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BALANCED SUBDIVISION AND ENUMERATION IN BALANCED SPHERES

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Abstract. The object of study is the affine space generated by the extended h-vectors of simplicial homology (d-1)-spheres which are balanced of a given type. The dimension of the space is computed by deriving a balanced version of the Dehn-Sommerville equations and exhibiting a set of balanced polytopes whose extended h-vectors span the space. To this end, a unique minimal complex of a given type is defined, along with a balanced version of stellar subdivision, and such a subdivision of a face in a minimal complex is realized as the boundary complex of a balanced polytope. For these complexes, we obtain an explicit computation of their extended h-vectors. The difficulties encountered in the attempt to determine those polytopes whose extended h-vectors give a basis for the space are also discussed.

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§0 Introduction

In general terms, this paper investigates balanced simplicial complexes. These are complexes whose vertices are labeled $1,2,\ldots,m$ in such a way that each maximal face has a_i vertices with label i, $1 \leq i \leq m$. In particular, we generalize to the balanced case some results of Bayer and Billera's for completely balanced complexes, that is, balanced complexes for which each label in a maximal face appears once. If Δ is a balanced simplicial complex then the integer $f_b(\Delta)$ counts the faces of Δ having b_i vertices with label i, where $b = (b_1,\ldots,b_m)$; these numbers are displayed in the extended f-vector $f(\Delta) = (f_b(\Delta))_{b \leq a}$, where the vector $a = (a_1,\ldots,a_m)$ describes the labeling on Δ. (Δ is then called "type a".) An invertible linear transformation is applied to $f(\Delta)$ to produce $h(\Delta)$, the extended h-vector of Δ. The particular object of study here is the affine space generated by the extended h-vectors of simplicial homology (d-1)-spheres which are balanced of a given type.

To place the work in context it should be noted that §1 provides a common generalization of classical results about simplicial polytopes and their affine span as described by the Dehn-Sommerville equations (see [11], [13], [19]), and the situation in the completely balanced case as determined by Bayer and Billera (see [4], [5]). In the classical case, $h_i(P) = h_{d-i}(P)$ for $0 \leq i \leq d$ and P a simplicial d-polytope. In the latter case, suppose P is a completely balanced simplicial d-polytope with labels $1,\ldots,d$. If $T \subseteq \{1,\ldots,d\}$,

$f_T(P) \equiv f_t(P)$ where t is the characteristic vector of T. Then for all $S \subseteq \{1,\ldots,d\}$, $h_S(P) = h_{\tilde{S}}(P)$ where $\tilde{S}$ is the complement of S in $\{1,\ldots,d\}$. We point out that for P a type a balanced simplicial d-polytope, $h_b(P) = h_{a-b}(P)$ for all $b \leq a$. Therefore, to recover the classical (unlabeled) result, take $a = (d)$ (all vertices receive the same label), and to specialize to the completely balanced case, take $a = (1,1,\ldots,1)$.

We now present the definitions and basic facts to be used throughout the paper. Let $\sigma \in \Delta$, a simplicial complex. Then the link of σ in Δ is defined to be $\ell k_\Delta \sigma = \{\tau \in \Delta: \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$. If $\sigma \neq \emptyset$, the deletion of σ in Δ is $\Delta \backslash \sigma \equiv \{\tau \in \Delta: \sigma \not\subseteq \tau\}$. If Δ_1 and Δ_2 are simplicial complexes with no vertices in common, then their join is defined to be $\Delta_1 * \Delta_2 = \{\sigma_1 \cup \sigma_2: \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$. Note that each of these operations yields a simplicial complex. For $\sigma \in \Delta$ and v a vertex not in Δ, the stellar subdivision of σ in Δ is the complex $st_\Delta(\sigma,v) \equiv (\Delta \backslash \sigma) \cup \bar{v} * \partial\bar{\sigma} * \ell k_\Delta \sigma$. Of historical interest is J.W. Alexander's "arithmetic" describing stellar subdivision on which a whole theory of stellar equivalence is based [1]. Whether such a theory can be developed for balanced complexes is a question of some interest, but beyond the scope of this paper (see [15].) Stellar subdivision may also be defined for polytopes. For a discussion in the most general (non-simplicial) case, see [9].

If Δ is a (d-1)-dimensional simplicial complex, then for $-1 \leq i \leq d-1$, we denote the number of faces of Δ having dimension i by $f_i(\Delta)$. The f-vector and h-vector of Δ are defined as in Billera [6]. We use the fact that the Euler relation for Δ is equivalent to $h_d(\Delta) = 1$.

(The same definitions can be made for polytopes, and are also found in [6].)

This paper deals with a certain class of simplicial complexes known as homology spheres. For the details about geometric complexes in the next definition, refer to [17]. A homology (d-1)-sphere is the abstraction of any geometric simplicial complex Δ such that for every face $\sigma \in \Delta$,

$$\tilde{H}_i(\ell k_\Delta \sigma;\ \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = \dim \ell k_\Delta \sigma \\ 0 & \text{else} \end{cases}$$

where $\ell k_\Delta \sigma$ is defined for Δ geometrically. Since $\ell k_\Delta(\emptyset) = \Delta$, Δ has the same homology as a (d-1)-sphere, so the Euler relation necessarily holds for these complexes.

Suppose Δ is a simplicial complex with vertex set V. Then a labeling of Δ is a partition $V = V_1 \cup \ldots \cup V_m$, and the vertices in V_i are said to be labeled i. If a pure complex Δ has a labeling such that each facet contains exactly a_i vertices labeled i, $a_i \in \mathbb{N}$, $1 \leq i \leq m$, then Δ is said to be balanced of type $a = (a_1,\ldots,a_m)$. Moreover, if $a_i = 1$ for $i = 1,\ldots,m$, then Δ is called completely balanced; in this case every facet of Δ contains exactly one vertex of each label. We will use C_a^d to denote the set of all type a balanced homology (d-1)-spheres. If P is a polytope, and its boundary complex ∂P is balanced of type a, we will also refer to P as being balanced of type a.

If $b = (b_1,\ldots,b_m) \in \mathbb{N}^m$, we define $f_b(\Delta)$ to be the cardinality of the set $\{\sigma \in \Delta: \sigma$ is a face of type $b\}$, and call $(f_b(\Delta))_{b\in\mathbb{N}^m}$ the <u>extended f-vector</u> of Δ. Note that $f_b(\Delta) = 0$ unless $b \leq a$, and that the extended f-vector is a refinement of the usual f-vector. Let $\underset{\sim}{t}$ represent the multivariable $(t_1,\ldots,t_m)$, and if $b = (b_1,\ldots,b_m) \in \mathbb{N}^m$, let $\underset{\sim}{t}^b$ denote $t_1^{b_1}\cdot\ldots\cdot t_m^{b_m}$. Then we develop an extended h-vector by defining the polynomial $f(\Delta,\underset{\sim}{t}) \equiv \sum_{b\leq a} f_b(\Delta)\underset{\sim}{t}^b$ and applying a linear transformation. For $w = (w_1,\ldots,w_m) \in \mathbb{N}^m$, define $h^{(w)}(\Delta,\underset{\sim}{t}) \equiv (1-\underset{\sim}{t})^w f(\Delta, \underset{\sim}{t}/1-\underset{\sim}{t})$ where the notation $(1-\underset{\sim}{t})^w$ means $\Pi_{i=1}^m (1-t_i)^{w_i}$. When $w = a$, the balanced type of Δ, we distinguish $h^{(a)}(\Delta,\underset{\sim}{t}) = (1-t)^a f(\Delta, \underset{\sim}{t}/1-\underset{\sim}{t})$ as $h(\Delta,\underset{\sim}{t})$, defining the coefficient of $\underset{\sim}{t}^b$ in $h(\Delta,\underset{\sim}{t})$ to be $h_b(\Delta)$. Then the <u>extended h-vector</u> of Δ is $(h_b(\Delta))_{b\in\mathbb{N}^m}$; when the context is clear we drop the notation $b \in \mathbb{N}^m$. As for f-vectors, the extended h-vector is a refinement of the usual h-vector

