, $\lfloor A_iA_j \rfloor$ is a chord of C. Since IIb is a forbidden configuration, $\lfloor A_iA_j \rfloor$ cannot lie on A_jA_{j+1}. Configuration 3 covers the situation in which $\lfloor A_iA_j \rfloor$ lies "above" A_jA_{j+1}. This is because IIc is forbidden, so $\lfloor A_iA_j \rfloor$ cannot lie in the open wedge bounded by A_jA_{j+1} and the infinite line $0A_j$, and Ia is forbidden, so $\lfloor A_iA_j \rfloor$ must be a single link. Configurations 4 and 5 cover the situation in which $\lfloor A_iA_j \rfloor$ lies "under" line A_jA_{j+1}. To see this, note that A_{i-1} and A_j cannot lie on opposite sides of $0A_i$ because III is forbidden, and L_i cannot lie above A_iA_j because IIa is forbidden. Then note that A_{i-1} can't be a straight joint because Ia is forbidden. Therefore, either $A_{i-1} = A_o \vee A_{i-1}$ is a fold, as in configuration 5, or A_{i-1} lies on C, as in configuration 4.

<u>Proof for the case of a multiple-link straight line final segment</u>:

Because Ia and Ib are forbidden, A_j cannot be immediately preceded by a multiple-link straight line. Hence either A_{j-1} is a fold or A_{j-1} lies on C. Let us consider these possibilities separately.

Configuration 6 covers the situation in which A_{j-1} is a fold. This is because L_j must lie in the closed wedge bounded by A_jA_m and $0A_j$ as Ia is forbidden, and furthermore, as we are about to see, L_j cannot lie on A_jA_m or on $0A_j$. If L_j lay on A_jA_m, then A_m would have to lie on A_o or between A_o and A_j because IIb is forbidden. But this would place A_m on a basic circle. If L_j lay on $0A_j$ with 0 between A_{j-1} and A_j, A_j would be fixed, and if A_{j-1} lay at 0 or between A_j and 0, A_j would not lie at the endpoint of an arc of R_j, implying that A_m did not lie on the boundary of S_m.

Several configurations are needed to cover the situation in which A_{j-1} lies on C. Certainly L_j lies in the closed wedge bounded by A_jA_m and $0A_j$ in that situation, as Ia is forbidden. There are two cases to consider, namely whether A_{j-1} is, or is not, the first joint past A_o on C.

Configuration 7 covers the situation in which A_{j-1} is the first joint past A_o on C. The possibility that the initial segment is a single link or has a fold at A_{j-2} is clearly covered. The only remaining possibility is that the initial segment is a multiple-link straight line $\lfloor A_oA_{j-1}\rfloor$. In that case $\lfloor A_oA_{j-1}\rfloor$ must lie on or below L_j, because Ia is forbidden. Again, configuration 7 applies.

Now suppose that A_{j-1} is not the first joint past A_o on C so that for some A_i, where $j-1>i>0$, $\lfloor A_iA_{j-1}\rfloor$ is a chord. Configuration 7 covers the possibility that L_j is a diagonal chord, so we may assume that L_j is off the diagonal from now on.

