SOME COMMENTS ON FUNCTIONAL SELF-REDUCIBILITY AND THE NP HIERARCHY

A. Borodin*

A. Demers

TR 76-284

July 1976

Department of Computer Science
Cornell University
Ithaca, New York 14853

---

* Allan Borodin on leave from the University of Toronto.

SOME COMMENTS ON FUNCTIONAL SELF-REDUCIBILITY AND THE NP HIERARCHY*

A. Borodin
A. Demers

Abstract:

In Valiant [11] and Schnorr [9], concepts of "functional self-reducibility" are introduced and investigated. We concentrate on the class NP and on the NP hierarchy of Meyer and Stockmeyer [7] to further investigate these ideas. Assuming that the NP hierarchy exists (specifically, assuming that $P \subsetneq NP = \Sigma_1^P \subsetneq \Sigma_2^P$) we show that, while every complete set in Σ_2^P is functionally self-reducible, there exist sets in Σ_2^P which are not functionally self-reducible.

I. Introduction

Following the work of Valiant [11] and Schnorr [9], we wish to motivate and define a property which we call "functional self-reducibility". As in Valiant's study of "evaluation vs. checking", our main concern lies in knowing if and when a functional problem can be more difficult that the corresponding set recognition problem. In this regard, we study a very general definition of self-reducibility. It should be noted that, as usual, reducibility implies relative complexity bounds but not conversely.

Schnorr [9] studies a notion of reducibility that is more restrictive than ours, in order to derive results concerning the existence of optimal programs for certain sets. For example, consider the satisfiability problem A and the "natural" functional problem f associated with A - namely, to compute a satisfying assignment if one exists. Schnorr observes that f is reducible to A in the following way. Let x be a propositional statement on k variables. We can determine a truth value for the variable v_1 by substituting "true" and "false" for v_1 in x. If x is satisfiable, then one of these substitutions (at least) must yield a satisfiable formula with k-1 variables; using the oracle A we can determine which one. Continuing in this way we can compute a satisfying assignment, using k oracle questions which are decreasing in length. It follows that, if one defines T(n) to be the maximum time complexity on inputs of length n, then $T_f(n) = O(nT_A(n))$. Obviously, $T_A = O(T_f(n))$. Thus, using a result of Levin [6] that f has optimal program (for Kolmogorov-Uspensky machines) on the set $\{x|x \in A\}$, A must have a "nearly" optimal algorithm. (It is still possible, say, for A to have a complexity sequence T, T/logT, $T/\log^2 T$,....).

Schnorr also states that "any polynomial complete problem in NP can be encoded such that the self-reducible property holds". Thus, using Levin [6] again, such sets have optimal programs. Certain sets which are not seemingly complete - e.g., the graph isomorphism problem - also have this property, but it remains an open question whether all sets in NP have this property.

Our interest lies in the NP hierarchy and the study of self-reducibility. It follows from Valiant [11], Baker, Gill and Solovay [1], Schnorr [9], or the development here that if P = NP then every set is complete and hence self-reducible. We henceforth assume that the hierarchy exists (in particular, that $\Sigma_0^P = P \subsetneq NP = \Sigma_1^P \subsetneq \Sigma_2^P$).

II. Definitions and Notation

For our purpose, the precise model of computation is unimportant, but we might as well choose Turing machines. As usual P (respectively NP) is the class of languages (say over $\Sigma = \{0,1\}$) acceptable in deterministic (resp. nondeterministic) polynomial time. More generally, let us recall the definition of the NP hierarchy (Meyer and Stockmeyer [7]).

Definition 1:

(a) P(B) (respectively NP(B)) is the class of languages accepted by deterministic (resp. nondeterministic) oracle machines which run in polynomial time,

(b) If $C = \{B_i\}$ is a class of sets, then
$P(C) = \bigcup_i P(B_i)$

$NP(C) = \bigcup_i NP(B_i)$

(c) $\Sigma_0^P = \Pi_0^P = \Delta_0^P = P$

$$\left.\begin{aligned} \Sigma_{k+1}^P &= NP(\Sigma_k^P) \\ \Pi_{k+1}^P &= \text{co-}NP(\Sigma_k^P) \\ \Delta_{k+1}^P &= P(\Sigma_k^P) \end{aligned}\right\} \qquad k \geq 0$$

In particular, $\Sigma_1^P = NP(P) = NP$

This definition is due to Meyer & Stockmeyer. It corresponds to the arithmetic hierarchy in recursive function theory, where

Σ_{k+1} = those sets recursively enumerable in Σ_k

Δ_{k+1} = those sets recursive in Σ_k.

