# ON THE GENERATION OF PRIME IMPLICANTS[+]

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## 1. Introduction

The main purpose of this paper is to show that certain normal forms for Boolean functions as well as a number of the most basic methods for generation of prime implicants can very easily be derived using certain notations, basic theorems and proof techniques. We hope to make available material in a short and readable way, which is otherwise scattered over a large number of original papers and, in our opinion, not properly presented in textbooks on Switching Theory.

The next section is used to introduce our notion of Boolean functions, Boolean formulas and subfunctions. The latter is similar to the one used in [5]. Theorem 2.1 proves to be central for this section, a weaker form of it can also be found in [5]. From this theorem are easily derived as corollaries the Shamon-decompositions (which were well known in the 19th century!) as well as the minterm- and maxterm-normal forms for Boolean functions.

Section 3 is devoted to the discussion of various algorithms for generation prime implicants of Boolean functions. We did not concentrate on variations of basic methods, which were introduced

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for better implementations, as for instance in [11,12,17,18,44], nor did we want to deal with the well known topological terminology in [29,30,44], which is nicely summarized in [43]. Also multiple output prime implicants or multi-level prime implicants were not our deal.

The basic result of this section is Theorem 3.1 and its collary. Brown in [5] mentions a theorem of Blake [4], which is based on the same idea. Our theorem is proven to be the basis for all product-methods [1,3,5,7,8,14,21,22,41] whereas its corollary, independently found in [26,27], is basic for all consensus-algorithms [13,15,17,24,25,26,40,41] and variations of them [2,10,11,15,16,17,18,19]. Some tree-methods and other decomposition-methods [6,7,20,26,32] can also be proven by Lemma 3.1.

The last section gives some hints on the complexity of the algorithms discussed. Worst-case considerations are shown to be related to the celebrated (P = NP)-question.

Although we have emphasized the tutorial nature of this paper, we want to point out that a number of things are new. First of all practically all proofs are new. Theorem 2.1 and Theorem 3.1, to our best knowledge, have not been reported in this general form elsewhere. The method of exclusion in Section 3,f is a fair generalization of an equally named method by Tison [40], the proof of which is reported to having been subject to some doubts [42].

Our list of references is by no means complete. To compute a complete list of the enormous number of papers on the generation of prime implicants is not an easy task and must be left to someone else.

## 2. Boolean functions and formulas.

In this section we want to introduce briefly the basic notions of Boolean functions and formulas as well as their interrelationship. We will use what we think to be a rather elegant notation to state and prove some fundamental representation theorems.

Def. 2.1: A total mapping $f:\{0,1\}^n \to \{0,1\}$ is called a *Boolean function*. n is called the argument length of f.

The set {0,1} carries a simple algebraic structure, which is easily extended to the set of Boolean functions in the usual way.

| a | b | a+b | a·b | $\bar{a}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 |

{0,1} together with these operations is a Boolean lattice.

Def. 2.2: Let f and g be Boolean functions with the same argument length n. Then

$(f+q)(b) := f(b)+g(b)$

$(f\cdot g)(b) := f(b)\cdot g(b)$

$\bar{f}(b) := \overline{f(b)}$

for all $b \in \{0,1\}^n$.

We have not introduced new symbols for the operations "+", "·" and "-" on Boolean functions because of their close relation to those on $\{0,1\}$ and since confusion is unlikely to arise.

Boolean functions of equal argument length together with the operations of Def. 2.2 form a Boolean lattice, the greatest and smallest elements of which are denoted by 1 and 0 respectively, where $1(b)=1$ and $0(b)=0$ for all $b \quad \{0,1\}^n$.

Boolean functions can be described by Boolean formulas which are defined as follows:

<u>Def. 2.3</u>: <u>Boolean formulas</u> are the strings generated by the following grammar:

Startsymbol : S

Nonterminal : B

Terminals : 0, 1, $x_1,\ldots,x_n$, +, ·, -, (, ).

Productions:: $S \rightarrow B$

$B \rightarrow (B)\cdot(B)$

$B \rightarrow (B)+(B)$

$B \rightarrow (\bar{B})$

$B \rightarrow x_i$ , for $i = 1,\ldots,n$

For instance $(((x_1))(x_2)) + ((x_1)(x_2)))$ is a Boolean formula, which is convenient to be written as $\bar{x}_1x_2 + x\bar{x}_2$ with the understanding that "·" binds stronger than "+". The set of Boolean formulas itself forms an algebra, the free $<\cdot,+,->$-algebra over $\{x_1,\ldots,x_n\}$, since if A and B are Boolean formulas so are (A)+(B),

$(A)\cdot(B)$ and $(\bar{A})$.

This free algebra is homomorphically mapped onto the Boolean lattice of Boolean functions, if we assign "a meaning" to each formula in the following way.

<u>Def. 2.4:</u> The Boolean formula A is mapped onto the Boolean functions $\langle A\rangle$, where $\langle A\rangle$ is defined recursively as follows:

Let A and B be Boolean formulas, then

$$\langle(A) + (B)\rangle := \langle A\rangle + \langle B\rangle$$

$$\langle(A) \cdot (B)\rangle := \langle A\rangle \cdot \langle B\rangle$$

$$\langle(\bar{A})\rangle := \langle\bar{A}\rangle$$

$$\langle 0\rangle := |0|$$

$$\langle 1\rangle := 1$$

$$\langle x_j\rangle(b_1,\ldots,b_n) := b_j \text{ , for all } b_j \; \{0,1\} \text{ and } i = 1,\ldots,n$$

We say f is represented by A if $f = \langle A\rangle$. By definition, every Boolean formula represents some Boolean function. We will show, that every Boolean function is represented by some Boolean formula also.

It is convenient for our purposes to identify certain formulas.

<u>Def. 2.5</u>: Let $B_1$, $B_2$, $B_3$, $B_4$ be arbitrary Boolean formulas.

1. $(B_1)\cdot(B_2) \equiv (B_2)\cdot(B_1)$

   $(B_1)+(B_2) \equiv (B_2)+(B_1)$

   $((B_1)\cdot(B_2))\cdot(B_3) \equiv (B_1)\cdot((B_2)(B_3))$

$$((B_1)+(B_2))+(B_3) \equiv (B_1)+((B_2)+(B_3))$$
$$(B_1)+(B_1) \equiv B_1$$
$$(B_1)\cdot(B_1) \equiv B_1$$
$$B_1 \equiv B_1$$

2. If $B_1 \equiv B_2$ , then
$$(B_3)\cdot(B_1) \equiv (B_3)\cdot(B_2)$$
$$(B_3)+(B_1) \equiv (B_3)+(B_2)$$
$$(\overline{B_1}) \equiv (\overline{B_2})$$

3. If $B_1 \equiv B_2$ and $B_2 \equiv B_3$ according to 1. or 2., then $B_1 \equiv B_3$.

It can be shown that "$\equiv$" is a congruence relation on the set of Boolean formulas. Hence operations can be defined on the equivalence classes, such that the new algebra is a homomorphic image of the algebra of Boolean formulas and the Boolean lattice of Boolean functions is still a homomorphic image of it. This simply means that we did not identify Boolean formulas by def. 2.5, which represent different Boolean functions.

In the sequel we call the equivalence classes defined above formulas for short. Associativity saves us brackets and we need not write out repetitions.

