Reducing Multiple Object Motion Planning to Graph Searching*

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ABSTRACT

In this paper we study the motion planning problem for multiple objects where an object is a 2-dimensional body whose faces are line segments parallel to the axes of $\mathbf{R}^2$ and translations are the only motions allowed. Towards this end we analyze the structure of configuration space, the space of points that correspond to positions of the objects. In particular, we consider CONNECTED, the set of all points in configuration space that correspond to configurations of the objects where the objects form one connected component. We show that CONNECTED consists of faces of various dimensions such that if there is a path in CONNECTED between two 0-dimensional faces (vertices) of CONNECTED then there is a path between them along 1-dimensional faces (edges) of CONNECTED. It is known that if there is a motion between two configurations of CONNECTED then there is a path in CONNECTED between the configurations. Thus by the result of this paper the existence of a motion between two vertices of CONNECTED implies a motion corresponding to a path along edges of CONNECTED. Hence we have reduced the motion planning problem from a search of a high dimensional space to a graph searching problem.

Searching the graph of vertices and edges of CONNECTED for a path has a prohibitive worse-case complexity because of the large number of vertices and edges. However, if the search generates edges and vertices only as they are needed, a practical and efficient algorithm may be possible using some effective heuristic.

From this result it is shown that motion planning for rectangles in a rectangular boundary is in PSPACE. Since it is known that the problem is PSPACE-hard we conclude it is a PSPACE-complete problem.

Keywords: motion planning, robotics, polynomial space, PSPACE, algorithm

0. Introduction

Motion planning is an important problem in robotics and computer-aided manufacture. Probably the most influential work in motion planning is due to Lozano-Perez and Wesley [6] whose paper brought to the academic community the realization that motion planning was a rich mathematical area with practical applications. They considered the motion of a single object in the presence of obstacles by shrinking the object to a point and growing the obstacles in such a manner that the original object could be moved from A to B if and only if the point could be moved from A to B in the presence of the enlarged obstacles. The enlarged obstacles were called configuration space obstacles thereby introducing the notion of representing a configuration by a point in a space. The points inside a configuration space obstacle represented illegal configurations where the moving object overlaps an obstacle.

The next major contribution was a series of papers by Schwartz and Sharir [9 - 11]. They provided a precise framework for general motion planning sufficient to encompass not only coordinated motion of multiple rigid objects but also of objects whose shape could change such as a robot arm. By applying techniques from the theory of reals they proved that even in this general setting motion planning was decidable. In fact, for problems with a fixed bounded number of degrees of freedom they gave a polynomial time algorithm. The consideration of multiple objects introduced a new aspect to motion planning; namely, the coordination of motion. Simply calculating trajectories was no longer sufficient.

The complexity of motion planning has been considered by a number of researchers. Reif [13] showed that motion planning for a spider like object in a 3-dimensional cave was PSPACE-hard. Hopcroft, Joseph, and Whitesides [3] showed that the motion planning problem for linkages, even in two dimensions, was PSPACE-hard. Hopcroft, Schwartz, and

Sharir [4] showed that the coordinated motion planning problem for rectangles inside a 2-dimensional rectangular box was PSPACE-hard even if translation was the only allowable motion. However, there was a fundamental difficulty in establishing that these problems were in PSPACE. The difficulty is that there could conceivably be a motion but not a motion that could be described in polynomial space. For example, what if the only possible motions involved moving objects to positions that were not even algebraically related. Although the work of Schwartz and Sharir [10] showed this not to be the case, it did not rule out moving objects to algebraic positions of high degree, too high to represent in an obvious manner in a polynomial amount of space.

Hopcroft and Wilfong [5] considered motion as a special case of a transformation. In their setting an object was a parameterized mapping from a canonical object to a region of 3-space. A motion was a path in the space of parameters and thus included not only translations and rotations but growth and parameterized continuous deformations. They proved the basic theorem that any motion of n objects between two configurations where they were in contact could be transformed to one in which the objects remained in contact throughout the motion. This reduced motion planning from searching an entire space to searching the surface of some generalized configuration space obstacle. An appealing method of searching the surface of the obstacle would be to move to lower and lower dimensional surfaces until a 0-dimensional surface or vertex was reached. If the vertices of the object in configuration space were connected by paths consisting solely of vertices and edges then the search of the geometrical surface could be reduced to the search of a purely combinatorial structure, i.e., the graph consisting of the vertices and edges of the surface. Although the number of vertices of the graph may be astronomical, the entire graph need not be constructed. By means of some heuristic, a search could proceed by generating an edge only when it was to be traversed in the search. If in practical cases, where objects are being moved in a work space, only a small number of vertices need be generated, then a practical and effective algorithm might be possible although the worst-case complexity would be prohibitive. The search process envisioned is somewhat analogous to linear programming where only a small number of vertices of a polytope are examined.

The process of searching by following features of the configuration space obstacle also removes one of the difficulties of establishing the computational complexity of motion planning. It appears to bound the algebraic complexity more closely than the method of Schwartz and Sharir [10] of applying the theory of reals.

In this paper we study the motion of two dimensional objects with linear surfaces. These objects will be allowed to translate but not to rotate. The arrangements of the objects are in an obvious one to one relationship with a point in a high dimensional Euclidean space, called configuration space. We define classes of configurations by specifying the faces of the objects that touch one another. The classes that correspond to configurations that are such that the objects form a connected arrangement are actually various dimensional hyperplanes in configuration space. The classes that contain only one configuration are called vertex configurations and the classes that correspond to one dimensional hyperplanes are called edges. We show that if there is any motion of the objects between two vertex configurations then there is a motion that follows edges from vertex to vertex. This result is not true for more general situations in which rotation is allowed.