We also have the inclusions

$$\Sigma_k^P \cup \Pi_k^P \subseteq \Delta_{k+1}^P \subseteq \Sigma_{k+1}^P \cap \Pi_{k+1}^P \qquad k \geq 0.$$

One can only conjecture that these inclusions are proper; indeed, it is completely open whether the hierarchy exists at all.

Meyer and Stockmeyer [7] motivate and Wrathall [12] provides another characterization (again in complete analogy to the situation in recursive function theory) in showing.

> $A \in \Sigma_k^P$ iff there exists a polynomial $p(x)$ and a $(k+1)$-ary relation $R(x,y_1,\ldots,y_k) \in P$ such that $A = \{x \mid (\exists y_1)(\forall y_2)\ldots(Q_k y_k)\ R(x,y_1,\ldots,y_k)\}$, where Q_k is $\exists(\forall)$ if k is odd (even), and $Y_1,\ldots,y_k$ range over all words of length at most $p(|x|)$.

In particular,

> $A \in \Sigma_1^P = NP$ iff there exist R and p such that $A = \{x \mid (\exists y)\ |y| \leq p(|x|) \wedge R(x,y)\}$.

In this form, the relation R defines a set A_R. Of course, many different relations can define the same set. Indeed, if $A \in \Sigma_k^P$, we can define A as a Σ_ℓ^P set for any $\ell \geq k$. (If $A \in \Sigma_k^P - \Sigma_{k-1}^P$, then a vague but interesting question concerns how "different" the relations $R(x,y_1,\ldots,y_k)$ defining $A_R = A$ can be; see Open Problems 1 and 2).

The relation R also defines a total mapping

$$f_R(x) = \begin{cases} \text{some } y_1 & (\forall y_2)\ldots(Q_R y_R) \\ & \quad R(x,y_1,\ldots,y_k) \\ \text{"no"} & \text{if no such } y_1 \text{ exists.} \end{cases}$$

That is, f_R must compute an acceptable y_1 or say that none exists. Note that the output y_1 is not necessarily uniquely defined. Indeed, we believe the following to be open problems:

Open Problem 1: (for $k \geq 1$)

(a) $(\exists A \in \Sigma_k^P - \Sigma_{k-1}^P)(\exists R)$:

$A = \{x \mid (\exists \text{ unique } y_1)(\forall y_2)\ldots(Q_k y_k)\ R(x, \ldots, y_k)\}$

(b) $(\forall A \in \Sigma_k^P - \Sigma_{k-1}^P)(\exists R)$:

$A = \{x \mid (\exists \text{ unique } y_1)\ \ldots\}$.

The literature is full of excellent natural examples of problems in Σ_1^P = NP, most of which are complete but some seemingly not. The definitive works are obviously Cook [2] and Karp [4]. For the hierarchy in general, the best examples can be found in Stockmeyer [10]. For all these examples, there seems to be no way to uniquely define y_1 while keeping R polynomial.

We can now define functional self-reducibility for any set $A \in \Sigma_k^P$.

Definition 2:

(a) We say that (A,R) (or just R) is functionally self-reducible if

(i) $A = A_R$, and

(ii) $f_R \leq_T^P A$ - i.e., f_R is polynomially reducible (in the sense of Turing or Cook [2]) to A.

(b) $A \in \Sigma_k^P - \Sigma_{k-1}^P$ is <u>functionally</u> <u>self-reducible</u> if for every k+1-ary polynomially computable relation R defining A (as a Σ_k^P set) $f_R \leq_T^P A$.

<u>Open Problem 2</u>:

Suppose R_1 and R_2 both define $A \in \Sigma_k^P - \Sigma_{k-1}^P$ as a Σ_k^P set. Does it follow that R_1 is functionally self-reducible iff R_2 is functionally self-reducible?