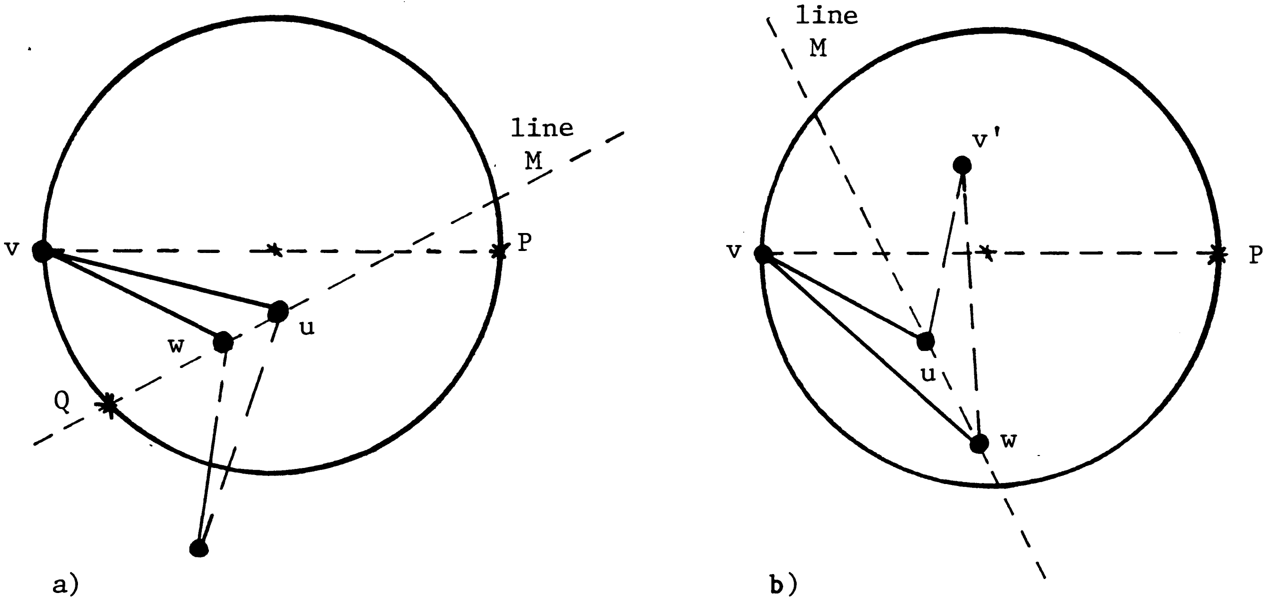
If chord $\lfloor A_iA_{j-1}\rfloor$ lies above L_j, then it must be a single link since Ia is forbidden, and furthermore, this link cannot lie in the open wedge bounded by L_j and $0A_{j-1}$ because IIa is forbidden. Hence, configuration 8 covers the situation in which $\lfloor A_iA_{j-1}\rfloor$ lies above L_j.

Chord $\lfloor A_iA_{j-1}\rfloor$ cannot lie on top of L_j because L_j and $\lfloor A_iA_{j-1}\rfloor$ could be rotated about the coincident joints A_j and A_i to form a fold at A_{j-1}, violating Theorem 3.

The last possibility is that $\lfloor A_i A_{j-1} \rfloor$ lies beneath L_j. Since i>0, there is a link L_i to consider. Since IIa is forbidden, L_i cannot lie on or beneath chord $\lfloor A_i A_{j-1} \rfloor$. Therefore, since III is forbidden, L_i must lie either on $0A_i$ or in the open wedge bounded by $0A_i$ and $\lfloor A_i A_{j-1} \rfloor$. Since Ia is forbidden, A_{i-1} cannot be straight, so either A_{i-1} is a fold and configuration 9 applies, or A_{i-1} lies on C and configuration 10 applies. ⊔

The basic idea for handling configurations 1-10 of Theorem 5 is this: In each case, we will show that there are only a constant number (independent of the arm) of possibilities for A_j, the last joint before A_m on C. Then a constant number of circles, at most 8 for each choice of A_j, can be added to the basic and supplementary circles to form a collection that covers the boundary of S_m. This is because A_m must lie on a circle of radius either $\sum_{k=j+1}^{m} l_k$ or $l_{j+1} - \sum_{k=j+2}^{m} l_j$ about A_j, and A_j must lie at one of the endpoints of R_j, of which there are at most four. The possibilities for A_j will be determined by inequalities involving the link lengths that can only be satisfied in a few ways. We have already seen a simple example of this in configuration 7. There, either A_j is the higher numbered endpoint of the last link of longest length before A_m, v A_j is the next joint after that endpoint. For several of the configurations, somewhat more complicated length inequalities will generate the possibilities for A_j. The next three lemmas will be used to find the additional inequalities that are needed.