Consider the situation in Figure 1 where rotations are allowed. This is an example of a vertex configuration but the only other vertex configuration that can be reached from this one along edges is the one resulting from moving the objects as a rigid piece to the left. Notice that when corners a and b of objects A and B come apart, object A is free to rotate by sliding c along L and d along M, and object B is free to rotate and move corner

e along N. Such configurations lie on a higher dimensional surface and not on an edge. A similar phenomenon occurs if any other constraint is moved.

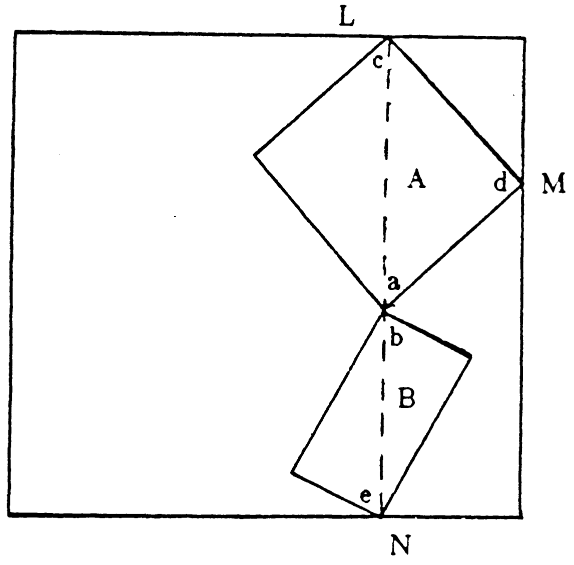


Figure 1. Vertices not connected by edges when rotations allowed.

The edge connectedness property provides a simple strategy for planning the motion of such objects. Call a subset of the objects that form a vertex configuration a part. A motion along an edge corresponds to moving a part so that a face of the part moves along the face of the remaining part. The strategy is to find a subset of the objects that is a part and for which the remaining objects form a part and move the face of one part along the face of the other part. Continue this process of moving along edges in configuration space until the required vertex configuration is reached.

Using the result about edge connectedness of vertex configurations it is shown that deciding whether there exists a motion between two configurations is a problem in PSPACE. This is done by showing that a vertex can be stored in polynomial space since it is the solution of a linear system of equations. Thus, a nondeterministic algorithm using polynomial space can guess the path vertex by vertex and the result follows from the fact that PSPACE=NSPACE. Thus, deciding if a motion of rectangles within a rectangular boundary exists is in PSPACE. Combining this result with the result of [4] we conclude that the problem of deciding if a motion of rectangles in a rectangular boundary exists is PSPACE-complete.

1. Partitioning configuration space

In this section the basic definitions will be presented and a result from algebraic topology known as the Mayer-Vietoris Theorem is stated. Using these concepts the existence of certain paths in the boundary of a surface in configuration space is shown.

A *path* in a set S is a homeomorphism $p:[0,1] \rightarrow S$. An *object* is a two dimensional, path connected, compact region with a finite number of faces that are closed line segments parallel to one of the two axes of $\mathbf{R}^2$. Each object is the closure of its interior but it is not necessarily convex. Fix some point on each object and call it the *origin* of the object. Unless otherwise stated we will assume there are $n+1$ objects. The objects are denoted by $A_0, A_1, \ldots, A_n$. Object A_0 has its position fixed and the others are allowed to translate but not to rotate. Thus the location of object A_i is completely determined by the position (x_i, y_i) of its origin in $\mathbf{R}^2$. A *configuration* is a vector $(x_1, y_1, \ldots, x_n, y_n) \in \mathbf{R}^{2n}$. The arrangement of a set of objects determines a configuration and a configuration determines the arrangement of the objects. Thus when we speak about a configuration, we may be talking about a point in $\mathbf{R}^{2n}$ or an arrangement of the objects but the two notions are interchangeable. The set of all configurations, $\mathbf{R}^{2n}$, is called *configuration space*. For notational purposes we use lower case v's to denote points in configuration space.

Two objects are said to *overlap* if the interiors of the objects have a nonempty intersection. Let $NONOVERLAP$ denote the set of all configurations in which no two objects overlap. Two objects *touch* if the intersection of the objects is nonempty but the objects do not overlap.

For every configuration v we define the ***graph of v***, denoted by G_v, to be the graph with a node corresponding to each object of v and an edge between nodes if the corresponding objects intersect in v. We say that a configuration is a ***connected configuration*** if the graph of the configuration is connected. Define $CONNECTED$ to be the set of all connected configurations in NONOVERLAP.

We say that the face a_i of object A_i ***touches*** face a_j of object A_j if the intersection of a_i and a_j is nonempty and a_i and a_j are a top and bottom face respectively, or a left and right face respectively. Thus objects A_i and A_j touch if and only if there are faces of A_i and A_j that touch. For every pair of faces that touch there is a corresponding linear equation that is satisfied.

A ***description*** is a predicate that is a conjunction of clauses of the form "face a_i of object A_i touches face a_j of object A_j." Such a clause will be satisfied if the corresponding linear equation is satisfied and a_i and a_j intersect. With each description D we associate a linear system $M_D v = c_D$ where for each clause of D the corresponding linear equation is in the system and $v = (x_1, y_1, \ldots, x_n, y_n)$.

We say that a configuration v ***exactly satisfies*** a configuration D if v satisfies D and the only faces that touch in v are those implied by D. For each configuration v there is a description D that v exactly satisfies. Thus when we speak of a matrix M_v we mean the matrix M_D where D is the description that v exactly satisfies. Throughout this paper we will consider only descriptions D such that any configuration that satisfies D is a connected configuration.