Two useful interpretations of the number $h_b(\Delta)$ have been supplied by Stanley. See [20] for the details of both. First, $h_b(\Delta)$ is the vector space dimension of the bth summand in the usual $\mathbb{N}^m$-grading on the Stanley-Reisner ring of Δ mod a homogeneous system of parameters. Secondly, if Δ is balanced of type a and $\sigma_1,\ldots,\sigma_s$ is a shelling of Δ, then $h_b(\Delta)$ counts the number of times it happens in the shelling process that the unique minimal face of σ_i $(1 \leq i \leq s)$ which is not contained in $\sigma_1 \cup \ldots \cup \sigma_{i-1}$ is type b.

We close this section by mentioning some special notation. If S is a set, $|S|$ will denote the cardinality of S. For $K \in \mathbb{Q}$, $\lfloor K \rfloor$ means the greatest integer $\leq K$, and we use $\lceil K \rceil$ to denote the least integer $\geq K$.

§1 Symmetry of the Extended h-vector

This section specifies certain linear equations which the coordinates of every extended h-vector satisfy. In addition to these, each h-vector satisfies the balanced version of Euler's relation, and this collection of equations will later be shown to characterize the space $\mathrm{aff}(h_b(C_a^d))$. The symmetric nature of the extended h-vector is displayed in the following theorem, whose proof is a straightforward generalization of a proof of Bayer and Billera's [4; p. 235], which in turn is based on one of Stanley's [22]. For details, see [15].

Theorem 1: Let $\Delta \in C_a^d$, $a = (a_1,\ldots,a_m)$. Then $h_b(\Delta) = h_{a-b}(\Delta)$ for all $b \leq a$. ■

This theorem is enough to determine an upper bound on the dimension of $\mathrm{aff}(h_b(C_a^d))$. In what follows, let $K = \prod_{i=1}^{m} (a_i+1)$.

Corollary 2: $\dim \mathrm{aff}(h_b(C_a^d)) \leq \lfloor \frac{K-1}{2} \rfloor$.

Proof: For all $\Delta \in C_a^d$ we have $h_{\underset{\sim}{0}} = 1$ and $h_b(\Delta) = h_{a-b}(\Delta)$ for all $b \leq a$. These are independent linear equations and there are $\lfloor \frac{K}{2} \rfloor + 1$ of them, so they determine an affine subspace of dimension $K - (\lfloor \frac{K}{2} \rfloor + 1) = \lfloor \frac{K-1}{2} \rfloor$. Finally, $\mathrm{aff}(h_b(C_a^d))$ is contained in this subspace, and so $\dim \mathrm{aff}(h_b(C_a^d)) \leq \lfloor \frac{K-1}{2} \rfloor$. ■

§2 The Minimal Type a Complex

It is known that if S is an unlabeled d-simplex, $h(\partial S)$ consists of $d+1$ 1's followed by the appropriate number of 0's [4]. This section offers a description of certain polytopes whose boundary complexes, in the balanced case, act as analogues to ∂S. The extended h-vectors of these complexes are computed, and are shown to consist of 1's.

First, we describe a basic construction. Suppose P_i is a d_i-polytope in $\mathbb{R}^{d_i}$ with $\underset{\sim}{0} \in \inf P_i$, $i = 1,2$. Let $\overline{P}_1 = P_1 \times \{\underset{\sim}{0}\}$ and $\overline{P}_2 = \{\underset{\sim}{0}\} \times P_2$ be embeddings of P_i in $\mathbb{R}^d$, $d = d_1+d_2$, and define $P_1 \# P_2 \equiv \text{conv}(\overline{P}_1 \cup \overline{P}_2)$. In the case in which P_1 and P_2 are simplicial polytopes, it is straightforward to verify that

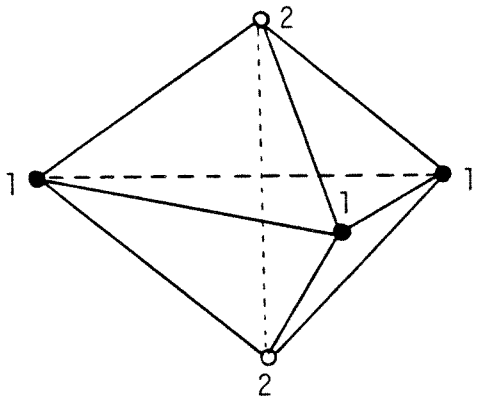
$$\partial(P_1 \# P_2) = \partial P_1 * \partial P_2.$$

Suppose $a = (a_1,\ldots,a_m)$ and let $d \equiv \sum_{i=1}^{m} a_i$. For $1 \leq i \leq m$, let S_i be an a_i-simplex in $\mathbb{R}^{a_i}$ with $\underset{\sim}{0}$ in its interior, and with vertices labeled i. Assume $S_1,\ldots,S_m$ are embedded in $\mathbb{R}^d$, as above and define $\hat{\Delta} \equiv \partial(S_1 \# S_2 \# \ldots \# S_m) = \partial S_1 * \partial S_2 * \ldots * \partial S_m$. It follows that $\hat{\Delta}$ is type a, for a maximal simplex of ∂S_i has exactly a_i vertices. We call $\hat{\Delta}$ the <u>minimal type a complex</u>.

<u>Examples</u>

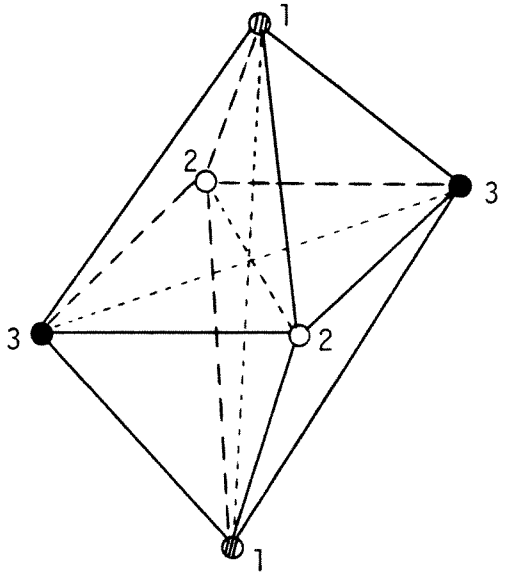
1) $a = (2,1)$

   $\hat{\Delta} = \partial S_1 * \partial S_2$, where S_1 is a 2-simplex labeled 1, and S_2 is a 1-simplex labeled 2.



2) $a = (1,1,1)$

$\hat{\Delta} = \partial S_1 * \partial S_2 * \partial S_3$, where for $1 \leq i \leq 3$ S_i is a 1-simplex labeled i.