Let us now come back to the question whether every Boolean function is represented by some Boolean formula. We need the following definitions and simple facts.

<u>Def. 2.6</u>: a) $\tilde{x}_{i_1} \ldots \tilde{x}_{i_k}$ is a <u>product-term</u>, if

$i_j \neq i_\ell$ for $j \neq \ell$ and $\tilde{x}_{i_j} \quad \{x_{i_j}, \bar{x}_{i_j}\}$

b) $\tilde{x}_{i_1} + \ldots + \tilde{x}_{i_k}$ is a <u>sum-term</u>, if

$i_j \neq i_\ell$ for $j \neq \ell$ and $\tilde{x}_{i_j} \quad \{x_{i_j}, \bar{x}_{i_j}\}$

We add 0 and 1 to be special product-terms as well as special sum-terms.

c) $\tilde{x}_{i_1} \ldots \tilde{x}_{i_k}$ is a <u>minterm</u> in the symbols

$x_1, \ldots, x_n$, if it is a product-term and

$\{i_1, \ldots, i_k\} = \{1, \ldots, n\}$

d) $\tilde{x}_{i_1} + \ldots + \tilde{x}_{i_k}$ is a <u>maxterm</u> in the symbols

$x_1, \ldots, x_n$, if it is a <u>sum-term</u> and $\{i_1, \ldots, i_k\} =$

$\{1, \ldots, n\}$

e) Let $P = \tilde{x}_{i_1} \ldots \tilde{x}_{i_k}$ and $Q = \tilde{x}_{j_1} \ldots \tilde{x}_{j}$ be product terms . P is a <u>shortening</u> of Q, if $\{i_1, \ldots, i_k\} \subseteq \{j_1, \ldots, j_\ell\}$ and $\tilde{x}_{i_m} = \tilde{x}_{j_n}$ for all $i_m = j_n$.

P is a <u>proper shortening</u> of Q, if it is a shortening and $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_\ell\}$.

Any product term P is a shortening of 0 and 1 is a shortening of every product term.

Product-terms, sum-terms, minterms and maxterms so defined are special Boolean formulas and as such represent Boolean functions. We will now associate with product-terms another Boolean function.

<u>Def. 2.7</u>: Notation: $x_i^0 := \bar{x}_i$ , $x_i^1 := x_i$ .

Let $t = x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$ be a product-term and

$b = (b_1,\ldots,b_n)\ \ \{0,1\}^n$. We define

$\langle t=1\rangle(b) := (b_1',\ldots,b_n')$ , where $b_{i_j}' := e_j$ for

$j = 1,\ldots,r$ and $b_k' := b_k$ for $k \mid \{i_1,\ldots,i_r\}$.

The functions $\langle t\rangle$ and $\langle t=1\rangle$ are different but closely related.

<u>Lemma 2.1:</u> Let t be a product-term. Then

$$\langle t\rangle(b) = 1 \quad \text{iff} \quad \langle t=1\rangle(b) = b$$

<u>Proof</u>: We have $t = x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$ , $b = (b_1,\ldots,b_n)$ and $\langle t=1\rangle(b) = (b_1',\ldots,b_n')$. The following statements are pairwise equivalent: a) $\langle t\rangle(b) = 1$; b) $\langle x_{i_j}^{e_j}\rangle(b) = 1$ for all $j = 1,\ldots,r$; c) $b_{i_j} = e_j$ for $j = 1,\ldots,r$; d) $b_{i_j} = b_{i_j}'$ for $j = 1,\ldots,r$; e) $\langle t=1\rangle(b) = b$.

Before stating the main theorem of this section we have to introduce the notion of subfunctions.

<u>Def. 2.8</u>: Let f be a Boolean function and t a product-term. The <u>subfunction of f with respect to t</u> is defined by

$f_t := f \circ \langle t=1 \rangle$ ,

where $f \circ \langle t=1 \rangle$ means: apply $\langle t=1 \rangle$ first and then f to the result.

<u>Theorem 2.1:</u> Let f be a Boolean function and let $t_1,\dots,t_k$ be product-terms such that

$$\langle \sum_{i=1}^{k} t_i \rangle = 1. \quad \text{Then}$$

a) $$f = \sum_{i=1}^{k} \langle t_i \rangle f_{t_i}$$

b) $$f = \prod_{i=1}^{k} (\langle \overline{t_i} \rangle + f_{t_i})$$

<u>Proof:</u> For any $b \in \{0,1\}^n$ there is an i such that $\langle t_i \rangle (b) = 1$ because of $\langle \sum_{i=1}^{k} t_i \rangle = 1$. For this i we have $\langle t_i = 1 \rangle (b) = b$ by the previous lemma and therefore $f_{t_i}(b) = f(\langle t_i = 1 \rangle (b)) = f(b)$. For every $b \in \{0,1\}^n$ we define $I_b := \{i \mid \langle t_i \rangle (b) = 1\}$ and $J_b := \{i \mid \langle t_i \rangle (b) = 0\}$.

a) $$\left(\sum_{i=1}^{k} \langle t_i \rangle f_{t_i}\right)(b) = \sum_{i \in I_b} \langle t_i \rangle (b) f_{t_i}(b) + \sum_{i \in J_b} \langle t_i \rangle (b) f_{t_i}(b)$$

$$= \sum_{i \in I_b} f_{t_i}(b) = f(b)$$

b) $$\prod_{i=1}^{k} (\langle \overline{t_i} \rangle (b) + f_{t_i}(b)) =$$

$$= \prod_{i \in I_b} (\langle \overline{t_i} \rangle (b) + f_{t_i}(b)) \prod_{i \in J_b} (\langle \overline{t_i} \rangle (b) + f_{t_i}(b))$$

$$= \prod_{i \in I_b} f_{t_i}(b) = f(b) \;.$$

Two very well known formulas can very easily be derived from the general one given in this theorem.

<u>Lemma 2.2:</u> (Shannon-decomposition)

Let f be a Boolean function and x one of its variables. Then

a) $f = \langle \bar{x} \rangle f_{\bar{x}} + \langle x \rangle f_x$

b) $f = (\langle \bar{x} \rangle + f_x)(\langle x \rangle + f_{\bar{x}})$

<u>Proof:</u> This is a special case of Theorem 2.1 if we show $\langle \bar{x} \rangle = \langle \tilde{x} \rangle$ and $\langle x + \bar{x} \rangle = 1$. Both are trivial.

<u>Lemma 2.3</u>: Let f be a Boolean function in the variables $x_1,\ldots,x_n$.

a) (Minterm-normalform)

$$f = \langle \sum_{(e_1,\ldots,e_n)\in\{0,1\}^n} x_1^{e_1} \ldots x_n^{e_n} f(e_1,\ldots,e_n)\rangle$$

b) (Maxterm-normalform)

$$f = \langle \prod_{(e_1,\ldots,e_n)\in\{0,1\}^n} (x_1^{\bar{e}_1}+\ldots+x_n^{\bar{e}_n} + f(e_1,\ldots,e_n))\rangle$$

<u>Proof</u>: In order to apply Theorem 2.1, we only have to show the following three simple facts.