Define H_D to be the set of all configurations that satisfy the linear system corresponding to D. That is $H_D = \{v \mid M_D v = c_D\}$. Thus if $k = \mathrm{rank}(M_D)$ then H_D is a $2n-k$-dimensional hyperplane. Notice that a configuration can satisfy the system of linear equations without satisfying the predicate D. In Figure 2 the objects have faces that are aligned and so satisfy the linear equation but the faces do not touch.

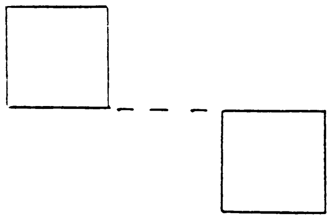
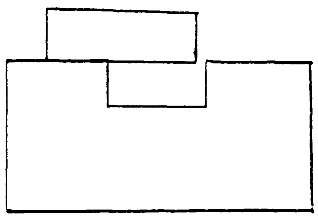


Figure 2. Objects satisfy the linear system but do not touch.

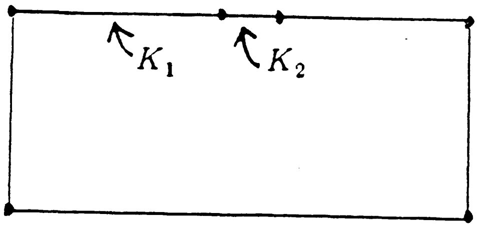
Let E_D be the set of all configurations in NONOVERLAP that exactly satisfy description D. Notice that E_D is not necessarily path connected. In Figure 3 the description D states that the bottom faces of A and C touch the top face of B. However there is one path connected component of E_D where A is to the left of C and another path connected component of E_D where A is to the right of C. We will denote a path connected component of E_D by P_D. Let K_D be the closure of P_D in H_D. K_D is called a *face* of CONNECTED of dimension d if $\text{rank}(M_D) = 2n - d$.

Figure 3. E_D with two path connected components.

Notice that a face as defined here may not conform to what one would usually call a face. For example, in Figure 4 consider faces K_1 and K_2. Although they are colinear surfaces with nonempty intersection, they are two distinct faces by the definitions used in this paper.



Objects



CONNECTED

Figure 4. A surface of CONNECTED divided into more than one face.

The Mayer-Vietoris theorem from algebraic topology is used to show that a face and its boundary have the same number of path connected components. The theorem shows that a sequence of groups is an exact sequence. An exact sequence $A \xrightarrow{h_1} B \xrightarrow{h_2} C \xrightarrow{h_3} \cdots \xrightarrow{h_n} \{0\}$ is such that the image of h_i, $Im(h_i)$, is the kernel of h_{i+1}, $ker(h_{i+1})$. The notation H_1 and H_0 in the theorem is as follows:

(i) $H_0(S)$ is the zeroth homology group of S where

$$H_0(S) \simeq \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{\text{m copies}} \text{ if } S \text{ has } m \text{ path connected components.}$$

(ii) $H_1(S)$ is the first homology group of S where $H_1(S) \simeq \{0\}$ if $H_1(S)$ is contractible to a point.

Note that a set S is contractible to a point if there is a continuous function $f : S \times [0,1] \rightarrow S$ such that

$$\left.\begin{array}{l} f(x,0)=x \\ f(x,1)=y \end{array}\right\} \quad \forall x \in S$$

$$f(y,t)=y \qquad \forall t \in [0,1].$$

Theorem 1.1 (Mayer-Vietoris): Let A and B be two closed sets. Then the sequence

$$H_1(A \cup B) \xrightarrow{h_1} H_0(A \cap B) \xrightarrow{h_2} H_0(A) \oplus H_0(B) \xrightarrow{h_3} H_0(A \cup B) \xrightarrow{h_4} \{0\}$$

is an exact sequence.

Proof: See Massey [8]. □

Ultimately we wish to show that path connected components of CONNECTED are edge connected. For every configuration v there is a description D such that faces touch in v if and only if they touch according to D. In other words, v exactly satisfies D (i.e. $v \in E_D$). Since a configuration cannot exactly satisfy more than one description, the sets of configurations that exactly satisfy descriptions partition CONNECTED. That is, CONNECTED can be partitioned into E_D's. As noted earlier not all E_D's are path connected and since distinct path connected components must be disjoint, we can further partition CONNECTED into the path connected components of the sets of configurations that exactly satisfy descriptions. That is, we partition CONNECTED into P_D's.

We will then show that if there is a path in K_D, a face of CONNECTED, where K_D is the closure of a path connected component P_D, between two configurations in $K_D - P_D$, the boundary of the face, then there is a path between them in the boundary. This will be done by using the Mayer-Vietoris Theorem which requires the following facts:

(1) H_D is contractible to a point and is path connected

(2) K_D is closed in H_D

(3) $H_D - P_D$ is closed in H_D and path connected.

(1) follows from the fact that H_D is a hyperplane and (2) is immediate because of the definition of K_D. Lemmas 1.2 to 1.6 are used to establish (3). Starting with Lemma 1.7 we will show that each path connected component of a face of CONNECTED contains exactly one path connected component of its boundary.

First it will be shown that the set of configurations E_D that exactly satisfy a description is open in the set of configurations that satisfy the linear system of the description. We will then conclude that those configurations that satisfy the linear system of the description but are not in some fixed path connected component of the configurations that exactly satisfy the description form a closed set.

Lemma 1.2: E_D is open in H_D.