Remarks: 1) $\hat{\Delta}$ is minimal in the sense that it has the smallest number of vertices of any simplicial polytope whose boundary complex is type a. For, if P is a simplicial polytope and ∂P is type a, the number of vertices of P with label i must be at least a_i, by definition of "type a". Suppose for some i that the number of vertices with label i is exactly a_i. Then each facet of P contains all the vertices with label i, that is, a vertex v with label i is contained in all the facets of P. But then the combinatorial dual P^* has a facet which contains all the vertices of P^*, which is impossible. Thus $\hat{\Delta}$ is minimal among type a simplicial polytopes in terms of number of vertices.

For a given type, how many vertices is this? If $a = (a_1,\ldots,a_m)$ and $a_i \neq 0$ for all $1 \leq i \leq m$, then clearly $f_0(\hat{\Delta}) = \sum_{i=1}^{m} (a_i+1)$. An adjustment must be made if $a_i = 0$ for some i. (This situation occurs in the next section.) In the construction of $\hat{\Delta}$, $a_i = 0$ indicates a 0-simplex v, labeled i. v is placed on the origin in the embedding, so when the convex hull of all the simplices is taken to make $\hat{\Delta}$, v disappears into the interior of the polytope. Thus $a_i = 0$ contributes no vertices to $\hat{\Delta}$. So in general,

$$f_0(\hat{\Delta}) = \sum_{i=1}^{m} (a_i+1) - |\{i: a_i = 0\}|.$$

2) It is also true that $\hat{\Delta}$ is unique among the boundary complexes of type $a = (a_1,\ldots,a_m)$ polytopes having a_i+1 vertices labeled i, $1 \leq i \leq m$. For if there exists another such complex Γ, it can be embedded as a subcomplex of $\hat{\Delta}$. Then since Γ is the boundary complex of a polytope, "invariance of domain" forces $\Gamma = \hat{\Delta}$. See [16], [12].

The rest of this section is devoted to the computation of $h_b(\hat{\Delta})$ for $b \leq a$.

Proposition 3: Let Δ_1 and Δ_2 be balanced complexes. Then $\Delta_1 * \Delta_2$ is balanced, and $h_b(\Delta_1 * \Delta_2) = \sum_{v+w=b} h_v(\Delta_1) \cdot h_w(\Delta_2)$.

Proof: Choose an order for the set of distinct labels appearing in Δ_1 or Δ_2, and write the vectors describing the type of each complex according to this order, using 0's where necessary.

Suppose Δ_1 is type $a = (a_1, \ldots, a_m)$ and Δ_2 is type $a' = (a_1', \ldots, a_m')$. A maximal simplex in $\Delta_1 * \Delta_2$ is the union of a maximal simplex in Δ_1 with a maximal simplex in Δ_2, and it's clear that $\Delta_1 * \Delta_2$ is balanced of type $a+a'$.

In what follows, let $\underset{\sim}{t}$ denote the multivariable $(t_1, \ldots, t_m)$ and let $(1-\underset{\sim}{t})^v$ denote the product $(1-t_1)^{v_1} \cdot \ldots \cdot (1-t_m)^{v_m}$ for $v = (v_1, \ldots, v_m)$. Since $f_b(\Delta_1 * \Delta_2) = \sum_{v+w=b} f_v(\Delta_1) \cdot f_w(\Delta_2)$, we have $f(\Delta_1 * \Delta_2, \underset{\sim}{t}) = f(\Delta_1, \underset{\sim}{t}) \cdot f(\Delta_2, \underset{\sim}{t})$. Then

$$\begin{aligned} h(\Delta_1 * \Delta_2, \underset{\sim}{t}) &= (1-\underset{\sim}{t})^{a+a'} f(\Delta_1 * \Delta_2, \underset{\sim}{t}/1-\underset{\sim}{t}) \\ &= (1-\underset{\sim}{t})^{a} f(\Delta_1, \underset{\sim}{t}/1-\underset{\sim}{t}) \cdot (1-\underset{\sim}{t})^{a'} f(\Delta_2, \underset{\sim}{t}/1-\underset{\sim}{t}) \\ &= h(\Delta_1, \underset{\sim}{t}) \cdot h(\Delta_2, \underset{\sim}{t}). \end{aligned}$$

Equating coefficients of these polynomials gives

$$h_b(\Delta_1 * \Delta_2) = \sum_{v+w=b} h_v(\Delta_1) \cdot h_w(\Delta_2). \qquad \blacksquare$$

Corollary 4: If $\Delta_1,\ldots,\Delta_n$ are balanced complexes, then $\Delta_1*\ldots*\Delta_n$ is balanced, and $h_b(\Delta_1*\ldots*\Delta_n) = \sum_{v_1+\ldots+v_n=b} h_{v_1}(\Delta_1)\cdot\ldots\cdot h_{v_n}(\Delta_n)$, v_i a vector, $1 \leq i \leq n$. ∎

Corollary 5: Let $\Delta_1,\ldots,\Delta_n$ be balanced complexes with disjoint labeling sets; say Δ_i is type $\underset{\sim}{a}_i = (a_{i1},\ldots,a_{im_i})$ for $1 \leq i \leq n$. Then $\Delta_1*\ldots*\Delta_n$ is balanced of type $a = (\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_n)$ and for all $b \leq a$, $h_b(\Delta_1*\ldots*\Delta_n) = h_{\underset{\sim}{b}_1}(\Delta_1)\cdot\ldots\cdot h_{\underset{\sim}{b}_n}(\Delta_n)$, where $\underset{\sim}{b}_i = (b_{i1},\ldots,b_{im_i})$. ∎

Proposition 6: If $\hat{\Delta}$ is the minimal complex of type $a = (a_1,\ldots,a_m)$, then for all $b \leq a$, $h_b(\hat{\Delta}) = 1$.

Proof: Recall $\hat{\Delta} = \partial S_1*\ldots*\partial S_m$, where S_i is an a_i-simplex, and the vertices in S_i are labeled i. Let $b = (b_1,\ldots,b_m)$. Then $h_b(\hat{\Delta}) = h_b(\partial S_1*\ldots*\partial S_m) = h_{b_1}(\partial S_1)\cdot\ldots\cdot h_{b_m}(\partial S_m)$ by Corollary 5. Since $h(\partial S_i) = (\underbrace{1,\ldots,1}_{a_i+1},0,\ldots,0)$, and $b_i \leq a_i$ for all i, $h_{b_i}(\partial S_i) = 1$ for $1 \leq i \leq m$. Thus $h_b(\hat{\Delta}) = 1$. ∎

§ 3 A Balanced Stellar Subdivision

One technique used often in generating new complexes from existing ones is stellar subdivision. In this section we introduce a balanced version of the concept, and examine how such an operation affects the extended h-vector.

Suppose Δ is a balanced simplicial complex of type $a = (a_1,\ldots,a_m)$, and $\sigma \in \Delta$ is a face of type $x \leq a$. Let $\hat{\Delta}$ denote the minimal type x complex, and $\hat{\sigma}$ a maximal face in $\hat{\Delta}$. Then define the balanced stellar subdivision of σ in Δ to be

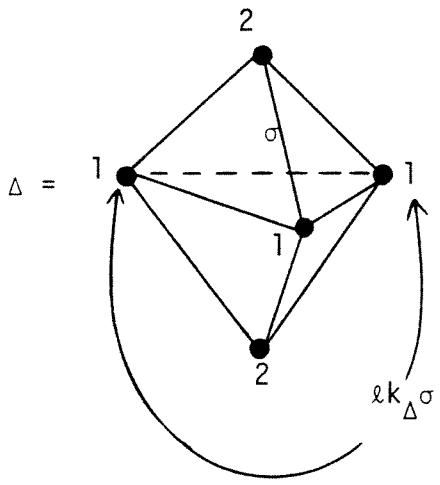
$$st_{bal}(\Delta,\sigma) \equiv (\Delta\backslash\sigma) \cup [\ell k_{\Delta}\sigma * (\hat{\Delta}\backslash\hat{\sigma})],$$

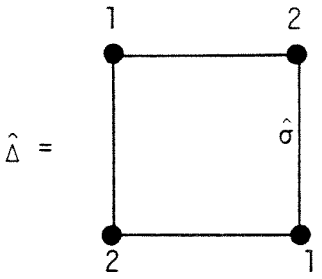
where σ and $\hat{\sigma}$ are identified by a label-preserving bijection of their vertices.