Lemma 6: Consider a configuration of straight lines of links $\lfloor vu \rfloor$ and $\lfloor vw \rfloor$ joined at v and constrained to move inside circle C, where v is on C, vu does not lie on the diagonal through v, and $\lfloor vw \rfloor$ lies beneath vu. Joints u and w may or may not lie on C. (Figure 13 shows that the mirror image of $\lfloor vu \rfloor$ and $\lfloor vw \rfloor$ with respect to the line through u and w need not lie inside C.) If $|\lfloor vw \rfloor| \geq 2|\lfloor vu \rfloor|$, then the location of either endpoint of the configuration can be kept fixed while $\lfloor vu \rfloor$ and $\lfloor vw \rfloor$ are moved to their mirror images with respect to the line determined by the initial locations of u and w. During this motion, the distance from the moving endpoint to C can be kept within its initial value.



a) b)

Figure 13. $\lfloor uv \rfloor$ lies off the diagonal. In a), the mirror image falls outside C whereas in b), the image lies inside C.

<u>Proof</u>: It is easy to see that the mirror image of the configuration with respect to the line uw lies within the circle with v off C if, and only if, v and 0 lie on opposite sides of uw, so that uw intersects 0v at a point strictly between v and 0. We will first show that $|\lfloor vw \rfloor| \geq 2|\lfloor vu \rfloor|$ forces this situation to occur, and then we will show that the configuration can be moved to its mirror image while either endpoint is held fixed and the other is kept within the proper distance from C.

Assume now that $|\lfloor vw \rfloor| \geq 2|\lfloor vu \rfloor|$. We want to show that uw intersects 0v at a point strictly between v and 0. For notational convenience, we will think of 0 as lying at the origin of a Cartesian coordinate system with v on the negative horizontal axis at $(-r,0)$. P will denote the point $(r,0)$. Note that 0v is the horizontal axis.

Suppose that uw has non-negative slope. If it intersects the horizontal axis 0v, it does so to the right of v. Then if uw is parallel to 0v or if it intersects 0v at P or to the right of P, the perpendicular from v to uw meets uw on or outside C. Consequently $|\lfloor vw \rfloor| < |\lfloor vu \rfloor|$, a contradiction.

Consider a line M with positive slope that intersects 0v at 0 or between 0 and P. If u and w are placed on this line to minimize the ratio of $|\lfloor vu \rfloor|$ to $|[vw\rfloor|$, then u lies at the intersection of M with the perpendicular from v and, since vw lies under vu, w lies at the point Q where M meets C. (See Figure 13.) Hence we want to minimize the sine of the angle between M and vQ, where Q is on C. For any given Q, the optimum choice for M is the vertical line through Q if Q is in the fourth quadrant, and is the line 0Q if Q is in the third quandrant. Among the vertical lines and the lines through 0, the minimum ratio is achieved by the vertical line through 0. Hence, the ratio of $|\lfloor vu \rfloor|$ to $|\lfloor vw \rfloor|$ is at least $1/\sqrt{2}$, so $|\lfloor vw \rfloor| \leq \sqrt{2}\,|\lfloor vu \rfloor|$, a contradiction.

Next, suppose that uw has negative slope, so it intersects 0v to the left of P. If the intersection lies outside C, then $|\lfloor vu \rfloor| > |\lfloor vw \rfloor|$, a contradiction. The intersection cannot occur at v since vu and vw do not lie on top of one another. If the point of intersection lies at 0 or to the right of 0, then $|\lfloor vu \rfloor| \geq r$ and $|\lfloor vw \rfloor| < 2r$, so $|\lfloor vw \rfloor| < 2|\lfloor vu \rfloor|$, a contradiction. This completes the proof that $|\lfloor vw \rfloor| \geq 2|\lfloor vu \rfloor|$ implies that uw intersects 0v at a point strictly between v and 0 and hence, that the mirror image of the configuration lies inside C with v not on C.

Now we need to show that when $|\lfloor vw \rfloor| \geq 2|\lfloor vu \rfloor|$, the configuration can reach its mirror image inside C by motions that keep the moving endpoint within its initial distance from C.