If we assume $P \neq NP$, then it is not difficult to define Σ^* (clearly a Σ_0^P set) as a Σ_2^P set by a relation $R(x,y,z)$ so that f_R is not polynomially computable. Thus, in defining functional self-reducibility, one cannot demand that <u>every</u> relation defining set A be self-reducible. On the other hand, we do not know how to define Σ^* as a Σ_1^P set so that the R defining Σ^* is not self-reducible. (This seems to require the assumption that $\Sigma_1^P \cap \Pi_1^P \supsetneq \Sigma_0^P$; see Proposition 5 in Valiant [11]). Unfortunately, our definition cannot deal well with Σ_0^P sets defined as Σ_1^P sets, even though such sets may occur naturally. For example, the "natural" way to specify the composite numbers is as a Σ_1^P set; however, it may turn out that the composite numbers are in P, but producing a nontrivial factor is not polynomially computable (see Miller [8]).

III. Complete Sets are Functionally Self-Reducible

The following development shows that any set A which is complete in Σ_k^P is functionally self-reducible. The proof is based on the observation that the predicate $R(x,y)$ = "y is an accepting computation of x" is self-reducible. This observation can be found in Baker, Gill, and Solovay [1] and in Valiant [11]. For Σ_1^P, the theorem can be viewed as a restatement of Schnorr's [9] remark that "any polynomial complete set can be encoded such that the self-reducible property holds".

Theorem 1

Let A be complete in Σ_k^P $(k \geq 1)$. Then A is functionally self-reducible.

Proof:

Let $A = \{x \mid (\exists y_1) \ldots (Q_k y_k)\ R(x,y_1,\ldots,y_k)\}$. We wish to show that f_R can be computed in polynomial time given the oracle A.

We know that A is the set accepted by some nondeterministic oracle machine M which uses an oracle $B \in \Sigma_{k-1}^P$. Furthermore, we can insist that M asks only one oracle question and then decides membership in A. That is, M nondeterministically chooses a y_1 and then consults the oracle B, where

$$B = \{\langle x_1 y_1\rangle \mid (\exists y_2)(\forall y_3)\ldots(\bar{Q}_k y_k)\ \neg R(x,y_1,\ldots,y_k)\}$$

If $\langle x_1 y_1\rangle \notin B$, then M accepts; otherwise, M rejects. (For

$k=1$, $B \in \Sigma_0^P$ can be taken as $\{\langle x,y\rangle \mid \neg R(x,y)\}$.)

Now, consider the relation $R_M(x', y_1', \ldots, y_k')$ defined as follows:

(i) x' is of the form $1^t\ ID_j$

(ii) y' is of the form $m_1 \ldots m_t\ ID_{accept}$

(iii) from ID_j, M can proceed in t steps by the sequence of choices $m_1, \ldots, m_t$ to the accepting ID_{accept}. The last step consists of the appropriate oracle call.

By definition, the set A_M defined by R_M is a Σ_k^P set. It is easy to see that $A \leq_T^P A_M$, and since by hypothesis A is complete in Σ_k^P, we have $A \equiv_T^P A_M$.

Now observe that R_M is functionally self-reducible, even in the strong sense of Schnorr [8]. We can show that R is self-reducible - i.e., $f_R \leq_T^P A$, as follows:

(i) Transform x to $x' = 1^{P(n)} ID_0$.

(ii) Compute $f_{R_M}(x')$ using the oracle A; here is where we use the completeness of A, for we replace the oracle calls to A_M by calls to A.

(iii) Recover f_R from f_{R_M} in the obvious way.

In the case of $\Sigma_1^P = NP$, our intuition about all the known complete sets confirms Theorem 1. Now, consider Σ_2^P. The first complete set for this class was given by Meyer and Stockmeyer [7]; namely

B_2 = {R|R is a well-formed propositional formula over the variables $y_{11},\ldots,y_{12}$, $y_{21},\ldots,y_{2j}$, and $(\exists y_{11})\ldots(\exists y_{12})(\forall y_{21})\ldots(\forall y_{2j})\ R(y_{11}\ldots y_{2j})$ }

Clearly, R is functionally self-reducible. On the other hand, Stockmeyer [10] shows that INEQ = {<E,F>|E and F are "integer expressions" and the sets of integers defined by E and F are different.} is complete in Σ_2^P. In this case, the "natural" function to be computed is an integer which exhibits the inequality. It is not as obvious here that any relation defining INEQ should be self-reducible.