1) Let $b = (e_1,\ldots,e_n)$. Then $\langle x_1^{e_1}\ldots x_n^{e_n} = 1\rangle(b) = b$ and therefore $\langle x_1^{e_1} \ldots x_n^{e_n}\rangle(b) = 1$. Thus $\langle \sum_{(e_1,\ldots,e_n)\ \{0,1\}^n} x_1^{e_1} \ldots x_n^{e_n}\rangle(b) = 1$. Since b was arbitrary we conclude

$$\langle \sum_{(e_1,\ldots,e_n)\in\{0,1\}^n} x_1^{e_1} \ldots x_n^{e_n}\rangle = 1 .$$

2) Let $t = x_1^{e_1} \ldots x_n^{e_n}$. Then $f_t(b) =$ $= f(\langle x_1^{e_1} \ldots x_n^{e_n} = 1\rangle(b)) = f(e_1,\ldots,e_n)$ for any $b \in \{0,1\}^n$.

3) $\overline{\langle x_1^{e_1} \dots x_n^{e_n}\rangle} = \overline{\langle x_1^{e_1}\rangle} + \dots + \overline{\langle x_n^{e_n}\rangle} = \langle x^{\bar{e}_1} + \dots + x_n^{\bar{e}_n}\rangle$

Lemma 2.3 also solves a question we raised earlier.

Corollary: For every Boolean function there is a Boolean formula representing it.

The notations and methods developed in this section will be used to examine various algorithms to generate all the prime implicants of a given Boolean function f.

## 3. Algorithms to generate prime implicants.

In this section various well known algorithms to generate prime implicants for Boolean functions given in some sum-of-products form are presented and their correctness is proved in a rather uniform manner.

Def. 3.1: Let f be a Boolean function and P a product-term.

a) P is an implicant of f, if $\langle P\rangle(b) = 1$ implies $f(b) = 1$ for all $b \in \{0,1\}^n$. Notation: $\langle P\rangle \to f$.

b) P is a prime implicant of f, if $\langle P\rangle \to \langle Q\rangle \to f$ implies $P = Q$ for all product terms Q.

P is a prime implicant of f simply if it is an implicant of f and no proper shortening of P has this property. The following will turn out to be a basic theorem for proving several algorithms for generating prime implicants correct. As a notation we use P*Q to denote the product term we get from product-terms P and Q by concatenation and then removing repetitions and contradictions. For example $x_1x_2*x_2x_3 = x_1x_2x_3$ but $x_1\bar{x}_2*x_2x_3 = 0$.

<u>Theorem 3.1</u>: Let $f = \prod_{i=1}^{\ell} f_i$ and P be a prime implicant of f. Then $P = P_1* \ldots *P_\ell$ where $P_i$ is a prime implicant of $f_i$, $i=1,\ldots,\ell$.

<u>Proof</u>: From $\langle P\rangle(b)=1$ we conclude $f(b)=1$ and $f_i(b)=1$ for $i=1,\ldots,\ell$. Therefore P is an implicant for all $f_i$'s. But then there exist prime implicants $P_i$ of $f_i$ such that $\langle P\rangle\rightarrow\langle P_i\rangle$, $i=1,\ldots,\ell$ which induces $\langle P\rangle\rightarrow\langle P_1*\ldots*P_\ell\rangle$. If we now can show $\langle P_1*\ldots*P_\ell\rangle\rightarrow f$, we can conclude $P = P_1*\ldots*P_\ell$ by the definition of prime implicants.

If $\langle P_1*\ldots*P_\ell\rangle(b)=1$, then $\langle P_i\rangle(b)=1$ and hence $f_i(b)=1$ for $i=1,\ldots,\ell$. But this immediately gives $f(b)=1$ and hence $P_1*\ldots*P_\ell$ is an implicant of f.

A simple corollary of this theorem will also prove to be very helpful. It shows how prime implicants of a function f depend on the prime implicants of its subfunctions with respect to product-forms x and $\bar{x}$, where x is a variable of the function. This result will be used later for straight forward inductions on the number of variables.

<u>Lemma 3.1</u>: Let f be a Boolean function, x one of its variables and P a prime implicant of f. Then one of the following statements is valid:

1) $P = \bar{x}P_0$ and $P_0$ is a prime implicant of $f_{\bar{x}}$.

2) $P = xP_1$ and $P_1$ is a prime implicant of $f_x$.

3) $P = P_0 * P_1$ and $P_0$ is a prime implicant of $f_{\bar{x}}$ and $P_1$ a prime implicant of $f_{\bar{x}}$.

Proof: We know $f = (\langle\bar{x}\rangle + f_x)(\langle x\rangle + f_{\bar{x}})$. To apply Theorem 3.1, we only have to show that if Q is a prime implicant of $(\langle\bar{x}\rangle + f_x)$ then either $Q = \bar{x}$ or Q is a prime implicant of $f_x$ and the related statement for $(\langle x\rangle + f_{\bar{x}})$. If $Q \neq \bar{x}$, then $Q \neq \bar{x}Q'$ and $Q \neq xQ''$ since otherwise Q could not be a prime implicant. Therefore $\langle Q\rangle(b) = (\langle Q\rangle \circ \langle x=1\rangle)(b)$ for all b. Assume $\langle Q\rangle(b) = 1$. Then we know $(\langle Q\rangle \circ \langle x=1\rangle)(b) = 1$ also and $\langle\bar{x}\rangle(\langle x=1\rangle(b)) = 0$. But since Q is an implicant of $(\langle\bar{x}\rangle + f_x)$ we conclude $f_x(\langle x=1\rangle(b)) = 1$. Therefore $(f \circ \langle x=1\rangle \circ \langle x=1\rangle)(b) = (f \circ \langle x=1\rangle)(b) = f_x(b) = 1$ and Q is an implicant of $f_x$. Clearly it must be a prime implicant since implicants of $f_x$ also are implicants of $(\langle\bar{x}\rangle + f_x)$

Both results of this section imply simple algorithms to generate all prime implicants. We will give a recursive "schema" for algorithms which can be derived from Theorem 3.1.

Algorithm 0:

(1) Represent f as $f = \prod_{i=1}^{k} f_i$ .

(2) $\ell = 1$

(3) Are the prime implicants of $f_\ell$ known?

no: Apply Algorithm 0 to $f_\ell$.

(4) Remember the prime implicants of $f_\ell$.

(5) $\ell = k$?

yes: goto (7)

(6) $\ell := \ell+1$, goto (3)

(7) Construct all possible $P_1 * \ldots * P_k$, where $P_\ell$ is prime implicant of $f_\ell$.

(8) Remove all product-terms for which there are proper shortenings in the list.

(9) Stop.

In fact a number of known algorithms more or less use this schema. There are two points in Algorithm 0 where variations come in.

a) How does one represent f as a product?

b) When are the prime implicants of $f_\ell$ "known"?

We will now introduce a number of methods to generate prime implicants, the first two of which will be instances of what is said above. The starting point for all methods will be a sum-of-products representation of the Boolean function under consideration.

a) Method of double product [3,4,21,22,29].