First, suppose that w is to be held fixed during the motions. Rotate the line of links $\lfloor vu \rfloor$ about v down to $\lfloor vw \rfloor$, moving u closer to C. Then force u to retrace its path back to its original location but move $\lfloor vw \rfloor$ to the other side of $\lfloor vu \rfloor$, so that $\lfloor vw \rfloor$ rotates to its mirror image position. Note that the distance between u and C never exceeds its initial value during these motions.

Finally, suppose that u is to be held fixed during the move to the mirror image. Note that $|\lfloor vu \rfloor| < r$ since $|\lfloor vw \rfloor| \geq 2|\lfloor vu \rfloor|$. Rotate $\lfloor vw \rfloor$ about v so that w moves away from 0 down to C. Then fold joint v while simultaneously rotating the configuration about u to keep w on C. Note that v is constrained by $\lfloor vu \rfloor$ to remain within distance $2|\lfloor vu \rfloor|$ of C so that w can stay on C until v folds. Then force w to retrace its path back to its original location but move $\lfloor vw \rfloor$ to the other side of $\lfloor vu \rfloor$. $\sqcup$

<u>Lemma 7</u>: Suppose that u, v, and w are joints lying on circle C that are connected by straight lines of links $\lfloor uv \rfloor$ and $\lfloor vw \rfloor$, and that either $\lfloor vw \rfloor$ lies under $\lfloor uv \rfloor$ or v=w so that $\lfloor vw \rfloor$ contains no links. Also suppose that $\lfloor wx \rfloor$ is a straight line of links connecting w to a joint x that lies on C above $\lfloor wv \rfloor$ and on the opposite side of 0w from u. Then if $|\lfloor uv \rfloor| + |\lfloor wx \rfloor| \leq d$, the configuration between u and x can be moved to its mirror image with respect to the line determined by the initial locations of u and x without ever moving u or removing x from C. (See Figure 14.)

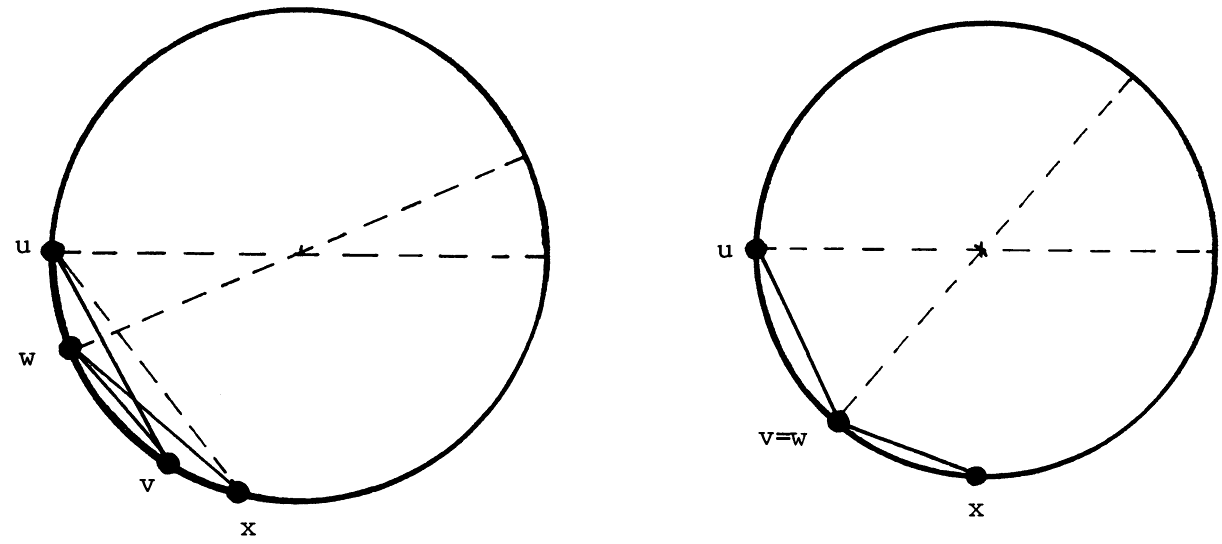


Figure 14. $|\lfloor uv \rfloor| + |\lfloor wx \rfloor| \leq d$.