Proof: Let $v \in E_D$. Then v exactly satisfies D and $v \in$ NONOVERLAP. Let $\epsilon>0$ be the minimum distance between any two right and left faces or top and bottom faces of objects such that the faces do not touch in v. Then any configuration v' in H_D where the objects are closer than $\epsilon/2$ from their positions in v is such that no faces touch in v' that do not in v. That is, there is an open ball B in H_D about v such that any faces that touch in v' also touch in v.

Suppose two objects overlap in $v' \in B$. By the definition of B no objects that do not touch in v can overlap. Since $B \subseteq H_D$ any $v' \in B$ must satisfy the linear equations corresponding to the clauses of D. Thus no objects that touch according to D can overlap in v'. But $v \in E_D$ and so no objects that touch in v can overlap in v'. That is $B \subseteq$ NONOVERLAP.

Suppose $v' \in B$ and there is a face a_1 of A_1 and a face a_2 of A_2 that touch according to D, and hence touch in v, but do not touch in v'. Since B is a ball, between v' and v there is a path contained in $B \subseteq H_D$. Let m be the motion corresponding to the path. Throughout m, A_1 and A_2 must be aligned along a_1 and a_2. Since a_1 and a_2 touch in v but not in v' and m is continuous, we conclude that there is a point in the motion when A_1 and A_2 touch at corners. That is, at this point there are two faces that touch but they do not touch in v. This contradicts the definition of B.

Therefore, for any $v' \in B$, faces touch in v' if and only if they touch according to D and $v' \in$ NONOVERLAP. That is $B \subseteq E_D$ and so E_D is open in K_D. □

Next we show that each path connected component of E_D is open in H_D.

Lemma 1.3: P_D is open in H_D.

Proof: H_D is locally pathwise connected and E_D is open in H_D. Thus any path connected component of E_D is open. See Willard [14]. □

We can now conclude one of the facts required to use the Mayer-Vietoris Theorem that is needed to establish that a path exists in the boundary of a face of CONNECTED between two configurations in the boundary whenever there is a path in the face between the configurations.

Corollary 1.4: $H_D - P_D$ is closed in H_D.

Proof: Immediate from Lemma 1.3. □

Let D be some description and S be a subset of the objects. Choose one of the objects of S (take it to be A_0 if $A_0 \in S$) to be considered the object of S whose position is fixed. As above we can construct a linear system from D for the objects in S. If the rank of the resulting coefficient matrix is full and S is maximal with respect to this property then we say that S is a *vertex object* of D.

In Figure 5 there are two vertex objects {0,1} and {2,3,4}. Notice that although taking $S = \{2,3\}$ results in a coefficient matrix of full rank, S is not maximal and so {2,3} is not a vertex object.

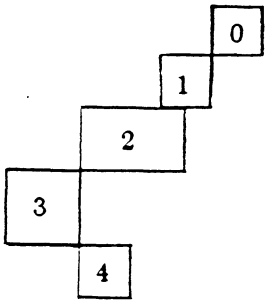


Figure 5. Two vertex objects.

Lemma 1.5: Let D be such that $\text{rank}(M_D) \leq 2n-2$. Then D has at least three vertex objects.

Proof: If D has only one vertex object then by definition $\text{rank}(M_D) = 2n$. Suppose D has exactly two vertex objects and one has m_1 objects and the other has m_2 objects where $m_1 + m_2 = n+1$ the total number of objects. Then the number of independent rows of M_D that correspond to two objects of the first vertex object touching is $2(m_1-1)$. Similarly we have $2(m_2-1)$ independent rows for the second vertex object. Since the two vertex objects must touch there must be an additional independent row. Thus $\text{rank}(M_D) \geq 2(m_1 + m_2 - 2) + 1 = 2(n-1) + 1 = 2n-1$. Hence if $\text{rank}(M_D) \leq 2n-2$ then D has at least three vertex objects. □

Next we show that the set of configurations that satisfy the linear system of a connected description D but are not in the interior of a fixed face is path connected if the dimension of the space of configurations that satisfy the linear system of D is large enough.

Lemma 1.6: If rank$(M_D) \leq 2n-2$ then $H_D - P_D$ is path connected.

Proof: It will be shown that there is a fixed configuration v_0 in $H_D - P_D$ such that there is a path in $H_D - P_D$ from any v in $H_D - P_D$ to v_0. The existence of such a v_0 establishes that $H_D - P_D$ is path connected.

Let $v_0 \in H_D - P_D$ be a configuration such that if the positions of any two vertex objects are held fixed to their positions in v_0, no motion of the remaining vertex objects within H_D can result in a connected configuration and thus cannot result in a configuration in E_D.

We proceed in two steps. First, we show how to move within $H_D - P_D$ from any configuration in $H_D - P_D$ to one in $H_D - E_D$. Then we show how to move within $H_D - P_D$ from any configuration in $H_D - E_D$ to v_0.

Let $v \in H_D - P_D$. If $v \notin E_D$ then our first step is completed. Suppose $v \in E_D$ but $v \notin P_D$. Move a vertex object along an edge of another vertex object that it is touching until a new pair of faces touch. The resulting configuration is in $H_D - E_D$. The entire motion except the final configuration is in $E_D - P_D$ and so the motion is in $H_D - P_D$.

Next we show that any $v \in H_D - E_D$ can be moved in $H_D - E_D$ (and hence in $H_D - P_D$) to v_0. Let A and B be two vertex objects of D that touch according to D but in v they either touch along faces not specified by D, overlap one another, or do not touch along some face specified by D. Since A and B touch according to D their relative positions have only one degree of freedom in H_D. Fix the relative positions of A and B as they are in v. This adds one constraint to the system. Since rank$(M_D) \leq 2n-2$ there is still a collection of vertex objects that is unconstrained in one of the x or y directions.