If f is given in a product of sums form, then merely "multiplying out" and removing terms if shortenings also appear in the resulting sum-of-products form will result in the sum of all prime implicants. This follows from Theorem 3.1 and the fact that implicants of length one of functions $f \not\equiv 1$ are prime implicants. Converting a given sum-of-products form into a product-of-sums form can be done by using the rule $\langle a+bc \rangle = \langle (a+b)(a+c) \rangle$ repeatedly. The methods name stems from a way to do these operations, which is demonstrated by the following example.

| | |
|---|---|
| f is given by | $x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1\bar{x}_2x_4$ |
| replace "+" by "·" and vice versa | $(x_1+\bar{x}_2)(x_2+\bar{x}_3)(\bar{x}_1+\bar{x}_2+x_4)$ |
| multiply out | $\bar{x}_2\bar{x}_3 + x_1x_2x_4 + x_1\bar{x}_3x_4$ |
| replace "+" by "·" and vice versa | $(\bar{x}_2+\bar{x}_3)(x_1+x_2+x_4)(x_1+\bar{x}_3+x_4)$ |
| multiply out | $x_2\bar{x}_3 + x_1\bar{x}_2 + \bar{x}_2x_4 + x_1\bar{x}_3 + \bar{x}_3$ |

The same method sometimes is called the "method of double dualization". The "method of double inversion" is practically the same. If f is given by a sum-of-products form, then a sum-of-products form of $\bar{f}$ is computed by using deMorgan's law and after that the same method is used to compute a sum-of-products form of $f = \bar{\bar{f}}$ from $\bar{f}$. Again by Theorem 3.1 the result is the sum of all prime implicants of f.

There is a version which is less easy to detect namely the one of "sharp products", which we will formulate next.

<u>Def. 3.2</u>: Let Q and $P = x_{i_1}^{e_1} \ldots x_{i_k}^{e_k}$ be product terms. Then

$$Q \# P := \{Q * x_{i_j}^{-e_j} \mid j = 1,\ldots,k\}$$

Let $L = \{Q_1,\dots,Q_m\}$ be a list of product terms. $Q_k$ is called <u>removable in L</u>, if there is a shortening of $Q_k$ contained in L.

<u>Def. 3.3:</u> The operation $1\#L$ on $L = \{Q_1,\dots,Q_m\}$ is defined by the following algorithm

(1) $L_0 := \{Q_1,\dots,Q_m\}$, list of product terms.

(2) $L_1 := 1\#Q_1$

(3) $i := 1$.

(4) $L'_{i+1} := \{R\#Q_{i+1} \mid R \in L_i\}$

(5) Delete removable terms in $L'_{i+1}$, call the result $L_{i+1}$.

(6) $i+1 = m^2$

no: $i:=i+1$, goto (9)

(7) $1\#L_0 := L_m$.

Using $1\#L_0$ we are now able to formulate a short algorithm to generate prime implicants.

<u>Algorithm 1:</u>

(1) $L_0 := \{Q_1,\dots,Q_m\}$, list of product terms.

(2) $L_E := 1\#(1\#L_0)$.

For easier writing we use $\langle L\rangle$ for the Boolean function $\langle R_1+\dots+R_k\rangle$, where $L = \{R_1,\dots,R_k\}$.

Theorem 3.2: Let $L_E$ be the final list of an application of algorithm 1 to $L_0$. If P is a prime implicant of $<L_0>$, then $P \in L_E$.

Proof: We only have to show, that $1\#L_0$ has the same effect on $L_0$ as "replace "+" by "·" and vice versa, multiply out and delete removable terms". We do this by induction on the number of terms in $L_0$.

a) $L_0 = \{Q_1\} = \{x_{i_1}^{e_1} \ldots x_{i_k}^{e_k}\}$. In this case

$1\#L_0 = \{x_{i_1}^{e_1}, \ldots, x_{i_k}^{e_k}\}$ and everything is fine.

b) $L_0 = \{Q_1, \ldots, Q_m\}$ and $1\#\{Q_1, \ldots, Q_{m-1}\} = \{P_1, \ldots, P_k\}$. Then

$1\#L_0 = \{P_1\#Q_m, P_2\#Q_m, \ldots, P_k\#Q_m\}$ and we conclude by the definition of #.

Let us demonstrate the method by our previous example.

$L_0 = \{x_1\bar{x}_2, x_2\bar{x}_3, \bar{x}_1\bar{x}_2x_4\}$

$L_1 = \{x_1, \bar{x}_2\}$

$L_2 = \{x_1x_2, x_1\bar{x}_3, \bar{x}_2\bar{x}_3\}$

$L_3 = \{x_1x_2x_4, \bar{x}_2\bar{x}_3, x_1\bar{x}_3x_4\}$

Now the computation of $1\#L_0$ is completed.

$L_0' = \{x_1x_2x_4,\ \bar{x}_2\bar{x}_3,\ x_1\bar{x}_3x_4\}$

$L_1' = \{x_1,\ x_2,\ x_4\}$

$L_2' = \{x_1\bar{x}_2,\ x_1\bar{x}_3,\ x_2\bar{x}_3,\ \bar{x}_2x_4,\ \bar{x}_3x_4\}$

$L_3' = \{x_1\bar{x}_2,\ x_1\bar{x}_3,\ x_2\bar{x}_3,\ \bar{x}_2x_4,\ \bar{x}_3x_4\}$

This is the result of $1\#L_0' = 1\#(1\#L_0)$.

b) A tree method [5,26,27].

We now want to use Theorem 2.1 directly to generate prime implicants using the scheme of Algorithm 0.

a) We decompose $f = \prod_{i=1}^{k} (\langle\overline{t_i}\rangle + f_{t_i})$ where $\Sigma\langle t_i\rangle = 1$.

b) The prime implicants of $f_\ell$ are known if $f_\ell$ is represented in sum of products form, where the products consist of 1, 0, $x$ or $\bar{x}$.

The simplest set of terms in a) is $\{x,\bar{x}\}$ and in b) more complicated stopping rules may be used. Let us again use our previous example to illustrate the method.

1) $f = \langle x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1x_2x_4\rangle$

2) Choose $\{x_1,\bar{x}_1\}$

3) $f = \langle(x_1 + x_2\bar{x}_3 + \bar{x}_2x_4)(\bar{x}_1 + \bar{x}_2 + x_2\bar{x}_3)\rangle$

4) Choose $\{x_2,\bar{x}_2\}$

5) $f = \langle(\bar{x}_2 + x_1 + \bar{x}_3)(x_2 + x_1 + x_4)(\bar{x}_2 + \bar{x}_1 + \bar{x}_3)(x_2 + \bar{x}_1 + 1)\rangle$

6) $f = \langle(x_1 + \bar{x}_2x_4 + x_2\bar{x}_3 + \bar{x}_3x_4)(\bar{x}_2 + \bar{x}_1 + \bar{x}_3)\rangle$

$\quad = \langle\bar{x}_2x_4 + x_2\bar{x}_3 + \bar{x}_3x_4 + x_1\bar{x}_2 + x_1\bar{x}_3\rangle$

The method is called tree method because of a slightly different version which uses the notation of a tree.

Instead of Theorem 2.1 and Theorem 3.1 we may use their corollary, namely Lemma 3.1. Here we have to compute prime implicants of subfunctions and then go back to the function.