<u>Proof</u>: Suppose that $|\lfloor uv \rfloor| + |\lfloor wx \rfloor| \leq d$. Then ux lies between uv and Ou, so the mirror image of the configuration lies inside C with the images of w and v off C.

The original configuration can be moved to a position in which u, w, v, and x are collinear and v and w, if distinct, are folds. This can be done in the following way. If w=v, simply straighten the joint v while keeping u fixed and moving x around C. If $w \neq v$, fold w while keeping u fixed and moving w and x around C. (This causes $\lfloor uv \rfloor$ to rotate about u.) Next, keeping w folded, fold v while moving x around C and rotating $\lfloor uv \rfloor$ about u. This brings the joints to the desired collinear configuration. The fact that $|\lfloor uv \rfloor| + |\lfloor wx \rfloor| \leq d$ guarantees that these motions can be made and that x does not cross Ou.

Note that at each moment during the motions just described, the mirror image of the configuration with respect to the moving line ux lies inside C. Hence the motions induce legal motions of the mirror image of the initial configuration that carry the mirror image to the configuration in which u, v, w, and x are collinear. Hence the mirror image of the original configuration can be reached by moving to the collinear configuration, where the joints coincide with their images, and then reversing the induced motion of the mirror image. $\sqcup$

<u>Lemma 8</u>: Suppose that u, v, w and x satisfy the hypotheses of Lemma 7, except that x need not lie on C. Then at least one of the following conditions holds:

i) $|\lfloor uv \rfloor| > r$;

ii) $|\lfloor wx \rfloor| > r$;

iii) The configuration can be moved to its mirror image with respect to the line through the initial positions of u and x while u is held fixed and x is kept within its initial distance from C.

iv) The configuration between v and x can be moved to its mirror image with respect to the line through the initial positions of v and x while u and v are held fixed and x is kept within its initial distance from C.

<u>Proof</u>: We will show that iii) or iv) holds when i) and ii) do not. Note that if $|\lfloor uv \rfloor| \le r$ and $|\lfloor wx \rfloor| \le r$, then the mirror images of w and v with respect to the line through the initial locations of u and x lie strictly inside C.

First, suppose that $|\lfloor wx \rfloor| > |\lfloor wv \rfloor|$. Then $\lfloor wx \rfloor$ can be rotated about w so that x moves away from 0 until x reaches C. Note that this induces a legal motion of the mirror image of the configuration. By Lemma 7, the mirror image of the configuration between u and x with x on C can be reached. Finally, the reverse of the induced motions can be applied to carry the configuration the rest of the way to its initial mirror image.

Next, suppose that $|\lfloor wx \rfloor| \le |\lfloor wv \rfloor|$ and that $\lfloor wx \rfloor$ crosses $\lfloor uv \rfloor$. Rotate $\lfloor wx \rfloor$ about w until x reaches $\lfloor uv \rfloor$, and note that the mirror image with respect to ux stays inside C. Hence it suffices to show that the present configuration with x located on $\lfloor uv \rfloor$ can be moved to its mirror image with respect to $\lfloor uv \rfloor$. But u, v, and x already coincide with their mirror images, so we only need to move w to its image. To do this, fold w, moving x to $\lfloor vw \rfloor$. Then make x retrace its path, but with $\lfloor wx \rfloor$ crossed over to the other side of $\lfloor vw \rfloor$ so that w will reach the image of its initial position.