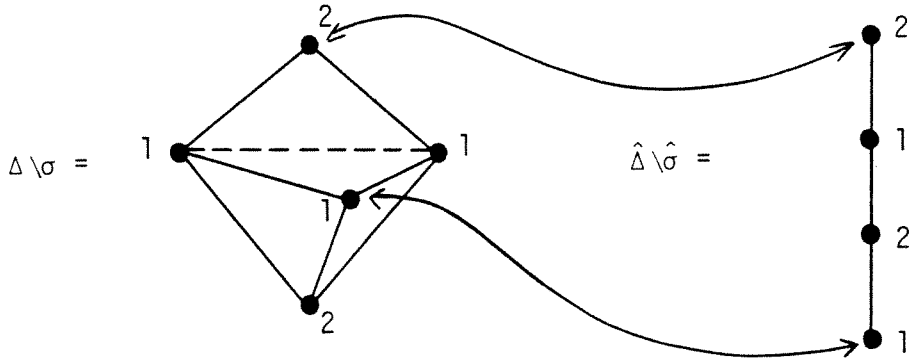
Examples

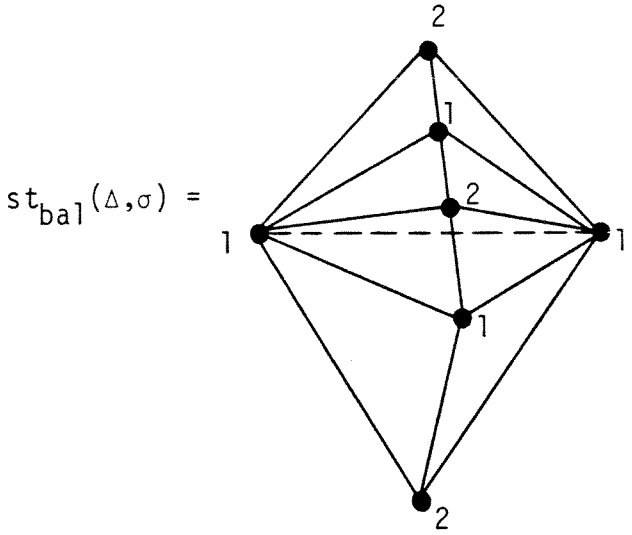
1) $a = (2,1)$

$x = (1,1)$



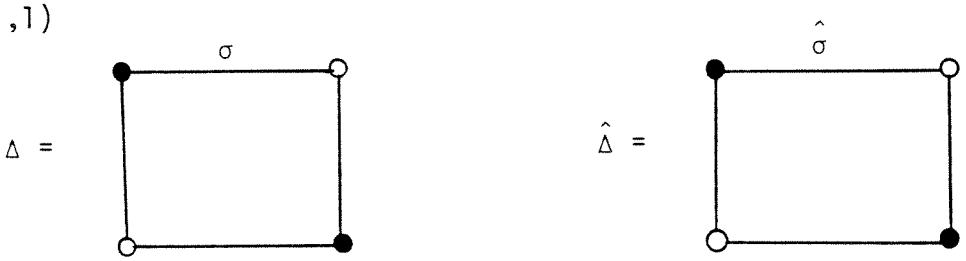




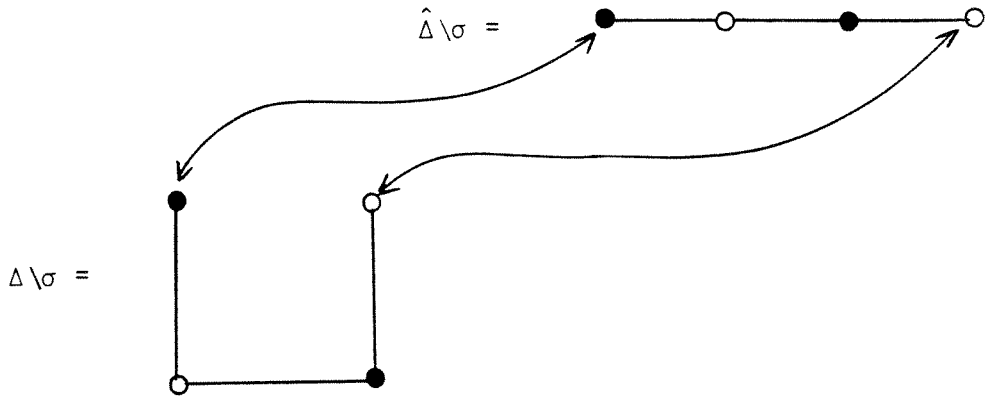


2) a = (1,1)

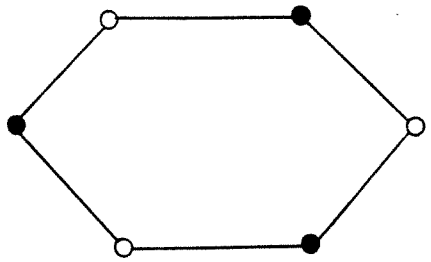
x = (1,1)



$\ell k_{\Delta}\sigma = \emptyset$



$st_{bal}(\Delta,\sigma)$ =



To show that $st_{bal}(\Delta,\sigma)$ is balanced of type a, we examine its maximal faces and see they are of two types:

(1) maximal faces of Δ not containing σ, and

(2) faces which are obtained by the union of a maximal face in $\ell k_{\Delta}\sigma$ with a maximal face in $\hat{\Delta}\backslash\hat{\sigma}$.

Faces of the first kind are clearly type a. Suppose τ is a face of the second kind, that is, $\tau = \gamma \cup \delta$ where γ is a maximal face in $\ell k_{\Delta}\sigma$ and δ is maximal in $\hat{\Delta}\backslash\hat{\sigma}$. Then δ is type x; $\gamma \cup \sigma$ is a face in Δ, and is maximal since γ is, so $\gamma \cup \sigma$ is type a. Since σ is type x, γ must be type a-x. Hence $\tau = \gamma \cup \delta$ is type $(a-x)+x = a$, showing $st_{bal}(\Delta,\sigma)$ is balanced of type a.

Another useful fact emerges from the foregoing discussion, which we record here for future reference.

Lemma 7: If Δ is a balanced complex of type a, and $\sigma \in \Delta$ is a face of type $x \leq a$, then $\ell k_{\Delta}\sigma$ is a balanced complex of type a-x. ∎

We now consider how, for Δ minimal and $\sigma \in \Delta$, $st_{bal}(\Delta,\sigma)$ may be realized as the boundary complex of a simplicial polytope.

Theorem 8: Suppose Δ is the minimal type $a = (a_1,\ldots,a_m)$ complex, and $\sigma \in \Delta$ is a face of type x. Then there exists a simplicial d-polytope P^x $(d \equiv \sum_{i=1}^{m} a_i)$ and a labeling of the vertices of P^x such that $\partial P^x = st_{bal}(\Delta,\sigma)$.