IV. A Σ_2^P Set Which is Not Functionally Self-Reducible

Consider the Σ_2^P set NMIN as defined in Meyer and Stockmeyer [7]: NMIN = {F|F is a propositional formula which is not minimal}. That is, F is in NMIN is there exists a shorter (say in terms of primitive propositional languages characters) equvalent formula. It is not know whether NMIN is complete for Σ_2^P. We also do not know whether NMIN - or the "natural" relation $R(x,y_1,y_2)$ = "y_1 is a shorter formula than x with the same truth value on assignment y_2" which defines NMIN - is functionally self-reducible.

Consider also the satisfiability problem S in Σ_1^P. Again it is not clear how to reduce S to NMIN, or NMIN to S. Although we cannot prove anything about S and NMIN, by Theorem

2 the possibility exists that S and NMIN are incomparable.

On the other hand, it is easy to reduce satisfiability to the "natural" functional problem associated with NMIN, namely, $f(F) = F'$ where F' is a shorter formula equivalent to F. Given F, we can keep computing shorter and shorter equivalent formulas until a minimal formula F_x is derived. F is satisfiable if and only if F_x is not $v \cap \neg v$. Theorem 3 shows that whenever $A \in \Sigma_2^P - \Sigma_1^P$ is sufficiently dense, we can reduce any $B \in \Sigma_1^P$ to some functional problem associated with A.

Using Theorems 2 and 3, we can show that there are sets $A \in \Sigma_2^P$ which are not functionally self-reducible.

The next result is based directly on the proof techniques of Ladner [5].

Theorem 2:

Suppose for some $B \notin \Sigma_0^P$ and $C \notin \Sigma_1^P$ we have $B \leq_T^P C$ but $C \not\leq_T^P B$. Then there exists $D \in P$ such that $A = C \cap D$ and B are incomparable; furthermore $A \notin \Sigma_1^P$.

Proof:

We follow Ladner and construct a polynomial time bounded transducer T such that $D = \{x \mid |T(x)| \text{ is even}\}$. T is trying to satisfy the following conditions:

1) $B \neq M_i(A)$

2) $A \neq M_i(B)$

3) $A \neq NP_i$

where $M_i(A)$ denotes the language recognized by the i^{th} deterministic polynomial time bounded Turing machine with oracle A, and NP_i is the language recognized by the i^{th} nondeterministic polynomial time bounded TM. To satisfy 1) we make D look empty. To satisfy 2) and 3) we make D look like Σ^*. ($\Sigma = \{0,1\}$.) T operates as follows:

$T(\lambda) = \lambda$

$T(x) = T(0^n)$ for $x \in \Sigma^n - \{0^n\}$

On input 0^n, T first tries, for n moves, to reconstruct the sequence $T(\lambda)$, $T(0)$, $T(00)$, ... as far as possible. Let $T(0^m)$ be the last value computed. Then

(a) If $|T(0^m)| = 4i$, we are trying to make $A \neq M_i(B)$. That is, for n moves T tries to find a z such that $A(z) \neq M_i(B)(z)$. If no such z is found, T prints 4i; otherwise it prints 4i+1. In doing this simulation, T only needs to look at those z of length $\leq$ m. Also, as part of the n moves allowed for the simulation we count the time required to compute each oracle question using some fixed, perhaps very inefficient deterministic algorithm for B.

(b) If $|T(0^m)| = 4i+1$, we are trying to make $B \neq M_{2i}(A)$. Again, for n moves T tries to find a z such that $B(z) \neq M_{2i}(A)(z)$. In doing so, T looks only at z and A oracle questions of length $\leq$ m. If no such z is found, T outputs 4i+1; otherwise 4i+2.

(c) If $|T(0^m)| = 4i+2$, we are trying to satisfy $A \neq NP_i$. For n moves, T tries to find a z such that $A(z) \neq NP_i(z)$. Here T must try all computation paths for $NP_i(z)$. If no such z is found, T outputs 4i+2; otherwise 4i+3.