Suppose that object C of this collection touches, according to D, object E of the vertex objects that are fixed relative to the fixed object made up of A and B. Then we can move all the objects keeping the same relative positions until E is in its position as specified by v_0. Since C is free to move while E is kept fixed we can move C to its position in v_0 keeping the positions of A and B fixed. By the definition of v_0 the rest of the vertex objects can now be moved within H_D-E_D to their positions in v_0. The entire motion is in H_D-E_D because either A and B do not touch exactly as specified by D or the objects are not connected.

Suppose all vertex objects that are supposed to touch according to D touch correctly. Then there must be two vertex objects A and B such that A and B do not touch according to D but touch or overlap in v. If adding the constraints that A and B retain their relative positions leaves a degree of freedom, then proceed as in the previous case. If not, then by Lemma 1.5 there must be a vertex object C that touches A and has its x-coordinate dependent on the x-coordinate of A and its y-coordinate dependent on the y-coordinate of B (or vice versa). Without loss of generality assume that the lowest point on the face of C that touches A is at least as high as the lowest face of B. Then since A and B are path connected and they have no constraints in their relative positions B can be moved so that it is always intersecting A until the lowest face of B touches the highest face of A. Thus either C is no longer touching A or C and A touch at a new pair of faces. In either case we are back to the first situation.

Hence it has been shown that there is a path in H_D-P_D from any $v \in H_D-P_D$ to a fixed $v_0 \in H_D-P_D$. Thus we conclude that H_D-P_D is path connected. □

The previous proof depended on the fact that there were at least three vertex objects. The result for the two dimensional objects holds if D has as few as two degrees of freedom.

With three dimensional objects we might only have two vertex objects when there are two degrees of freedom but the result holds if there are at least three degrees of freedom. We will later conclude that the path connected components of CONNECTED have their 0-dimensional faces (vertices) connected by 1-dimensional faces (edges) in the case of objects in $\mathbf{R}^2$. For objects in $\mathbf{R}^3$ with planar faces all the proofs in this paper hold except the previous one because of the possibility of only two vertex objects when we have two degrees of freedom. However we can conclude that edges of path connected components of CONNECTED are connected by 2-dimensional faces because there must be at least three vertex objects when there are three degrees of freedom and so $H_D - P_D$ is path connected if $\mathrm{rank}(M_D) \leq 3n-3$.

Now the Mayer-Vietoris theorem is used to show that the number of path connected components of the boundary of a face of CONNECTED is equal to the number of path connected components of the face of CONNECTED.

Theorem 1.7: $H_0(K_D - P_D) \simeq H_0(K_D)$ when $\mathrm{rank}(M_D) \leq 2n-2$.

Proof: $H_D - P_D$ and K_D are closed sets in H_D by Corollary 1.4 and the fact that $K_D = cl_{H_D}(P_D)$. $H_D = (H_D - P_D) \cup K_D$ is clearly contractible to a point and path connected. By Lemma 1.6, $H_D - P_D$ is path connected. Thus taking $A = H_D - P_D$ and $B = K_D$ in Theorem 1.1 we get that $H_0((H_D - P_D) \cap K_D) = H_0(K_D - P_D) \simeq H_0(K_D)$. □

We have now shown that a face of CONNECTED and its boundary have the same number of path connected components. However, we further need that each path connected component of a face of CONNECTED contains exactly one path connected component of its boundary. This would follow immediately if we could show that the face was path connected. However we have been unable to prove this and so the following two lemmas are needed.

Lemma 1.8: Each path connected component of $K_D - P_D$ intersects exactly one path connected component of K_D.

Proof: This follows from the fact that $K_D - P_D \subseteq K_D$. □

Next it is proved that each path connected component of a face of CONNECTED contains a path connected component of its boundary provided that the objects do not form a vertex object.

Lemma 1.9: Each path connected component of K_D contains at least one path connected component of $K_D - P_D$ if rank$(M_D) \leq 2n-1$.

Proof: It is sufficient to show that for any configuration v_1 in K_D there is a configuration v_2 in $K_D - P_D$ and a path in K_D from v_1 to v_2.

Thus the result is trivial if $v_1 \in K_D - P_D$. Otherwise $v_1 \in P_D$. Since rank$(M_{v_1}) =$ rank$(M_D) \leq 2n-1$ there is a vertex object B_1 of v_1 with a face b_1 that touches a face b_2 of another vertex object B_2 of v_1. Moving B_1 in P_D so that b_1 moves across b_2 until some face of B_1 touches some face that was not touching in v_1 results in a configuration v_2 where $v_2 \notin E_D$ and so $v_2 \notin P_D$. Clearly $v_2 \in K_D$ by definition of v_2. Thus the Lemma follows. □

The preceding results can be combined to conclude that there is a one-to-one correspondence between the path connected components of a face of CONNECTED and those of its boundary. This is stated in the following theorem.

Theorem 1.10: Each path connected component of K_D contains exactly one path connected component of $K_D - P_D$.

Proof: The Theorem follows from Theorem 1.7 and Lemmas 1.8 and 1.9 by a simple counting argument. □

As a consequence of Theorem 1.10 we can now conclude that a path in the boundary of a face exists between configurations in the boundary if there is a path in the face between the configurations.

Theorem 1.11: If $\mathrm{rank}(M_D) \leq 2n-2$ and there is a path in K_D between two configurations in $K_D - P_D$, then there is a path in $K_D - P_D$ between these configurations.