The root of the tree is f. If $\tilde{f}$ is a node of the tree and the prime implicants of $\tilde{f}$ are not known, then the tree is continued by

$$\begin{tikzcd} & \tilde{f} \arrow[dl, "\bar{x}"'] \arrow[dr, "x"] & \\ \tilde{f}_{\bar{x}} & & \tilde{f}_x \end{tikzcd}$$

where x is a variable of $\tilde{f}$. The prime implicants of $\tilde{f}$ are known, if $\tilde{f}$ is given by a single term. To use our previous example:

$f = \langle x_1\bar{x}_2 + x_2\bar{x}_3 + \bar{x}_1\bar{x}_2x_4\rangle$

$\quad = \langle x_1A + \bar{x}_1B\rangle$ where

$A = \bar{x}_2 + x_2\bar{x}_3$

$B = x_2\bar{x}_3 + \bar{x}_2x_4$

$<A> = <\bar{x}_2 \cdot 1 + x_2 \cdot C>$

$<B> = <\bar{x}_2 \cdot D + x_2 \cdot E>$

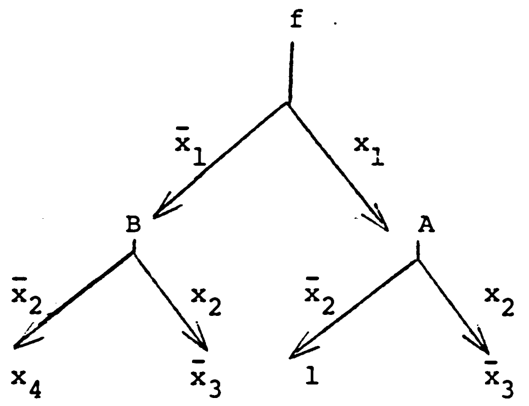
where

$C = \bar{x}_3$

$D = x_4$

$E = \bar{x}_3$

Hence we have the following tree:




Now we work back to generate the prime implicants of A, B and finally f.

A: $\{\bar{x}_2\} * \{1\} \cup \{x_2\} * \{\bar{x}_3\} \cup \{\bar{x}_3\} * \{1\} = \{\bar{x}_2, x_2\bar{x}_3, \bar{x}_3\}$

after deletions: $\{\bar{x}_2, \bar{x}_3\}$

B: $\{\bar{x}_2\} * \{x_4\} \cup \{x_2\} * \{\bar{x}_3\} \cup \{x_4\} * \{\bar{x}_3\} = \{\bar{x}_2 x_4, x_2\bar{x}_3, \bar{x}_3 x_4\}$

f: $\{\bar{x}_1\} * \{\bar{x}_2 x_4, x_2\bar{x}_3, \bar{x}_3 x_4\} \cup \{x_1\} * \{\bar{x}_2, \bar{x}_3\} \cup \{\bar{x}_2 x_4, x_2\bar{x}_3, \bar{x}_3 x_4\} * \{\bar{x}_2, \bar{x}_3\}$

$= \{\bar{x}_1\bar{x}_2 x_4, \bar{x}_1 x_2\bar{x}_3, \bar{x}_1\bar{x}_3 x_4, x_1\bar{x}_2, x_1\bar{x}_3, \bar{x}_2 x_4, x_2\bar{x}_3, \bar{x}_3 x_4\}$

after deletions: $\{x_1\bar{x}_2, x_1\bar{x}_3, \bar{x}_2 x_4, x_2\bar{x}_3, \bar{x}_3 x_4\}$.

Improvements of the method are possible if we allow rules to modify the lists appearing at the nodes or if we use more complicated stopping rules.

c) Method of iterated consensus [4,15,24,31,41].

Although the previous methods are very simple to understand and prove, their execution may take too much time. The method of iterated consensus uses a more complicated basic operation to overcome this difficulty.

Def. 3.4: a) Let $P = xP'$ and $Q = \bar{x}Q'$ be product-terms.

$\tilde{c}(P,Q) := P'*Q'$

b) Let P and Q be product-terms.

$$c(P,Q) := \begin{cases} \tilde{c}(P,Q), \text{ if it exists for some } x \\ 0, \text{ otherwise} \end{cases}$$

c(P,Q) is called the consensus of P and Q.

Lemma 3.3: $\langle P+Q\rangle = \langle P+Q+c(P,Q)\rangle$.

Proof: We have to show $\langle c(P,Q)\rangle \rightarrow \langle P+Q\rangle$. Assume $\langle c(P,Q)\rangle(b)=1$. Then there is a variable x such that $P = xP'$ and $Q = \bar{x}Q'$ and $\langle P'\rangle(b)=1$ as well as $\langle Q'\rangle(b)=1$. Now either $\langle x=1\rangle(b)=b$ or $\langle \bar{x} = 1\rangle(b)=b$. In the first case we get $\langle P\rangle(b)=1$ in the second $\langle Q\rangle(b)=1$.

Let $L = \{Q_1,\ldots,Q_m\}$ be a list of product terms. The consensus $c(Q_i,Q_j)$ is called <u>useful in L</u>, if it is not removable in $L \cup \{c(Q_i,Q_j)\}$.

<u>Algorithm 3</u>:

(1) $L_0 := \{Q_1,\ldots,Q_m\}$, list of product-terms.

(2) $i:=0$

(3) Is there a useful consensus $c(Q_k,Q_\ell)$ in $L_i$?
no: goto 7

(4) $L'_{i+1} := L_i \cup \{c(Q_k,Q_\ell)\}$

(5) Delete removable terms in $L'_{i+1}$, call the result $L_{i+1}$.

(6) $L_{i+1} = L_i$?
no: $i:=1+1$, goto 3

(7) $L_E := L_i$, stop.

The algorithm stops on every list $L_0$, simply because no new variables are introduced and there are only finitely many product terms for a given set of variables. The same holds true for other algorithms to follow. Hence we prove only "partial correctness", i.e. we prove that all prime implicants are generated under the assumption that the algorithm halts.

<u>Theorem 3.3</u>: Let $L_E = \{P_1,\ldots,P_n\}$ be the final list of an application of Algorithm 3 to the list $L_0 = \{Q_1,\ldots,Q_k\}$. If P is a prime implicant of $f = \langle Q_1+\ldots+Q_k \rangle$, then $P \in L_E$.

<u>Proof</u>: By Lemma 3.3 we know that $f = \langle P_1 + \ldots + P_n \rangle$. Hence it is sufficient to show that every list which is invariant under the algorithm contains exactly the prime implicants of its associated function.

We prove by induction on the number of variables in $L_E$.

a) $\{0\},\{1\},\{x\},\{\bar{x}\}$ are the only cases of final lists with one or less variables. They contain all prime implicants of the associated functions.

b) Let the theorem be true for lists containing up to n variables, let $L_E$ be a list with n+1 variables and let $P = xP'$ be a prime implicant. Now we sort $L_E$ according to x and get

$$L_E = \{xA_1,\ldots,xA_k,\ A_{k+1},\ \ldots,\ A_\ell,\ \bar{x}C_1,\ldots,\bar{x}C_m\}$$

where $A_{k+1},\ldots,A_\ell$ do not contain x or $\bar{x}$.
P' is a prime implicant of $\langle A_1 + \ldots + A_\ell \rangle$ by Lemma 3.1 and from $P' \in \{A_1,\ldots,A_\ell\} = L_1$ we can conclude $P \in L_E$. Assume $P' \notin L_1$. Since $L_1$ contains only n variables this means the is a useful consensus $c(A_i,A_j)$ in $L_1$. But then $xc(A_i,A_j)$ is a useful consensus in $L_E$, which is a contradiction. The argument for the case $P = \bar{x}P'$ runs similarly. If $P = 1$ then P itself is prime implicant of $f_x$ <u>and</u>

$f_{\bar{x}}$ for any variable x. Hence either the argument runs as above, or $L_E \supseteq \{x,\bar{x}\}$, whence there is a useful consensus again.