(d) If $|T(0^m)| = 4i+3$, we are trying to satisfy $B \neq M_{2i+1}(A)$ (as in case b). If T can find a x such that $B(z) \neq M_{2i=1}(A)(z)$ then it outputs $4i+4 = 4(i+1)$; otherwise 4i+3.

If range $(T) = 1^*$, then the set A must be as desired. By cases, we can show that T cannot "get stuck" on 4i, 4i+1, 4i+2, or 4i+3 for any i.

Corollary:

(Assuming $\Sigma_2^P - \Sigma_1^P \neq \phi$) Let B be in $\Sigma_1^P - \Sigma_0^P$ and not complete for Σ_1^P. Then there exists $A \in \Sigma_2^P - \Sigma_1^P$ such that A and B are incomparable.

Proof:

In Theorem 2, choose C to be complete in Σ_2^P. Then $B \leq_T^P C$ (since $\Sigma_2^P \supseteq \Sigma_1^P$) and $C \not\leq_T^P B$ (since B is not complete for Σ_2^P). $A = C \cap D$ is still in $\Sigma_2^P - \Sigma_1^P$, and by the theorem A and B are incomparable.

Remark:

If we know $\Sigma_2^P - \Delta_2^P \neq \phi$ (as in recursive function theory) then in the Corollary we would not have to require that B is not complete for Σ_1.

Open Problem 3:

Assuming $\Sigma_k^P - \Sigma_{k-1}^P \neq \phi$, $(\exists A \in \Sigma_k^P - \Sigma_{k-1}^P)(\forall B \in \Sigma_{k-1}^P)\ B \not\leq_T^P A$?

As constructed in Theorem 2, the set A may be sparse; that is,

$$\lim_{n\to\infty} \frac{\#\{x \mid x \in A,\ |x| < n\}}{\#\{x \mid x \in \Sigma^*,\ |x| \leq n\}} = 0.$$

However, we can construct A join H, for any set H, by letting $A' = A \nabla H = \{0x \mid x \in A\} \cup \{1y \mid y \in H\}$. Clearly A' is dense if H is dense; moreover, if A satisfies the Corollary and H is in P, then A' satisfies the Corollary also. We may as well let H be Σ^*, though in what follows all we need is that H (and thus A') has an unbounded regular subset.

Lemma:

(Hopcroft [3]). If A has an unbounded regular subset, then there exist r, u, v, and s in Σ^* such that u and v do not commute and $r\{u,v\}^*s \subseteq A$.

The impact of this lemma is that every word in $\{0,1\}^*$ can be encoded as a word in A. That is, $b_1 \ldots b_k$ is encoded as $k(b_1 \ldots b_k) = r\, z_1 \ldots z_k s$, where $z_i = uv$ (if $b_i = 0$) or vu (if $b_i = 1$). Clearly k is injective, and any sets B and $kB = \{kx \mid x \in B\}$ are of essentially the same complexity.

Given incomparable sets A and B (as in the Corollary), we can now construct a relation $\tilde{R}$ defining A as a Σ_2^P set such that $B \leq_T^P f_{\tilde{R}}$.

Theorem 3:

Let $B \in \Sigma_1^P$, $A \in \Sigma_2^P$, and $B \subseteq A$. (Note: by using the previous lemma, we can replace the condition "$B \subseteq A$" by "A has an unbounded regular subset"). Then there exists a polynomial computable predicate $\tilde{R}(x,y,z)$ defining A as a Σ_2^P set so that $B \leq_T^P f_{\tilde{R}}$.

Proof:

Let R define B as a Σ_1^P, and let R' define A as a Σ_2^P set; that is,

$$A = \{x \mid (\exists y)(\forall z)\ R'(x,y,z)\}$$

$$B = \{x \mid (\exists y)\ R(x,y)\}.$$

Without loss of generality, we may assume

(1) $R(x,\lambda)$ is false for all x. Otherwise, define $\hat{R}(x,by) = R(x,y)$ for any $b \in \Sigma$, and $\hat{R}(x,\lambda) =$ false. Clearly $\hat{R}$ defines B as a Σ_1^P set just as R does.

(2) $\langle y_1,y_2\rangle$ represents a pairing of y_1 and y_2.