Proof: Let $v_1, v_2 \in K_D - P_D$ and p be a path in K_D between v_1 and v_2. Thus, v_1 and v_2 are in the same path connected component of K_D and hence by Theorem 1.10, v_1 and v_2 are in the same path connected component of $K_D - P_D$. That is, there is a path in $K_D - P_D$ between v_1 and v_2. □

2. Paths in lower dimensional faces

In this section we will show that the boundary of a face K_D of CONNECTED consists of faces of CONNECTED of dimension less than that of the face K_D. Thus, by the previous section, if there is a path in a face of CONNECTED between two configurations in the boundary of the face, then there is a path between the two configurations contained in faces of CONNECTED of dimension less than that of the face K_D. We begin by showing that a configuration in a face satisfies the description that defines the face.

Lemma 2.1: Let v be a configuration in K_D. Then v satisfies D and so $\mathrm{rank}(M_v) \geq \mathrm{rank}(M_D)$.

Proof: Suppose v does not satisfy D. Then there must be two faces that touch according to D but do not touch in v. Let $\epsilon > 0$ be the distance between the two closest such faces.

Any configuration in H_D where each object has been moved less than $\epsilon/2$ from its position in v also does not satisfy D. Thus there is a ball B in H_D about v such that $B \cap P_D = \phi$. Since K_D is the closure of P_D in H_D, v is not in K_D, a contradiction. Hence each v in K_D satisfies D and thus M_v has at least the rows of M_D. Therefore $\text{rank}(M_v) \geq \text{rank}(M_D)$. □

We continue to classify those configurations in the boundary of a face of CONNECTED. Towards this end we show a simple fact about CONNECTED.

Lemma 2.2: CONNECTED $\cap H_D$ is closed in H_D.

Proof: Both CONNECTED and H_D are closed. □

Recall that a face of CONNECTED, K_D, is the closure of a path connected component P_D of the set of configurations that exactly satisfy a description D. We now show that a configuration v in the boundary of a face, K_D, of CONNECTED lies in a face of CONNECTED with dimension less than that of the face K_D.

Lemma 2.3: If $v \in K_D - P_D$ then $\text{rank}(M_v) > \text{rank}(M_D)$.

Proof: By Lemma 2.1, $\text{rank}(M_v) \geq \text{rank}(M_D)$. Suppose $\text{rank}(M_v) = \text{rank}(M_D)$. Since $P_D \subseteq$ CONNECTED $\cap H_D$ and by Lemma 2.2, CONNECTED $\cap H_D$ is closed in H_D, we conclude that $K_D = cl_{H_D}(P_D) \subseteq cl_{H_D}(\text{CONNECTED} \cap H_D) = \text{CONNECTED} \cap H_D$. Thus v is in CONNECTED.

Suppose $v \in Q_D$ where Q_D is a path connected component of E_D other than P_D. By Lemma 1.3, Q_D is open in H_D and so there is an open ball B about v such that $B \subseteq Q_D$. However, v is in K_D, the closure of P_D in H_D. This implies that $B \cap P_D \neq \phi$. Thus $Q_D \cap P_D \neq \phi$. This contradicts the assumption that P_D and Q_D are distinct path connected components of E_D. Thus $v \notin E_D$.

Let D_1 be a description such that $v \in E_{D_1}$. Then $H_{D_1} \subseteq H_D$. Suppose $H_{D_1} \neq H_D$. Then $\text{rank}(M_v) = \text{rank}(M_{D_1}) > \text{rank}(M_D)$.

Suppose that $H_{D_1} = H_D$. Since $v \in E_{D_1}$ and by Lemma 1.2, E_{D_1} is open in H_{D_1} and hence in H_D, we know that there is an open ball B about v in $E_{D_1} \subseteq H_{D_1} = H_D$. Thus any configuration in B exactly satisfies D_1 and hence does not exactly satisfy D. Thus $B \cap P_D = \phi$ and so $v \notin cl_{H_D}(P_D) = K_D$ which is a contradiction. Hence if $v \in K_D - P_D$ then $v \notin E_D$ and $\text{rank}(M_v) > \text{rank}(M_D)$. □

It is now possible to prove that the boundary of a face of CONNECTED, $K_D - P_D$, is exactly those configurations satisfying D that lie in faces of CONNECTED of dimension less than that of the face K_D.

Lemma 2.4: $K_D - P_D = K_D \cap \{v \mid \text{rank}(M_v) > \text{rank}(M_D)\}$.

Proof: Let $v \in K_D - P_D$. By Lemma 2.3, $\text{rank}(M_v) > \text{rank}(M_D)$ and so $K_D - P_D \subseteq K_D \cap \{v \mid \text{rank}(M_v) > \text{rank}(M_D)\}$.

Let $v \in K_D \cap \{v \mid \text{rank}(M_v) > \text{rank}(M_D)\}$. Since $\text{rank}(M_v) > \text{rank}(M_D)$ we know that $v \notin E_D$ and hence $v \notin P_D$. Thus $K_D \cap \{v \mid \text{rank}(M_v) > \text{rank}(M_D)\} \subseteq K_D - P_D$ and so $K_D - P_D = K_D \cap \{v \mid \text{rank}(M_v) > \text{rank}(M_D)\}$. □

Combining the results of Theorem 1.11 and Lemma 2.4 allows us to conclude that if there is a path in a face of CONNECTED of dimension d between two configurations in faces of CONNECTED of dimension less than d then there is a path between them contained in faces of CONNECTED of dimension less than d.

3. Edge connectedness

In order to further examine the structure of CONNECTED, we introduce the notion of a complex. Recall that a face of CONNECTED is the closure in H_D of a path connected component P_D of E_D for some D. The face has dimension d where $2n-d$ is the rank of M_D. A *d-complex*, C_d, is the union of all faces of CONNECTED of dimension d or less. The faces of CONNECTED in C_0 and C_1 are called *vertices* and *edges* respectively.