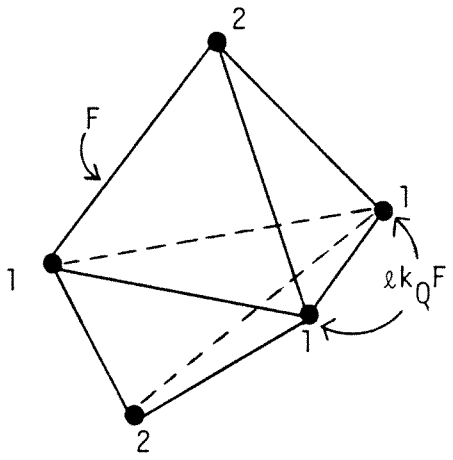
Proof: We give here the basic idea of the proof, which generalizes that of [5; Theorem 3.1]. See [15] for the details in the general case.

Let Q be the minimal polytope of type a; let $x = (x_1,\ldots,x_m)$. For $x = \underset{\sim}{0}$, the theorem is satisfied by letting P^x be Q. So we may assume that $x_1,\ldots,x_r > 0$, $x_{r+1} = \ldots = x_m = 0$, for some $r > 0$.

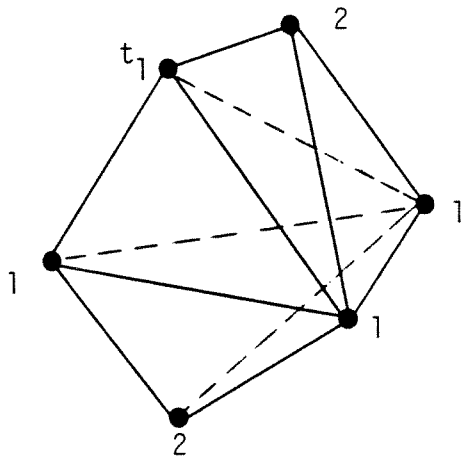
Starting with any face F_1 of type x, form the polytope $Q_1 = st_Q(F_1,t_1)$, where t_1 is a new vertex labeled 1. Unless $r = 1$, Q_1 is no longer balanced, as it has a face F_2 with a_1+1 vertices of label 1. Form $Q_2 = st_{Q_1}(F_2,t_2)$ where t_2 has label 2. If $r > 2$, then Q_2 will have a face F_3 with a_2+1 vertices of label 2. Continue until defining Q^r which can be taken to be the polytope P^x.

Example

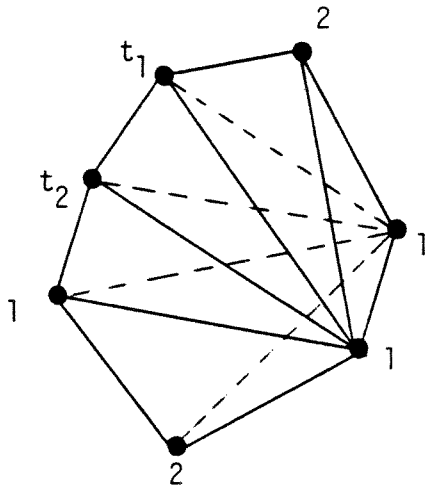
$a = (2,1)$, $x = (1,1)$



Q (solid)



Q_1



Q_2

In fact, the construction in the proof shows that if Δ is <u>any</u> balanced simplicial type $a = (a_1,\ldots,a_m)$ complex and $\sigma \in \Delta$ is a type x face involving r labels, $1 \leq r \leq m$, then $st_{bal}(\Delta,\sigma)$ is obtained by a sequence of r ordinary stellar subdivisions. A similar procedure can be followed to show that any pure labeled complex can be stellar subdivided into a type a balanced complex (maintaining existing labels) for any vector a not obviously excluded by the existing labels (i.e. the type allows for at least as many labels as have been already assigned). See [15; Proposition 3.3].

There remains the computation of the extended h-vector of $st_{bal}(\Delta,\sigma)$.

<u>Lemma 9:</u> If Δ is the minimal type $a = (a_1,\ldots,a_m)$ complex, and $\sigma \in \Delta$ is a face of type $x \leq a$, then $\ell k_\Delta \sigma$ is the minimal type $a-x$ complex.

<u>Proof</u>: By Lemma 7, $\ell k_\Delta \sigma$ is a balanced (simplicial) complex of type $a-x$. We first show that $f_0(\ell k_\Delta \sigma) = f_0(\Delta')$, where Δ' is the minimal type $a-x$ complex. Remark (1) preceding Proposition 3

established that $f_0(\Delta') = \sum_{i=1}^{m} (a_i - x_i + 1) - |\{i: a_i = x_i\}|$. Without loss of generality, assume that $a_i \neq 0$ for each i, since otherwise $a_i = x_i = 0$ and there would be no vertices labeled i in either Δ or Δ'. Now, suppose $\sigma = \sigma_1 * \sigma_2 * \ldots * \sigma_m$, where σ_i consists of all the vertices of σ having label i. Then

$$\ell k_\Delta \sigma = \ell k_{\partial S_1} \sigma_1 * \ldots * \ell k_{\partial S_m} \sigma_m$$

where the S_i are as in §2. Thus

$$f_0(\ell k_\Delta \sigma) = \sum_{i=1}^{m} f_0(\ell k_{\partial S_i} \sigma_i).$$

If $a_i = x_i$ then $\ell k_{\partial S_i} \sigma_i = \emptyset$ and if $a_i > x_i$ then $f_0(\ell k_{\partial S_i} \sigma_i) = a_i + 1 - x_i$. So $f_0(\ell k_\Delta \sigma) = f_0(\Delta')$ and thus $\ell k_\Delta \sigma$ is combinatorially equivalent to Δ' by the uniqueness of Δ'.

Theorem 10: If Δ is a minimal type $a = (a_1, \ldots, a_m)$ complex, and $\sigma \in \Delta$ is a face of type $x \leq a$, then for all $b \leq a$

$$h_b(st_{ba1}(\Delta, \sigma)) = 1 + |\{(y,z): y,z \in \mathbb{N}^m,\ y+z = b,\ 0 \leq y \leq a-x,\ 0 \lneq z \lneq x\}|.$$

Proof: First, if Δ_1 and Δ_2 are balanced complexes of type a, then $f(\Delta_1 \cup \Delta_2, \underset{\sim}{t}) = f(\Delta_1, \underset{\sim}{t}) + f(\Delta_2, \underset{\sim}{t}) - f(\Delta_1 \cap \Delta_2, \underset{\sim}{t})$. So

$$h(\Delta_1 \cup \Delta_2, \underset{\sim}{t}) = (1-\underset{\sim}{t})^a f(\Delta_1 \cup \Delta_2, \underset{\sim}{t}/1-\underset{\sim}{t})$$

$$= (1-\underset{\sim}{t})^a f(\Delta_1, \underset{\sim}{t}/1-\underset{\sim}{t}) + (1-\underset{\sim}{t})^a f(\Delta_2, \underset{\sim}{t}/1-\underset{\sim}{t})$$
$$- (1-\underset{\sim}{t})^a f(\Delta_1 \cap \Delta_2, \underset{\sim}{t}/1-\underset{\sim}{t})$$

$$= h(\Delta_1,\underset{\sim}{t}) + h(\Delta_2,\underset{\sim}{t}) - h^{(a)}(\Delta_1 \cap \Delta_2, \underset{\sim}{t}).$$