Finally, suppose that $|\lfloor wx \rfloor| \le |\lfloor wv \rfloor|$ and that $\lfloor wx \rfloor$ does not cross $\lfloor uv \rfloor$. Rotate $\lfloor wx \rfloor$ about w so that x moves closer to C until x reaches $\lfloor wv \rfloor$. Then force x to retrace its path, but with $\lfloor wv \rfloor$ crossed over to the other side of $\lfloor wx \rfloor$. $\sqcup$

The heart of the proof of the next theorem, which is the main result of this section, uses Lemmas 6-8 to show that configurations 1-10 of Theorem 5 generate only a constant number of circles to be added to the collection covering the boundary of S_m.

<u>Theorem 6</u>: If A_m is a non-fixed joint of an arm confined to move inside a circle C, then the boundary of S_m can be covered by a finite collection of circles. This collection contains at most 148 circles and can be determined in p(m) time, where p is a polynomial in m.

<u>Proof</u>: It follows from Theorem 5 that the boundary of S_m can be covered by a collection of circles consisting of the basic circles (at most 4), the supplementary circles (at most 4), and circles of radius $\mathbf{l}_m$ centered at the

endpoints of R_{m-1} (at most 4) together with circles covering configurations 1-10. In configurations 1-5, A_m lies on a circle of radius $\mathbf{l}_{j+1} - |\lfloor A_{j+1}A_m \rfloor|$ centered at the endpoint of an arc of R_j, where A_j is the last joint on C between A_o and A_m. In configurations 6-10, A_m lies on a circle of radius $|\lfloor A_j A_m \rfloor|$, where A_j has the same definition. Thus each possibility for j in each of the configurations gives rise to at most four new circles to add to the collection because R_j has at most four endpoints. Therefore, it suffices to show that the total number of possibilities for A_j is small, and that these possibilities can be determined in polynomial time. We will do this one configuration at a time.

In what follows, we will say that a link index is feasible with respect to a set of inequalities if the corresponding link provides a solution.

Configuration 1: If $\lfloor A_o A_j \rfloor$ lies on L_{j+1} then A_m must lie on a basic cirlce. If $\lfloor A_o A_j \rfloor$ does not lie on L_{j+1}, then Lemma 6 implies that $|\lfloor A_o A_j \rfloor| < 2\mathbf{l}_{j+1}$. Of course $\mathbf{l}_{j+1} > |\lfloor A_{j+1} A_m \rfloor|$. If there are solutions to these inequalities, let s be the smallest feasible choice for index j. It is easy to check that there can be at most one feasible choice for j that is larger than s. Thus, there are all together three possibilities for A_j in configuration 1.

Configuration 2: The last link of longest length between A_o and A_m must be either $L_j \vee L_{j+1}$, so there are at most two choices for A_j.

Configuration 3: Lemma 8 implies that $\mathbf{l}_j + \mathbf{l}_{j+1} > r$, and of course $d > \mathbf{l}_{j+1} > |\lfloor A_{j+1} A_m \rfloor|$. If there are solutions to these inequalities, let z be the largest feasible choice for index j+1. Then note that there can be at most three feasible cHoices for j+1 that are smaller than z, giving a total of four choices for A_j.

Configuration 4: The number of choices for the pair of indices $i,j+1$ is at most m choose 2, and of course $\mathbf{l}_{j+1} > |\lfloor A_{j+1}A_m \rfloor|$ and $\mathbf{l}_i > |\lfloor A_iA_j \rfloor|$.

First, suppose that $\mathbf{l}_i > r$, and if there are solutions to the inequalities including this new one, let s be the smallest index such that for some t, the pair s,t is a feasible choice for $i,j+1$. Note that there can be at most three links beyond L_s that are longer than r. Hence there are at most four feasible choices for i. But choosing a value for i leads us back to configuration 1, so there are at most <u>twelve</u> choices for j if $\mathbf{l}_i > r$.