We shall show that two vertices connected by a path in NONOVERLAP are connected by a path consisting solely of edges and vertices of CONNECTED. In the previous sections it was shown that there is a path in C_d between two configurations in the boundary of one face in C_{d+1} if there is a path between them in the face. Now we wish to use this fact to show that a path in C_d exists between two configurations in C_d if there is a path in C_{d+1} between them, even if the path goes through more than one face of C_{d+1}. The proof proceeds by showing in Lemma 3.2, that configurations in the intersection of two faces of dimension d, lie in faces of dimension $d-1$ or less. This result is used to show in Theorem 3.3 that any two configurations in C_d, $d>0$, that are connected by a path in C_{d+1} are connected by a path in C_d. An inductive argument is then used in Theorem 3.6 to show that there exists a path consisting of vertices and edges. To begin the induction, we use a result from [5] to argue that a path in NONOVERLAP implies a path in CONNECTED and Lemma 3.5 that establishes that CONNECTED equals C_n.

We now proceed to establish these results. Lemma 3.1 is a technical lemma concerning the intersections of closed sets. The reader may wish to skip immediately to Lemma 3.2.

Lemma 3.1: Let A and B be closed sets and let $p:[0,1] \rightarrow P$ be a path in $A \cup B$ from $x \in A$ to $y \in B$. Then $P \cap A \cap B \neq \phi$.

Proof: Suppose $P \cap A \cap B = \phi$. Then $p^{-1}(P \cap A) \cap p^{-1}(P \cap B) = \phi$. Since P is in $A \cup B$, $p^{-1}(P \cap A) \cup p^{-1}(P \cap B) = [0,1]$. Since $p^{-1}(P \cap A)$ and $p^{-1}(P \cap B)$ are closed we conclude that $[0,1]$ is not connected, a contradiction. Therefore, $P \cap A \cap B \neq \phi$. □

Suppose there is a path p in C_{2n-k} between two configurations v and w in C_{2n-k-1}. Let $K_{D_1}, K_{D_2}, \ldots, K_{D_t}$ be the sequence of faces that the path p intersects between v and w. Thus $K_{D_i} \neq K_{D_{i+1}}$. By Lemma 3.1 for each i, $1 \leq i \leq t-1$ there must be a $t_i \in [0,1]$ such that $p(t_i) \in K_{D_i} \cap K_{D_{i+1}}$. Let $p(t_i) = v_i$ and p_i be the path on K_{D_i}. To be able to apply Theorem 1.11 to the section of the path in a particular K_{D_i} we must show that $\mathrm{rank}(M_{v_i}) > k$. Lemma 3.2 will establish this fact.

Lemma 3.2: Let K_{D_1} and K_{D_2} be two faces of dimension at most $2n-k$. That is K_{D_1} and K_{D_2} are contained in C_{2n-k}. Let v be in their intersection. Then $\mathrm{rank}(M_v) > k$. In other words $v \in C_{2n-k-1}$.

Proof: Suppose $D_1 = D_2$. Since $K_{D_1} \neq K_{D_2}$ it must be that $P_{D_1} \neq P_{D_2}$ where K_{D_i} is the closure of P_{D_i} in H_{D_i} for $i = 1,2$. Thus P_{D_1} and P_{D_2} are two distinct path connected components of E_{D_1} and so $P_{D_1} \cap P_{D_2} = \phi$. In particular, $v \notin P_{D_1} \cap P_{D_2}$ and so $v \in K_{D_1} - P_{D_1}$ or $v \in K_{D_2} - P_{D_2}$. In either case, by Lemma 2.3, $\mathrm{rank}(M_v) > k$.

Suppose $D_1 \neq D_2$. Then v cannot exactly satisfy both D_1 and D_2 and so $v \notin P_{D_1} \cap P_{D_2}$ and hence $v \in K_{D_1} - P_{D_1}$ or $v \in K_{D_2} - P_{D_2}$. Then, again by Lemma 2.3, $\mathrm{rank}(M_v) > k$. □

If the dimension of K_{D_i} is less than $2n-k$ then p_i is contained in C_{2n-k-1}. Suppose the dimension of K_{D_i} is $2n-k$. By Lemma 3.2, $\mathrm{rank}(M_{v_{i-1}}) > k$ and $\mathrm{rank}(M_{v_i}) > k$. Since $\mathrm{rank}(M_{D_i}) = k$, $v_i, v_{i+1} \in K_{D_i} - P_{D_i}$ by Lemma 2.4. Thus p_i is a path in K_{D_i} between two configurations in $K_{D_i}-P_{D_i}$ and so by Theorem 1.11 there is a path in $K_{D_i}-P_{D_i}$ between them. By Lemma 2.4 we conclude that there is a path in C_{2n-k-1} between v_{i-1} and v_i. Thus there is a path from v to w in C_{2n-k-1}.

Theorem 3.3: If there is a path in C_{2n-k} from v to w where $v, w \in C_{2n-k-1}$ then there is a path in C_{2n-k-1} from v to w where $k \leq 2n-2$.

Proof: See preceding discussion. □

We are going to want to apply Theorem 3.3 inductively and so we will first show two results about CONNECTED to provide a starting point for the induction.

Lemma 3.4: If v is a connected configuration of $n+1$ objects, then then $\text{rank}(M_v) \geq n$.

Proof: The proof is by induction on the number of objects. Suppose v is a connected configuration of 2 objects. Then there is at least one face of the object that is free to move that touches a face of the fixed object. Thus M_v has at least one nonzero row and so $\text{rank}(M_v) \geq 1$.