Now define

$$\begin{aligned}\tilde{R}(x,\tilde{y},\tilde{z}) &= \tilde{R}(x, \langle y_1,y_2\rangle, \langle z_1,z_2\rangle)\\ &= R'(x,y_1,z_1) \wedge ((y_2 \neq \lambda \wedge R(x,y_2)) \vee (y_2 = \lambda \wedge \neg R(x,z_2))).\end{aligned}$$

We first need to show that $\tilde{R}$ defines A.

1) ($A_{\tilde{R}} \subseteq A_{R'} = A$). We must show

$$(\exists y)(\forall z)\ R'(x,y,z) \Rightarrow (\exists \tilde{y})(\forall \tilde{z})\ \tilde{R}(x,\tilde{y},\tilde{z}).$$

Case 1: $x \in B \subseteq A$. Then there exists $y_2 \neq \lambda$ such that $R(x,y_2)$ is true, since $x \in B$. Similarly, $(\exists y_1)(\forall z)\ R'(x,y_1,z)$. Thus, letting $\tilde{y} = \langle y_1, y_2 \rangle$ we have $(\forall \tilde{z})\ \tilde{R}(x,\tilde{y},\tilde{z})$ as required.

<u>Case 2</u>: $x \in A-B$. Then $(\exists y_1)(\forall z_1)\ R'(x,y_1,z_1)$, but $(\forall z_2\ \neg R(x,z_2)$ (i.e., $x \notin B$). Letting $\tilde{y} = \langle y_1, \lambda \rangle$, we have $(\forall \tilde{z})\ \tilde{R}(x,\tilde{y},\tilde{z})$.

2) ($A = A_{R'} \subseteq A_{\tilde{R}}$). Suppose $x \in A$. Then $(\exists \tilde{y})(\forall \tilde{z})\ \tilde{R}(x,\tilde{y}_1,\tilde{z})$. Let $\tilde{y} = y_1, y_2$; then $(\forall \tilde{z})\ R'(x,y_1,z_1)$, and thus $(\forall z_1)\ R'(x,y_1,z_1)$, as desired.

We now show that $B \leq_T^P f_{\tilde{R}}$.

1) If $f_{\tilde{R}}(x)$ = "no" then $x \notin A$, and hence $x \notin B$.

2) If $f_{\tilde{R}}(x) = \tilde{y} = \langle y_1, \lambda \rangle$ then $(\forall z)\ \neg R(x,z)$ (by definition of $f_{\tilde{R}}$ and $\tilde{R}$), thus $x \notin B$

3) If $f_{\tilde{R}}(x) = \tilde{y} = \langle y_1, y_2 \rangle$, $y_2 \neq \lambda$, then $R(x,y_2)$ is true, and thus $x \in B$.

We can now prove the main result of this section; namely, that there is a non self-reducible set in Σ_2^P.

<u>Theorem 4</u>:

(Assuming $\Sigma_2^P - \Sigma_1^P \neq \phi$) There exists a set $A \in \Sigma_2^P$, and a relation $R(x,y,z)$ defining A as a Σ_2^P set, such that $f_R \not\leq_T^P A$.

<u>Proof</u>:

Using Theorems 2 and 3, we can construct sets $A \in \Sigma_2^P - \Sigma_1^P$ and $B \in \Sigma_1^P$, and a relation $R(x,y,z)$ defining A as a Σ_2^P set, such that $B \leq_T^P f_R$ but A and B are incomparable. By transitivity of $\leq_T^P$, $B \leq_T^P f_R$ and $B \not\leq_T^P A$ imply $f_R \not\leq_T^P A$, so A has the desired properties.

It is easy to translate the above results upward to show that for any $k > 1$ there exists a set $A \in \Sigma_k^P - \Sigma_{k-1}^P$ and a relation R defining A as a Σ_k^P set such that (A,R) is not self-reducible. Unfortunately, the existence of non-self-reducible sets in Σ_1^P remains an open question.