Next, suppose that $\mathbf{l}_i \leq r$ and that $\mathbf{l}_{j+1} > r$. Since $r \geq \mathbf{l}_i > |\lfloor A_iA_j \rfloor|$ and $\mathbf{l}_{j+1} > |\lfloor A_{j+1}A_m \rfloor|$, L_{j+1} must be the unique link of longest length between A_{i-1} and A_m. If there are solutions to the inequalities, including $\mathbf{l}_i \leq r$ and $\mathbf{l}_{j+1} \geq r$, let s be the smallest index such that for some t, the pair s,t is a feasible choice for $i,j+1$. We have just noted that choosing $i = s$ forces $j+1 = t$. Now note that any other feasible choice for i between s and t also forces $j+1 = t$. Since there can be at most one link in $\lfloor A_{j+1}A_m \rfloor$ that is longer than r, there can be at most one more choice for $j+1$. Hence there are at most <u>two</u> choices for j if $\mathbf{l}_i \leq r$ and $\mathbf{l}_{j+1} > r$.

Lemma 7 rules out the last possibility, that $\mathbf{l}_i \leq r$ and $\mathbf{l}_{j+1} \leq r$, so all together there are at most 12+2=14 choices for A_j.

Configuration 5: Let L denote the link of maximum length between A_o and A_m that has highest index. Then either $L = L_i$ or L belongs to $\lfloor A_iA_j \rfloor$ or $L = L_{j+1}$. The possibility that A_j is the lower numbered joint of the last link of longest length was already taken into account when configuration 1 was considered. The possibility that $L = L_i$ or that L belongs to $\lfloor A_iA_j \rfloor$ can be handled by returning to configuration 1 and letting the endpoint of L with the higher index play the role of A_o. This generates at most two possibilities for A_j because $\lfloor A_iA_j \rfloor$ cannot lie on L_{j+1} in configuration 5.

Configuration 6: According to Lemma 6, $|\lfloor A_jA_m \rfloor| < 2\mathbf{l}_i$, and of course, $\mathbf{l}_j > |\lfloor A_oA_{j-1} \rfloor|$. If there are solutions to these inequalities, let s be the smallest feasible choice for index j. Then it is easy to see that there can be at most one feasible choice for j that is greater than s. Hence there are at most two choices for A_j.

Configuration 7: There are two choices for A_j. A_j could be the higher indexed endpoint of the highest indexed link of longest length, or A_j could be the next joint after that.

Configuration 8: According to Lemma 7, $\mathbf{l}_{j-1} + \mathbf{l}_j > d$, and of course $\mathbf{l}_j > |\lfloor A_jA_m \rfloor|$. If these inequalities can be satisfied, then consider the largest feasible choice for j, and note that there can be at most one smaller feasible choice.

Configuration 9: The longest link that has the highest index is either $L_j \vee L_i$. The first possibility was handled in configuration 7. In the second possibility, A_j is uniquely determined by the fact that L_j is the longest link after A_i.

Configuration 10: According to Lemma 7, $\mathbf{l}_i + \mathbf{l}_j > d$. Of course, $\mathbf{l}_j > |\lfloor A_jA_m \rfloor|$, $\mathbf{l}_j > |\lfloor A_iA_{j-1} \rfloor|$, and $\mathbf{l}_i > |\lfloor A_iA_{j-1} \rfloor|$. If these inequalities can be satisfied, let s be the smallest feasible choice for i, and let t > s be a feasible choice for j when i = s. Note that t is determined by the fact that L_j must be the longest link between A_i and A_m. All feasible choices for i between s and t-1 force j = t. Since $\mathbf{l}_i + \mathbf{l}_j > d$, there can be at most one more feasible choice for i, namely i = t, and then the j is again uniquely determined. In all, there are two possibilities for j. ⊔

The total number of choices for A_j is at most 34, and since R_j may have as many as four endpoints, this generates at most 136 circles. There were at most 12 circles initially, so the total number of circles needed is at most 148. ⊔

The bound of 148 is very generous. The important point, though, is that the bound does not depend on the arm.

Reference

[1] J.E. Hopcroft, D.A. Joseph, and S.H. Whitesides, On the Movement of Robot Arms in Two-Dimensional Bounded Regions, Cornell University Department of Computer Science Technical Report TR 82-486, 1982.