Assume the result holds for configurations of n objects. Let v be a connected configuration of $n+1$ objects. As before, let G_v be the graph with a node for each object and an edge between nodes if the corresponding objects touch in v. Let T be a depth first spanning tree of G_v with the node corresponding to A_0, the fixed object, as the root. Then the leaves of T are not articulation nodes of G_v (see [1]). That is, removing an object corresponding to a leaf of T results in a connected configuration w of n objects. By the induction hypothesis, $\text{rank}(M_w) \geq n-1$. Without loss of generality assume that the object removed was A_n. Then $M_v = \begin{bmatrix} M_w & 0 \\ * & B \end{bmatrix}$ where B contains nonzeroes because A_n touches at least one of the other objects. Thus $\text{rank}(M_v) \geq \text{rank}(M_w) + 1 \geq n$ as required. □

Lemma 3.5: CONNECTED $= C_n$.

Proof: By Lemma 3.4, $v \in$ CONNECTED implies $\text{rank}(M_v) \geq n$. But $\text{rank}(M_v) \geq n$ means $v \in C_p$ for some $p \leq n$. Since $p \leq n$ implies $C_p \subseteq C_n$ we conclude that $v \in C_n$.

Let $v \in C_n$. Then $v \in K_D$ for some D with $\text{rank}(M_D) \geq n$. Let P_D be the path connected component of E_D such that $cl_{H_D}(P_D) = K_D$. By definition of E_D, $P_D \subseteq$ CONNECTED $\cap H_D$ and so we get that

$$K_D \subseteq cl_{H_D}(\text{CONNECTED} \cap \text{H}_\text{D}) = \text{CONNECTED} \cap H_D$$

by Lemma 2.2. Thus $K_D \subseteq$ CONNECTED and so $C_n \subseteq$ CONNECTED. Hence CONNECTED $= C_n$. □

We now are in a position to prove our main goal. That is, we will show that if there is a path in NONOVERLAP between two vertices of CONNECTED then there is a path contained in the edges of CONNECTED between them.

Theorem 3.6: Let v and w be vertices of CONNECTED. If there is a path in NONOVERLAP between v and w then there is a path contained in the edges of CONNECTED between v and w.

Proof: By [5] we know that there is a path in CONNECTED from v to w. Thus by Lemma 3.5 there is a path in C_n from v to w. Applying Theorem 3.3 inductively we conclude that there is a path in C_1 from v to w. In other words, there is a path along edges from v to w. □

Notice that by Lemma 3.2 the path along edges in Theorem 3.6 is actually a path that starts at a vertex, follows an edge to another vertex, then follows another edge to a vertex and so on. That is, the path is as shown in Figure 6b, not as in Figure 6a.

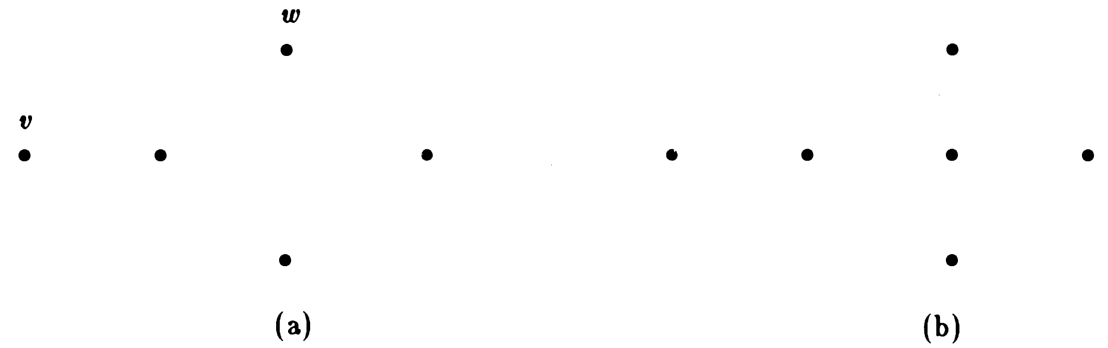


Figure 6. Edge path goes from vertex to vertex.

4. A PSPACE-complete motion problem

In this section we will show that the problem of determining whether a motion exists between two configurations of two dimensional rectangles within a rectangular enclosure with integer sizes where the fixed object is at an integer location is in PSPACE. To accomplish this, a nondeterministic method will be described and will be shown to require polynomial space. Since NPSPACE equals PSPACE we will conclude that the problem is in PSPACE.

First it must be shown how to get from an arbitrary configuration to a vertex configuration. Move the object which is closest to the fixed object in the x or y direction until it touches the fixed object. Considering these two objects as one fixed object, repeat this until a configuration in CONNECTED results. Now the motion in Lemma 1.9 can be repeated until a vertex configuration is encountered.

It is now sufficient to show how to determine if there is a motion between vertex configurations. From a vertex configuration, nondeterministically guess a subset of objects and face of an object to move this subset along. Check to see if this is a motion along an edge of CONNECTED. If so, move the subset of objects until a new pair of faces touch. By the previous results, this new configuration must be a vertex configuration. Continue this process until the desired vertex configuration is encountered.

Since the position of the objects in a vertex configuration constitute the solution to an integer linear system of equations, the positions of the objects in a vertex configuration can be stored in polynomial space (see [2]). Clearly finding the next faces to touch when moving objects in the direction of one of the coordinates axes can be done in polynomial space. Calculating the rank of the matrix M_D where D is the description resulting from moving the subset of objects to see if the motion is along an edge of CONNECTED also can be done in polynomial space. Thus the problem is in NSPACE and so in PSPACE.

In [4] it was shown that the problem of deciding whether there is a motion between two configurations of two dimensional rectangles in a two dimensional box is PSPACE-hard. Hence the problem is PSPACE-complete.

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