Now apply this to $st_{bal}(\Delta,\sigma) \equiv \Delta_1 \cup \Delta_2$, where $\Delta_1 = \Delta\backslash\sigma$, and $\Delta_2 = \ell k_\Delta \sigma * (\hat{\Delta}\backslash\hat{\sigma})$. Recall that σ and $\hat{\sigma}$ are identified by a bijection between vertex sets which preserves labeling; thus $\Delta_1 \cap \Delta_2 = (\Delta\backslash\sigma) \cap [\ell k_\Delta \sigma * (\hat{\Delta}\backslash\hat{\sigma})] = \ell k_\Delta \sigma * \partial\overline{\sigma}$. Note also that the complexes Δ_1 and Δ_2 are type a. So $h(st_{bal}(\Delta,\sigma), \underset{\sim}{t}) = h(\Delta\backslash\sigma, \underset{\sim}{t}) + h(\ell k_\Delta \sigma * (\hat{\Delta}\backslash\hat{\sigma}), \underset{\sim}{t}) - h^{(a)}(\ell k_\Delta \sigma * \partial\overline{\sigma}, \underset{\sim}{t})$. Since Δ is the union of the type a complexes $\Delta\backslash\sigma$ and $\overline{\sigma} * \ell k_\Delta \sigma$, and $(\Delta\backslash\sigma) \cap (\overline{\sigma} * \ell k_\Delta \sigma) = \ell k_\Delta \sigma * \partial\overline{\sigma}$, we also have $h(\Delta,\underset{\sim}{t}) = h(\Delta\backslash\sigma, \underset{\sim}{t}) + h(\overline{\sigma} * \ell k_\Delta \sigma, \underset{\sim}{t}) - h^{(a)}(\ell k_\Delta \sigma * \partial\overline{\sigma}, \underset{\sim}{t})$. Combining these two equations gives

$$h(st_{bal}(\Delta,\sigma), \underset{\sim}{t}) = h(\Delta,\underset{\sim}{t}) - h(\overline{\sigma} * \ell k_\Delta \sigma, \underset{\sim}{t}) + h^{(a)}(\ell k_\Delta \sigma * \partial\overline{\sigma}, \underset{\sim}{t})$$
$$+ h(\ell k_\Delta \sigma * (\hat{\Delta}\backslash\hat{\sigma}), \underset{\sim}{t}) - h^{(a)}(\ell k_\Delta \sigma * \partial\overline{\sigma}, \underset{\sim}{t})$$

$$= h(\Delta,\underset{\sim}{t}) + h(\ell k_\Delta \sigma,\underset{\sim}{t})[h(\hat{\Delta}\backslash\hat{\sigma}, \underset{\sim}{t}) - h(\overline{\sigma},\underset{\sim}{t})],$$

the last equality using the fact that the h-polynomial of a join is the product of the h-polynomials of the complexes joined, established in the proof of Proposition 3.

By shelling the single simplex $\overline{\sigma}$ and appealing to [20; Proposition 3.6], we see that $h(\overline{\sigma},\underset{\sim}{t}) = 1$. Similarly, shell $\hat{\Delta}$ so that $\hat{\sigma}$ is the last simplex in the shelling order. (See [7].) Since $\hat{\Delta}$ is minimal, Proposition 6 and [20; Proposition 3.6] yield

$$h_b(\hat{\Delta}\backslash\hat{\sigma}) = \begin{cases} 1 & \text{for all } b \leq x \\ 0 & \text{else.} \end{cases}$$

So

$$h_b(\hat{\Delta}\backslash\hat{\sigma}) - h_b(\overline{\sigma}) = \begin{cases} 1 & \text{for all } b < x,\ b \neq 0 \\ 0 & \text{else.} \end{cases}$$

Recalling that Δ is minimal of type a, and $\ell k_{\Delta}\sigma$ is minimal of type a-x gives

$$h_b(\Delta) = \begin{cases} 1 & \text{for all } b \leq a \\ 0 & \text{else} \end{cases}$$

and

$$h_b(\ell k_{\Delta}\sigma) = \begin{cases} 1 & \text{for all } b \leq a\text{-}x \\ 0 & \text{else.} \end{cases}$$

Hence for fixed $b \leq a$ the equation of h-polynomials yields $h_b(st_{bal}(\Delta,\sigma)) = 1 +$ (the number of ways b may be written as $y+z$ with $0 \leq y \leq a\text{-}x$ and $0 \lneq z \lneq x$) $= 1 + |\{(y,z): y,z \in \mathbb{N}^m,\ y+z = b,\ 0 \leq y \leq a\text{-}x,\ 0 \lneq z \lneq x\}|$. ∎

§4 An Affine Spanning Set for Balanced Spheres

We now turn our attention back to the problem of determining the dimension of $\mathrm{aff}(h_b(C_a^d))$. For convenience, the notation is reviewed here. C_a^d denotes the set of all type $a = (a_1,\ldots,a_m)$ balanced homology $(d-1)$-spheres and $(h_b(C_a^d))$ the set of all vectors $(h_b(\Delta))_{b\leq a} \in \mathbb{N}^K$ for $\Delta \in C_a^d$, where $K = \Pi_{i=1}^m (a_i+1)$.

It was shown in Corollary 2 that $\dim \mathrm{aff}(h_b(C_a^d)) \leq \lfloor \frac{K-1}{2} \rfloor$, and the aim here is to demonstrate <u>equality</u>. This is accomplished by obtaining $\lfloor \frac{K-1}{2} \rfloor + 1$ affinely independent vectors in $\mathrm{aff}(h_b(C_a^d))$. At the same time, by using ideas developed in the two previous sections, we exhibit a set of balanced simplicial polytopes whose extended h-vectors span $\mathrm{aff}(h_b(C_a^d))$.

For $x \leq a$ let P^x be the minimal type a complex subdivided over a face of type x. Define a lexicographic order on all $x \leq a$ as follows: $x < y$ if $x_i < y_i$ for some i, $1 \leq i \leq m$ and $x_j = y_j$ for all $j < i$. Define a $K\times K$ matrix M whose columns are indexed by $b \leq a$, written in this lexicographic order, and whose rows are indexed by $x \leq a$, same order. The (x,b)-entry of M is $h_b(P^x) = 1 + |\{(y,z): y,z \in \mathbb{N}^m,\ y+z = b,\ 0 \leq y \leq a-x,\ 0 \lneq z \lneq x\}|$. We will show that M has rank $\lfloor \frac{K-1}{2} \rfloor + 1$.

First, note that $h_b(P^{\underset{\sim}{0}}) = 1$ for all b, since $x = \underset{\sim}{0}$ precludes the possibility of any z. So subtract this row from all others to get the matrix M' whose (x,b)-entry is $h_b(P^x) - 1$ for all $x \leq a$, $x \neq \underset{\sim}{0}$, and whose $\underset{\sim}{0}$-row is all 1's; this operation doesn't change rank.

Next, consider the related matrix $M''-M_1-M_2$, all matrices indexed by row and column as before, where M'' is an adjustment of M' which allows $z = 0$ and $z = x$, M_1 corrects for the adjustment $z = 0$,

and M_2 for $z = x$. Then the (x,b)-entry of M'' is $|\{(y,z): y,z \in \mathbf{N}^m, y+z = b, 0 \leq y \leq a-x, 0 \leq z \leq x\}|$, the (x,b)-entry of M_1 is 1 if $0 \leq b \leq a-x$, 0 else, and the (x,b)-entry of M_2 is 1 if $x \leq b \leq a$, 0 else. M' differs from $M''-M_1-M_2$ only in the $\underset{\sim}{0}$ row, which in $M''-M_1-M_2$ is all -1's, so rank is the same.