The method of iterated consensus gives some freedom in Step 3 which useful consensus to add. An actual implementation of the algorithm has to give some rules which one is to be selected and depending on the rules given, there will be generated many or less many intermediate terms. We will see some algorithms where rather "good" rules are given.

**d) Method of Quine-McCluskey [15,23]**

In this section we will improve the iterated consensus for the case where the start list contains no product-terms but minterms.

**Def. 3.5:** a) Let P and Q be product-terms such that $P = xR$ and $Q = \bar{x}R$ for some variable x.
$\tilde{d}(P,Q) := R$

b) Let P and Q be product-terms

$$d(P,Q) := \begin{cases} \tilde{d}(P,Q), & \text{if it exists for some } x \\ 0, & \text{otherwise} \end{cases}$$

d(P,Q) is called simple consensus of P and Q.

**Algorithm 4:**

(1) $L_0 := \{Q_1,\dots,Q_m\}$, list of product-terms

(2) $i:=0$

(3) $L_{i+1} := L_i \cup \{d(R_j,R_k) \mid R_j,R_k \in L_i\}$

(4) $L_{i+1} = L_i$?
no: $i:=i+1$, goto (3)

(5) Delete all removable terms in $L_i$, call the result $L_E$

(6) Stop.

Theorem 3.4: Let $L_E = \{P_1,\ldots,P_n\}$ be the final list of an application of Algorithm 4 to the list $L_0 = \{Q_1,\ldots,Q_m\}$ consisting of minterms only. If P is a prime implicant of $f = \langle Q_1 + \ldots + Q_m \rangle$, then $P \in L_E$.

Proof: We can translate the proof for the method of iterated consensus almost word by word for the final list $L_i$ before cancelling removable terms using $d(R_j,R_k)$ instead of $c(Q_k,Q_\ell)$ there. We will give a second very short proof here.

Let there be n variables. It is sufficient to show that $L_i$ contains all implicants of length n-i.

a) i=0 is trivial, because we start with minterms.

b) Let it be true for $i \leq k$ and let R be an implicant of the length n-(k+1) where x and $\bar{x}$ are not contained in R. From that we know $xR \in L_k$ and $\bar{x}R \in L_k$. But $d(xR,\bar{x}R) = R$ and therefore $R \in L_{k+1}$.

e) Method of successive extraction [40]

In this method, consensus is taken with respect to one variable after the other, no repetition is necessary.

Algorithm 5:

(1) $L_1 = \{Q_1,\ldots,Q_m\}$, list of product-terms in n variables

(2) $i:=1$

(3) Sort $L_i$ to be

$L_i = \{\bar{x}_i A_1,\ldots,\bar{x}_i A_k,\ x_i B_1,\ldots,x_i B\ ,\ C_1,\ldots,C_p\}$

(4) $L'_{i+1} := L_i \cup \{A_j * B_q | j=1,\ldots k,\ q=1,\ldots,\ell\}$

(5) Delete removable terms in $L'_{i+1}$, call the result $L_{i+1}$.

(6) $i=n$?

no: $i:=i+1$, go to 3.

(7) $L_E := L_i$, stop.

Theorem 3.5: Let $L_E$ be the final list of an application of algorithm 5 to $L_1 = \{Q_1,\ldots,Q_m\}$. If P is a prime implicant of $f = \langle Q_1, + \ldots + Q_m \rangle$, then $P \in L_E$.

Proof: By induction on the number of variables in $L_1$.

a) If there is one variable, then $\{x\},\{\bar{x}\}$ and $\{x,\bar{x}\}$ are the only possibilities for $L_1$. In all three cases Algorithm 5 gives the correct result.

b) Let the theorem be true for up to n variables and

$L_1 = \{\bar{x}_{n+1}A_1,\ldots,\bar{x}_{n+1}A_k,x_{n+1}B_1,\ldots,x_{n+1}B_\ell,C_1,\ldots,C_p\}$

be a start list with n+1 variables. Now consider

$L_1' = \{A_1,\ldots,A_k,C_1,\ldots,C_p\}$ and $L_1'' = \{B_1,\ldots,B_\ell,C_1,\ldots,C_p\}$

which correspond to subfunctions according to $\bar{x}_{n+1}$ and $x_{n+1}$. Applying Algorithm 5 to $L_1'$ and $L_1''$ would result in $L_E' = \{P_1',\ldots,P_a',P_1,\ldots,P_b\}$, $L_E'' = \{P_1'',\ldots,P_c'',P_1,\ldots,P_b\}$ where $\{P_1',\ldots,P_a'\} \cap \{P_1'',\ldots,P_c''\} = \phi$. By assumption $L_E'$ and $L_E''$ contain all prime implicants of the subfunctions. But $L_1$ will be transferred after n steps into

$L_n = \{\bar{x}P_1',\ldots,\bar{x}P_a',\ xP_1'',\ldots,xP_c'',\ P_1,\ldots,P_b\}$

and by Lemma 3.1 finally $L_{n+1} = L_E$ will contain all prime implicants of $f = \langle Q_1 + \ldots + Q_m \rangle$.

f) Method of exclusion [40].

In this method, every term is used only once to form consensus with other terms. We use two different lists during the computation.

Algorithm 6:

(1) $A_0 = \phi$, $L_0 = \{Q_1,\ldots,Q_m\}$, lists of product-terms.

(2) i:=0

(3) Choose $R_i \in L_i$

(4) $A_{i+1}' := A_i \cup \{R_i\}$

$L_{i+1}' := L_i \cup \{c(R_i,S) \neq 0 \mid S \in L_i\}$

(5) Delete terms in $L'_{i+1}$, which are removable in $L'_{i+1} \cup A'_{i+1}$. Call the result $L_{i+1}$.

Delete terms in $A'_{i+1}$, which are removable in $L_{i+1} \cup A'_{i+1}$. Call the result $A_{i+1}$.

(6) $L_{i+1} = \phi$?
no: i:= i+1, goto (3).

(7) $L_E := A_{i+1}$, stop.

We want to show, that every prime implicant of $f = \langle L_0 \rangle$ is contained in $L_E$. The following lemma will be used.

<u>Lemma 3.4</u>: Let $A_i$, $L_i$ be the lists generated by Algorithm 6. If P is a prime implicant of $\langle A_i \cup L_i \rangle$ and $P \notin A_i$, then P is an implicant of $\langle L_i \rangle$.

<u>Proof</u>: We prove by induction on i, where the case i = 0 is trivial. If $P \notin A_{i+1}$, then $P \notin A_i$, since P is prime. By assumption P is an implicant of $\langle L_i \rangle$, i.e. for every $b \in \{0,1\}^n$ such that $\langle P \rangle(b)=1$ there is a $Q_b \in L_i$ such that $\langle Q_b \rangle(b)=1$. We have to find a $Q'_b \in L_{i+1}$ with the same property. If $Q_b \in L_{i+1}$, then $Q'_b = Q_b$. If $Q_b \notin L_{i+1}$, then there is a shortest $Q'_b \in L'_{i+1} \cup A'_{i+1}$ removing $Q_b$. We also know $\langle Q'_b \rangle(b)=1$. Assume $Q'_b \notin L_{i+1}$. Since it was shortest, it cannot have been removed and therefore $Q'_b \in A_{i+1} \subseteq A_i \cup L_i$. But then $Q_b \notin L_i$, because $Q'_b$ removed it one step earlier. This is a contradiction and hence $Q'_b \in L_{i+1}$.

