Long arithmetic progressions of primes: some old, some new

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This note updates what is known about certain "long" sequences of primes in arithmetic progression (PAPs). Following computations with the programs described in [6], the PAP of length n with minimum last term is now known for $n \leq 18$. Note that we do not consider PAPs with so-called "negative primes", unlike, e.g., [4].

Table 1 below is an update of Table 2 of [1]. It lists for each $m > 1$ the known PAP of length m with minimum last term. For the heuristic argument leading to the estimates in the last column, see [1]. The three separately-boxed PAPs are new, having been discovered by our programs, which have been running in the background since 6th October 1982 on two DEC VAX-11/780s in the Department of Computer Science at Cornell University. Also new is the knowledge that each PAP listed is indeed the one with minimum last term; previously this was known only for $m \leq 10$ (see [1]).

Table 1

m	PAP of length m with minimal last term	last term	estimated last term
2	2,3	3	2
3	3,5,7	7	2
4	5,11,17,23	23	2
5	5,11,17,23,29	29	29
6	$7+30k$ †	157	92
7	$7+150k$	907	497
8	$199+210k$	1669	1406
9	$199+210k$	1879	5086
10	$199+210k$	2089	24310
11	$110437+13860k$	249037	177300
12	$110437+13860k$	262897	829800
13	$4943+60060k$	725663	5582000
14	$31385539+420420k$	36850999	2.332×10^{7}
15	$115453391+4144140k$	173471351	1.137×10^{8}
16	$53297929+9699690k$	198793279	6.793×10^{8}
17	$3430751869+87297210k$	4827507229	5.774×10^{9}
18	$4808316343+717777060k$	17010526363	3.303×10^{10}
19	?	?	2.564×10^{11}
20	?	?	1.261×10^{12}

† In each case $k = 0, 1, 2, \cdots, m-1$.

Sierpiński defines $g(x)$ to be the maximum number of terms in a progression of primes not greater than x. The least x, $l(x)$, for which $g(x)$ takes the values 0,1,...,18 can be read off from Table 1 above. This corrects and extends the information in [2].

Table 2 is an adaption and extension of Table 1 on p.11 of [2]. It gives, for each $n \geq 12$, the first-discovered PAP with length n and the PAP of length n with smallest last term (a denotes the first term and d the common difference). We hope to extend this table with the relevant information for $n = 19$ in the forseeable future.

Table 2

n	d	a	$a+(n-1)d$	discovery
12	30030	23143	353473	V.A. Golubev, 1958 (see [4])
12	13860	110437	262897	E. Karst, 1967 (see [4])
13	60060	4943	725663	V.N. Seredinskij, 1963 (see [4])
14‡	223092870	2236133941	5136341251	S.C. Root, 1969 (see [3])
14	420420	31385539	36850999	P.A. Pritchard, 1983
15‡	223092870	2236133941	5359434121	S.C. Root, 1969 (see [3])
15	4144140	115453391	173471351	P.A. Pritchard, 1983
16	223092870	2236133941	5582526991	S.C. Root, 1969 (see [3])
16	9699690	53297929	198793279	S. Weintraub, 1976 (see [8])
17	87297210	3430751869	4827507229	S. Weintraub, 1977 (see [9])
18	9922782870	107928278317	276615587107	P.A. Pritchard, 1982 (see [7])
18	717777060	4808316343	17010526363	P.A. Pritchard, 1983

‡ This is an initial segment of Root's PAP of 16 terms.

As of writing, we know of no PAP of length 19 (or greater). The known PAPs of length 18 are given in Table 3 below.

Table 3: the known PAPs of length 18 (as of 1 January, 1984).

first term	common difference	last term
4808316343	717777060	17010526363
2518035911	7536659130	130641241121
98488875263	5169934770	186377766353
107928278317	9922782870	276615587107
51565746467	13889956080	287694999827

[1] Grosswald, E. and P. Hagis, Jr., Arithmetic progressions consisting only of primes, *Maths. of Comp.* **33** (148) (1979) 1343-1352.

[2] Guy, Richard K., *Unsolved Problems in Number Theory*, Springer-Verlag, New York, 1981.

[3] Karst, Edgar, 12-16 primes in arithmetical progression, *J. Recreational Math.* **2** (1969) 214-215.

[4] Karst, E., Lists of ten or more primes in arithmetical progressions, *Scripta Math.* **28** (1970) 313-317.

[5] Karst, E. and S.C. Root, Teilfogen von Primzahlen in arithmetischer Progression, *Anz. Oesterreich. Akad. Wiss. Math.-Naturwiss. Kl.* 1972, 19-20 (see also 178-179).

[6] Pritchard, P.A., A case study of number-theoretic computation: searching for primes in arithmetic progression, *Science of Computer Programming* **3** (1983) 37-63.

[7] Pritchard, P.A., Eighteen primes in arithmetic progression. *Maths. of Comp.* **41**, 164 (October 1983), 697.

[8] Weintraub, S., Primes in arithmetic progression, *B.I.T.* **17** (1977) 239-243.

[9] Weintraub, S., Seventeen primes in arithmetic progression, *Maths. of Comp.* **31** (140) (1977) 1030.