Further examination of the matrices M'', M_1, and M_2 reveals that each has an underlying structure which turns out to be the key to the determination of the rank of M. This structure is uncovered in the following series of lemmas.

Lemma 11: Let $a \geq 0$ be an integer. Then for integers $0 \leq x \leq a/2$ and $0 \leq b \leq a$, the function $N(x,b) \equiv |\{(y,z): y,z \in \mathbb{Z}, y+z = b, 0 \leq y \leq a-x, 0 \leq z \leq x\}|$ takes on values

$$N(x,b) = \begin{cases} b+1 & \text{if } 0 \leq b \leq x, \\ x+1 & \text{if } x+1 \leq b \leq a-x, \\ a-b+1 & \text{if } a-x+1 \leq b \leq a. \end{cases}$$

Proof: Set up a matrix of size $(a-x+1)\times(x+1)$, where the rows are indexed by $y = 0,1,2,\ldots,a-x$, and the columns by $z = 0,1,2,\ldots,x$. Note that since $x \leq a/2$ there are at least as many rows as columns. Define the (y,z)-entry to be $y+z$. Then the entries on a given reverse diagonal are all the same, and if b is the entry, $N(x,b)$ is the number of entries on that diagonal. So, if $0 \leq b \leq x$, the entries on the "b" diagonal are $b+0,(b-1)+1,\ldots,1+(b-1),0+b$, amounting to $b+1$ entries in all. Similarly, looking at the lower right corner of the matrix, if $a-x+1 \leq b \leq a$, the entries on the "b" diagonal are $x+(b-x),(x-1)+(b-x+1),\ldots,(x-(a-b))+(a-x)$, totaling $a-b+1$ entries.

In between, that is if $x+1 \leq b \leq a-x$, the number of entries on the "b" diagonal remains constant at $x+1$. Specifically the entries are $b+0,(b-1)+1,\ldots,(b-x)+x$. ∎

Example $a = 7$, $x = 2$

y \ z	0	1	2
0	~~0~~	~~1~~	~~2~~
1	~~1~~	~~2~~	~~3~~
2	~~2~~	~~3~~	~~4~~
3	~~3~~	~~4~~	~~5~~
4	~~4~~	~~5~~	~~6~~
5	~~5~~	~~6~~	~~7~~

b	N(x,b)	
0	1	} b+1
1	2	
2	3	
3	3	} x+1
4	3	
5	3	
6	2	} a-b+1
7	1	

Next let $n_i = a_i+1$, $1 \leq i \leq m$. We define $n_i \times n_i$ matrices A_i, for $i = 1,\ldots,m$. Index the rows and columns of A_i by $x_i = 0,1,\ldots,a_i$, and $b_i = 0,1,\ldots,a_i$, respectively, and let the (x_i,b_i)-entry be $N(x_i,b_i) \equiv |\{(y,z): y,z \in \mathbb{Z},\ y+z = b_i,\ 0 \leq y \leq a_i-x_i,\ 0 \leq z \leq x_i\}|$.

Lemma 12: A_i is of the form

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ & 2 & \cdot & 2 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & 2 & \cdot & 2 & \\ 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

where the innermost "ring" consists of $(\lfloor \frac{a_i}{2} \rfloor + 1)$'s.

Examples

1) For $a_i = 4$, $n_i = 5$,

$$A_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

2) For $a_i = 5$, $n_i = 6$,

$$A_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Proof of Lemma 12: The (a_i-x_i, b_i)-entry is the number of ways b_i may be written as y_i+z_i where $0 \le y_i \le a_i-(a_i-x_i)$ and $0 \le z_i \le a_i-x_i$. If the roles of y_i and z_i are interchanged it becomes evident that this is exactly the (x_i,b_i)-entry. So row x_i is the same as row (a_i-x_i) in A_i. Thus we only need consider $0 \le x_i \le \lfloor \frac{a_i}{2} \rfloor$.

The entries for such x_i have been computed in Lemma 11:

(1) if $0 \le b_i \le x_i$, the entry is b_i+1.

(2) if $x_i+1 \le b_i \le a_i-x_i$, the entry is x_i+1.

(3) if $a_i-x_i+1 \le b_i \le a_i$, the entry is a_i-b_i+1.

Using these, we now verify that column b_i is the same as column (a_i-b_i). Fix x_i, and let $b_i' \equiv a_i-b_i$.

(1) If $0 \le b_i \le x_i$, we show that the (x_i,b_i')-entry is b_i+1. We have $a_i-x_i \le a_i-b_i \le a_i$, so $a_i-x_i \le b_i' \le a_i$. If $b_i' = a_i-x_i$, the entry is x_i+1. But $a_i-b_i = a_i-x_i$, so $x_i+1 = b_i+1$. If $b_i' > a_i-x_i$, the entry is $a_i-b_i'+1 = a_i-(a_i-b_i)+1 = b_i+1$.

(2) If $x_i+1 \le b_i \le a_i-x_i$, we verify that the (x_i,b_i')-entry is x_i+1. But $x_i \le a_i-b_i \le a_i-x_i-1$, so $x_i \le b_i' \le a_i-x_i-1$. If $b_i' = x_i$, the entry is $b_i'+1 = x_i+1$. If $x_i < b_i' \le a_i-x_i-1$, the entry is x_i+1.

(3) If $a_i-x_i+1 \le b_i \le a_i$, we check that the (x_i,b_i')-entry is a_i-b_i+1. We have $0 \le a_i-b_i \le x_i-1$, or $0 \le b_i' \le x_i-1$, so the entry

is $b_i'+1 = a_i-b_i+1$. Therefore, column b_i and column (a_i-b_i) are identical.

So it suffices to restrict attention to the $(\lfloor\frac{a_i}{2}\rfloor + 1)\times(\lfloor\frac{a_i}{2}\rfloor + 1)$ submatrix in the upper left corner of A_i. Again by the computation in Lemma 11, the entry for $0 \le b_i \le x_i$ is b_i+1. The rest of the b_i's fall into the interval $x_i+1 \le b_i \le a_i/2$. Since $a_i/2 \le a_i-x_i$, the entry for these b_i's is x_i+1. Thus, row 0 is all 1's, and for $1 \le x_i \le \lfloor\frac{a_i}{2}\rfloor + 1$, row x_i is $1\ 2\ \ldots\ x_i\ x_i+1\ x_i+1\ \ldots\ x_i+1$. ∎

Let $A_1 \otimes A_2$ denote the tensor product of the matrices A_1 and A_2. That is, $A_1 \otimes A_2$ is the $n_1n_2 \times n_1n_2$ matrix consisting of rows (and columns) of n_1 square blocks of size n_2; the block in the $k^{\underline{th}}$ row and $\ell^{\underline{th}}$ column is A_2 multiplied by the scalar in the $k^{\underline{th}}$ row and ℓ^{th} column of A_1, $1 \le k,\ell \le n_1$. This product is associative, so the iterated tensor product of $A_1,\ldots,A_m$ is denoted $A_1 \otimes \ldots \otimes A_m$ (without parentheses), alternatively $\otimes_{i=1}^m A_i$. The rows and columns of $\otimes_{i=1}^m A_i$ are indexed by $x \le a$ and $b \le a$ respectively, written in the lexicographic order defined earlier. So the $((x_1,x_2,\ldots,x_m),(b_1,b_2,\ldots,b_m))$-entry of $\otimes_{i=1}^m A_i$ is given by $\Pi_{i=1}^m$ [(x_i,b_i)-entry of A_i]. For the basic facts about tensor products used here, refer to [14].