<u>Theorem 3.6</u>: Let $L_E$ be the final list of an application of Algorithm 6 to $L_0$. If P is a prime implicant of $\langle L_0 \rangle$, then $P \in L_E$.

<u>Proof</u>: If $L_i = \phi$ and $P \notin A_i$ for a prime implicant P of $\langle L_0 \rangle$, then by the previous lemma P is an implicant of $\langle \phi \rangle = 0$. This is a contradiction unless $L_0 = \phi$.

g) <u>Prime implicants of subfunctions [28]</u>.

In some applications Boolean functions depend on two types of variables, a set of "parameters" $a_n, \ldots, a_k$ and a set of "variables" $x_1, \ldots, x_n$. The problem is to find all prime implicants of functions in the variables $x_1, \ldots, x_n$ which can be obtained by fixing $a_1, \ldots, a_k$ to values from {0,1}. In the language used in this paper, this is to find all prime implicants of functions $f_t$, where t is a minterm in $a_1, \ldots, a_k$.

We will attack the more general problem of finding all prime implicants of $f_t$ for given Boolean function f and arbitrary product term t in its variables.

<u>Theorem 3.7</u>: Let f be a Boolean function and t a product term in the variables $x_1, \ldots, x_n$. If P is a prime implicant of $f_t$, then there is a prime implicant of QP of f, such that $\langle Q \rangle \circ \langle t=1 \rangle = 1$, i.e. Q is a shortening of t.

<u>Proof</u>: We will prove by induction on the length of t. If $t = x_i$ or $t = \bar{x}_i$ we consult Lemma 3.1 and obtain: if P is a prime implicant of $f_{x_i}$, then either is $x_i P$

a prime implicant of $f$, or there is a prime implicant $P_0$ of $f_{\bar{x}_i}$, such that $P * P_0$ is a shortening of $x_i P$ Since $P$ is prime, we conclude $P * P_0 = P = 1 \cdot P$ is a prime implicant of $f$ in this case. Thus our theorem holds for product terms of length 1. Let it hold for product terms up to length $k$ and let $t = t' \cdot x_i$ be a product term of length $k+1$.

We observe that $f_t = (f_{t'})_{x_i}$. Hence by the discussion above, if $P$ is a prime implicant of $f_t$, then either $x_i P$ or $P$ is a prime implicant of $f_{t'}$. By assumption we then have a shortening $Q'$ of $t'$ such that $Q' x_i P$ or $Q'P$ is a prime implicant of $f$. Hence $Q = Q'$ or $Q = x_i Q'$ is the desired term.

Using this result, we obtain the prime implicants of $f_t$ in the following way.

<u>Algorithm 7</u>:

(1) $L_0 := \{Q_1, \ldots, Q_m\}$, list of prime implicants of $\langle L_0 \rangle$.

(2) $L_1 := \{Q_j \mid j=1,\ldots,m,\ t * Q_j \neq 0\}$

(3) $L_2 := \{Q_j' \mid Q_j \in L_1,\ Q_j = Q_j' \tilde{Q}_j,\ t * \tilde{Q}_j = t,\ t * Q_j' = tQ_j'\}$

(4) Delete removable terms in $L_2$, call the result $L_E$.

Let us demonstrate the method by an example.

a) $L_0 = \{\bar{x}_2x_4, x_3\bar{x}_4, \bar{x}_2x_3, x_1x_2\bar{x}_4, \bar{x}_1x_3\}$

$t = \bar{x}_1\bar{x}_2$

$L_1 = \{\bar{x}_2x_4, x_3\bar{x}_4, x_3\} = \{x_4\cdot\bar{x}_2, x_3\bar{x}_4\cdot 1, x_3\cdot 1\}$

$L_2 = \{x_4, x_3\bar{x}_4, x_3\}$

$L_E = \{x_4, x_3\}$

b) $L_0$ as in a)

$t = x_1x_3$

$L_1 = \{\bar{x}_2x_4, x_3\bar{x}_4, \bar{x}_2x_3, x_1x_2\bar{x}_4\} = \{\bar{x}_2x_4\cdot 1, \bar{x}_4\cdot x_3, \bar{x}_2\cdot x_3, x_2\bar{x}_4\cdot x_1\}$

$L_2 = \{\bar{x}_2x_4\bar{x}_4, \bar{x}_2, x_2\bar{x}_4\}$

$L_E = \{\bar{x}_2, \bar{x}_4\}$

4. Remarks on running times.

Some experiments have been conduced with programmed versions of algorithms presented in this paper at the University of Dortmund. Although no attempt was made for serious measurement, almost all experiments uniformly indicated the following ranking from fast to slow: successive extraction, method of exclusion, tree-method, double product, method of exclusion.

Another method to evaluate the performance of algorithms is consideration of worst cases. This leads us to a celebrated open question in theoretical computer science, namely the so called (P = NP)—problem (for a presentation and discussion of this problem, see for instance: Aho, Hopcoft, Ullman, The Design and Analysis of Computer Algorithms, Addison Wesley, 1974). One version of it asks whether there is a "good" algorithm to decide a given sum-of-products form to denote the 1-function, i.e. to be a tautology. An algorithm is "good", if it takes a total number of reasonably elementary steps which is polynomial in the number of literals in the original list. Although there is no proof yet, it is generally believed that there is no "good" algorithm for the tautology problem.

Now, if there would be a "good" algorithm for generating prime implicants, we could use it for the tautology problem: 1 is the one and only prime implicant of the 1-function. On the other hand, if there is a "good" algorithm for the tautology-problem then there is at least a "good" algorithm to decide for any given product term whether it is an implicant. This is shown by our next lemma [34].

<u>Lemma 4.1</u>: Let $Q_1,\ldots,Q_m,t$ be product terms and $L_1$, $L_2$ as in Algorithm 7.

t is an implicant of $\langle Q_1 + \ldots + Q_m \rangle$ iff $\langle L_2 \rangle = 1$.

<u>Proof</u>: Assume $\langle L_2 \rangle = 1$ and $\langle t \rangle (b) = 1$. From $\langle L_2 \rangle = 1$ we conclude the existence of $Q_j' \in L_2$ such that $\langle Q_j' \rangle (b) = 1$. But now $\langle Q_j \rangle (b) = \langle Q_j' \tilde{Q}_j \rangle (b) = \langle Q_j' \tilde{Q}_j t \rangle (b) = \langle Q_j' \rangle (b) \cdot \langle t \rangle (b) = 1$ and therefore $\langle Q_1 + \ldots + Q_m \rangle (b) = 1$. Now assume t to be an implicant of $\langle Q_1 + \ldots + Q_m \rangle$ and $\langle L_2 \rangle \neq 1$. Then there is a b such that $\langle L_2 \rangle (b) = 0$, which means $\langle Q_j' \rangle (b) = 0$ for all $Q_j' \in L_2$.