Valiant [11, Proposition 5] shows that, assuming $(\Sigma_1^P \cap \Pi_1^P) - \Delta_1^P \neq \emptyset$, i.e., assuming $P \not\subseteq NP \cap co\text{-}NP$, there exists a total function which can be checked but not evaluated in polynomial time. In our terminology, he gives a polynomial binary relation $R(x,y)$ which defines Σ^* as a Σ_1^P set and which is not self-reducible. Using the techniques of this section, we can strengthen Valiant's result by showing the existence of a non-self-reducible binary relation defining a set which is properly in Σ_1^P.

Theorem 5:

Assume $\Sigma_1^P \cap \Pi_1^P - \Delta_1^P, \neq \emptyset$. Then there exists a polynomial binary relation $R(x,y)$ such that $A_R \in \Sigma_1^P - \Sigma_0^P$ and $f_R \not\leq_T^P A_R$.

Proof:

Choose $B \in (\Sigma_1^P \cap \Pi_1^P) - \Delta_1^P$. Then there exist polymonial relations R_B and $R_{\bar{B}}$ such that

$$B = \{x \mid (\exists y)\ R_B(x,y)\}$$

$$\bar{B} = \{x \mid (\exists y)\ R_{\bar{B}}(x,y)\}.$$

By Ladner [5, Theorem 1], there exists a set $A \in \Sigma_1^P - \Sigma_0^P$ such that $A <_T^P B$ but $B \not<_T^P A$. Let

$$A' = A \nabla \{0,1\}^* = \{0x \mid x \in A\} \cup \{1y \mid y \in \{0,1\}^*\}$$

Clearly $A \notin {}_0^P$, since $A \leq_T^P A'$; moreover $B \not\leq_T^P A'$. We next construct a relation R' defining A' such that $B \;{}_T^P\; f_{R'}$. Of course, let

$$A = \{x \; (\exists y) \; R_A(x,y)\}.$$

We define R' by

$$R'(ax,y) = \begin{cases} R_A(x,y) & \text{if } a=0 \\ R_B(x,y) \vee R_{\bar{B}}(x,y) & \text{if } a=1 \end{cases}$$

Clearly R' defines A' as a Σ_1^P set. Also, $B <_T^P f_{R'}$, since

$$x \in B \text{ iff } R_B(x,\; f_{R'}\; (1x)).$$

Since $B \not<_T^P A'$, it follows that $f_{R'} \not<_T^P A'$, and R' is the desired relation.

Like Theorem 4, this result can be generalized to any $k \geq 1$.

With Theorems 4 and 5 as evidence, we close with the following:

Conjecture:

There exists a set $A \in \Sigma_1^P - \Sigma_0^P$ and a polynomial relation R(x,y) defining A (as a Σ_1^P set) such that (A,R) is not self-reducible.

References

[1] T. Baker, J. Gill, and R. Solovay, "Relativization of the P = ? NP Question", Siam Journal on Computing 4:4, December 1975.

[2] S.A. Cook, "The Complexity of Theorem-Proving Procedures", Proc. Third STOC, 1971.

[3] J.E. Hopcroft, "On the Equivalence and Containment Problems for Context-free Languages", Mathematical Systems Theory 3:2, 1969.

[4] R.M. Karp, "Reducibility among Combinatorial Problems", in Complexity of Computer Computations, R.E. Miller and J.W. Thatcher, eds. Plenum Press, New York, 1972

[5] R.E. Ladner, "On the Structure of Polynomial Time Reducibility", JACM 22:1, January, 1975.

[6] L.A. Levin, "Universalnye Zadaci Perebora", Problemy Peredaci Informacii 9:3, 1973 (Russian).

[7] A. Meyer and L. Stockmeyer, "The Equivalence Problem for Regular Expressions with Squaring Requires Exponential Space", Proc. 13th SWAT, October 1972.

[8] G. Miller, "Riemann's Hypothesis and Tests for Primality", Proc. 7th STOC, May 1975.

[9] C.P. Schnorr, "Optimal Algorithms for Self-Reducible Problems", manuscript, September 1975.

[10] L.J. Stockmeyer, "The Polynomial Time Hierarchy", IBM Research Tech. Report RC 5379, April 1975.

[11] L. Valiant, "On the Relative Complexity of Checking and Evaluating", Univ. of Leeds Tech. Rept. LS29JT, October, 1974.

[12] C. Wrathall, "Complete Sets and the Polynomial Hierarchy", manuscript, Harvard Univ., 1975.