This property is conserved if we change b to $b' = \langle t{=}1 \rangle (b)$ by the construction of $L_2$. But now $\langle t \rangle (b') = 1$ and hence by assumption $\langle Q_1 + \ldots + Q_m \rangle (b') = 1$. On the other hand $\langle Q_\ell \rangle (b') = 0$ for $Q_\ell \notin L_1$ and $\langle Q_j \rangle (b') = \langle Q_j' \cdot \tilde{Q}_j \rangle (b') = 0$ for $Q_j \in L_1$ which is a contradiction.

The discussion above indicates, that most probably there is no "good" algorithm for generating all the prime implicants of a given Boolean function. It does not mean at all that it is easy to <u>prove</u> for a particular algorithm that it is <u>not</u> a "good" one. However Z. Galil has shown (Z. Galil, The Complexity of Resolution Procedures for Theorem Proving in the Propositional Calculus, Cornell University, Dept. of Computer Science, TR 75-239, 1975) that any algorithm using consensus as basic operation needs exponentially many steps, if a variable which is once eliminated in the computation of a particular prime implicant, is not introduced again in <u>that computation</u>.

References

[1] Bing, K., "On Simplifying Truth-Functional Formulas", J. Symb. Logic 21 (1956) pp. 253-254.

[2] Biswas, N.N., "Minimization of Boolean Functions", IEEE Trans. on Computers C-20 (1971), p. 925.

[3] Bredeson, J.G. and P.T. Hulina, "Generation of Prime Implicants by Direct Multiplication", IEEE Trans. on Computers C-20 (1971), pp. 475-476.

[4] Blake, A., "Canonical Expressions in Boolean Algebra", Ph.D. Diss., Dept. of Math., University of Chicago, (1937).

[5] Brown, F.M., "On a Convenient Division of Labor in the Generation of Prime Implicants", EE Dpt., University of Kentucky, Lexington, (1974).

[6] Choudbury, A.K., M.S. Basu, and S.R. Das, "On Adaptation of the Grouping Chart and Simplification of Multiple Output Switching Functions", Indian J. Physics 36 (1962) pp. 521-532.

[7] Das, S.R. and A.K. Choudbury, "Maxterm Type Expressions of Switching Functions and their Prime Implicants", IEEE Trans. on Computers C-14 (1965), pp. 920-923.

[8] Das, S.R. and N.S. Khabra, "Clause-Column Table Approach for Generating all the Prime Implicants of Switching Functions", IEEE Trans. on Computers C-21 (1972), pp. 1239-1246.

[9] Detering, L., "Algorithmen zur Bestimmung von Prinimplikatnen Diplomerbeit, Abt. Informatik, Universitat Dortmund, (1975).

[10] Hall, F.B., "Boolean Prime Implicants by the Binary Sieve Method", AIEE Trans. 80 (1962), pp. 709-713.

[11] Harris, B., "An Algorithm for Determining Minimal Representation of a Logic Function", IEEE Trans. on Computers C-6 (1957), pp. 103-108.

[12] Hotz, G., "Zur Reduktion von Schaltkreispolynomen im Hinblick auf die Verwendung von Rechen automaten", Elektronische Datenverarbeitung, Folge 5 (1960), pp. 21-27.

[13] Hwa, H.R., "A Method for Generating Prime Implicants of a Boolean Expression", IEEE Trans. on Computers C-23 (1974), pp. 637-641.

[14] Kuntzmann, J. "Un theoreme sur les composants premiers", Automatisme IX, 1(1964), pp. 18-19.

[15] McCluskey, E.J., Jr., "Minimization of Boolean Functions", Bell Sys. Techn. J. 35,6 (1956), pp. 1417-1444.

[16] McCluskey, E.J., Jr., "Minimal Sums for Boolean Functions Having Many Unspecified Fundamental Products", AIEE Trans. 81(1962), pp. 387-392.

[17] Matney, R.M., and C.H. Roth, "Parallel Computing Structures and Algorithms for Logic Design Problems", Electron. Res. Center, Austin, Texas (1969).

[18] Mott, H. and C.C. Carroll, "Numerical Procedures for Boolean Function Minimization", IEEE Trans. on Computers C-13 (1964), p. 470.

[19] Morreale, E., "Partitioned List Algorithms for Prime Implicant Determination from Canonical Forms", IEEE Trans. on Computers C-16 (1957), pp. 611-620.

[20] Morreale, E., "Recursive Operators for Prime Implicant and Irredundant Normal Form Determination", IEEE Trans. on Computers C-19(1970), pp. 504-509.

[21] Nelson, R.J., "Simplest Normal Truth Functions", J. Symb. Logic 20(1955), pp. 105-108.

[22] Nelson, R.J., "Weak Simplest Normal Truth Functions", J. Symb. Logic 20(1955), pp. 232-234.

[23] Quine, W.V., "The Problem of Simplifying Truth Functions", Am. Math. Monthly 59(1952), pp. 521-531.

[24] Quine, W.V., "A Way to Simplify Truth Functions", Am. Math. Monthly 62(1955), pp. 627-631.

[25] Quine, W.V., "On Cores and Prime Implicants of Truth Functions", Am. Math. Monthly 66(1956), pp. 755-760.

[26] Reusch, B., "On Boolean Function Minimization", Ges. f. Math und Datenverabeitung, Bonn, (1972).

[27] Reusch, B., "Generation of Prime Implicants from Subfunctions and a Unifying Approach to the Covering Problem", IEEE Trans. on Computers C-24(1975), pp. 924-930.

[28] Reusch, B., O. Heroz, and E. Riedemann, "Minimization of Boolean Functions with 'Don't cares' Subject to Constraints", Annual Hawai Conference (1974).

[29] Roth, J.P., "Algebraic Topological Methods for the Synthesis of Switching Systems I", Trans. Am. Math. Soc. 88,2(1958), pp. 301-326.

[30] Roth, J.P., "Algebraic Topological Methods in Synthesis II", Annals of the Harvard Comp. Lab. XXIX, (1959), pp. 57-73.

[31] Samson, E.W. and B.E. Mills, "Circuit Minimization: Algebra and Algorithms for New Boolean Canonical Expressions", Air Force Cambridge Research Center, TR 54-21, (1954).

[32] Slagle, J.R., C.C. Chang, and R.T.C. Lee, "A New Algorithm for Generating Prime Implicants", IEEE Trans. on Computers C-19(1970), pp. 304-311.

[33] Tison, P., "Circuits Combinatoires: 1. Theorie des Consensus, 2. Algorithmes de Recherche des Bases Premieres", Int. Conf. on Boolean Algebra, Grenoble, France (1965).

[34] Tison, P., "Generalized Consensus Theory and Application to the Minimization of Boolean Functions", IEEE Trans. on Computers C-16 (1967), pp. 446-456.

[35] Tison, P., private communication.

[36] Ulug, M.E. and B.A. Bowen, "A Unified Theory of the Algebraic Topological Methods for the Synthesis of Switching Circuits", IEEE Trans. on Computers C-23 (1974), pp. 255-267.

[37] Urbano, R.H. and R.K. Mueller, "A Topological Method for the Determination of the Minimal Forms of a Boolean Function", IEEE Trans. on Computers C-5 (1956), pp. 126-132.