Now denote by I_{n_i} the $n_i\times n_i$ identity matrix, and by J_{n_i} the matrix with the columns of I_{n_i} written in reverse order. Let E_i be the $n_i\times n_i$ upper triangular matrix of 1's, and let $C_i = E_iJ_{n_i}$, so C_i has the columns of E_i written in reverse order.

$$I_{n_i} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \qquad J_{n_i} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

$$E_i = \begin{bmatrix} 1 & & 1 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \qquad C_i = \begin{bmatrix} 1 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

Lemma 13: In the setting described above,

(i) $M'' = \otimes_{i=1}^{m} A_i$

(ii) $M_1 = \otimes_{i=1}^{m} C_i$

(iii) $M_2 = \otimes_{i=1}^{m} E_i$.

Proof: (i) The (x,b)-entry of M'' is $|\{(y,z): y,z \in \mathbf{N}^m, \ y+z = b, \ 0 \le y \le a-x, \ 0 \le z \le x\}| = \Pi_{i=1}^{m} N(x_i,b_i) = \Pi_{i=1}^{m} [(x_i,b_i)\text{-entry of } A_i] =$ (x,b)-entry of $\otimes_{i=1}^{m} A_i$.

(ii) The (x_i,b_i)-entry of C_i is

$$\begin{cases} 1 & \text{if } x_i+b_i \le a_i \\ 0 & \text{else,} \end{cases}$$

so the (x,b)-entry of $\otimes_{i=1}^{m} C_i$ is

$$\begin{cases} 1 & \text{if } x_i+b_i \le a_i \text{ for all } 1 \le i \le m \\ 0 & \text{else.} \end{cases}$$

But the (x,b)-entry of M_1 is

$$\begin{cases} 1 & \text{if } 0 \le b \le a-x \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } x+b \le a \\ 0 & \text{else} \end{cases} = (x,b)\text{-entry of } \otimes_{i=1}^{m} C_i.$$

(iii) The (x_i,b_i)-entry of E_i is

$$\begin{cases} 1 & \text{if } b_i \geq x_i \\ 0 & \text{else,} \end{cases}$$

so the (x,b)-entry of $\otimes_{i=1}^{m} E_i$ is

$$\begin{cases} 1 & \text{if } b_i \geq x_i \text{ for all } 1 \leq i \leq m \\ 0 & \text{else.} \end{cases}$$

Finally, the (x,b)-entry of M_2 is

$$\begin{cases} 1 & \text{if } x \leq b \leq a \\ 0 & \text{else} \end{cases} = (x,b)\text{-entry of } \otimes_{i=1}^{m} E_i.$$

■

Hence the problem of establishing equality in Corollary 2 is reduced to that of showing the matrix $(\otimes_{i=1}^{m} A_i) - (\otimes_{i=1}^{m} C_i) - (\otimes_{i=1}^{m} E_i)$ to have rank $\geq \lfloor \frac{K-1}{2} \rfloor + 1 = \lceil \frac{K}{2} \rceil$, where $K = \Pi_{i=1}^{m} (a_i+1) = \Pi_{i=1}^{m} n_i$.

Theorem 14: $(\otimes_{i=1}^{m} A_i) - (\otimes_{i=1}^{m} C_i) - (\otimes_{i=1}^{m} E_i)$ has rank $\geq \lceil \frac{K}{2} \rceil$.

Proof: $\operatorname{rank}[\otimes_{i=1}^{m} A_i - \otimes_{i=1}^{m} C_i - \otimes_{i=1}^{m} E_i]$

$$= \operatorname{rank}[(\otimes_{i=1}^{m} E_i)^{-1}(\otimes_{i=1}^{m} A_i - \otimes_{i=1}^{m} C_i - \otimes_{i=1}^{m} E_i)]$$

$$= \operatorname{rank}[(\otimes_{i=1}^{m} E_i^{-1})(\otimes_{i=1}^{m} A_i) - (\otimes_{i=1}^{m} E_i^{-1})(\otimes_{i=1}^{m} C_i) - I_K]$$

$$= \operatorname{rank}[\otimes_{i=1}^{m} (E_i^{-1}A_i) - \otimes_{i=1}^{m}(E_i^{-1}C_i) - I_K]$$

$$= \operatorname{rank}[\otimes_{i=1}^{m} (E_i^{-1}A_i) - J_K - I_K]$$

since

$$\otimes_{i=1}^{m} E_i^{-1}C_i = \otimes_{i=1}^{m} J_{n_i} C_i^{-1}C_i = \otimes_{i=1}^{m} J_{n_i} = J_{\Pi_{i=1}^{m} n_i} = J_K.$$

If K is even, $-J_K-I_K$ is

$$\begin{bmatrix} -1 & \cdots & 0 & \cdots & -1 \\ \vdots & \ddots & & \iddots & \vdots \\ 0 & & -1 \quad -1 & & 0 \\ \vdots & \iddots & -1 \quad -1 & \ddots & \vdots \\ -1 & \cdots & 0 & \cdots & -1 \end{bmatrix}_K$$

and if K is odd, $-J_K-I_K$ is

$$\begin{bmatrix} -1 & & & & -1 \\ & \ddots & 0 & \iddots & \\ & & -2 & & \\ & \iddots & 0 & \ddots & \\ -1 & & & & -1 \end{bmatrix}_K .$$

Whatever the case, denote this "correction" matrix by T.

The proof is completed by showing that the $\lceil \frac{K}{2} \rceil \times \lceil \frac{K}{2} \rceil$ submatrix in the upper left corner of $\oplus_{i=1}^{m} (E_i^{-1}A_i) + T$ has rank $\lceil \frac{K}{2} \rceil$. Recall that E_i is upper triangular with 1's, so $(E_i)^{-1}$ is obtained from I_{n_i} by replacing row j by [row j - row$(j+1)$] for $1 \leq j \leq n_i-1$. Thus for each $1 \leq i \leq m$, $E_i^{-1}A_i$ has the following form: for $1 \leq j \leq \left\lceil \frac{n_i-1}{2} \right\rceil$, row j is j 0's, followed by (n_i-2j) -1's, followed by j 0's; for $\left\lceil \frac{n_i-1}{2} \right\rceil < j \leq n_i-1$, row j is -(row(n_i-j)); row n_i consists of 1's.

Examples

1) For $a_i = 4$, $n_i = 5$

$$A_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad E_i^{-1}A_i = \left[\begin{array}{ccc:cc} 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

2) For $a_i = 5$, $n_i = 6$

$$A_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad E_i^{-1}A_i = \left[\begin{array}{ccc:ccc} 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

So the $\left\lceil \frac{n_i}{2} \right\rceil \times \left\lceil \frac{n_i}{2} \right\rceil$ submatrix in the upper left corner of $E_i^{-1}A_i$ is

$$\begin{bmatrix} 0 & & -1 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$$

if n_i is even, and

$$\begin{bmatrix} 0 & & -1 & \\ & \ddots & & \\ 0 & & 0 & \\ & & & 1 \end{bmatrix}$$

if n_i is odd.

Denote the $\lceil\frac{K}{2}\rceil\times\lceil\frac{K}{2}\rceil$ upper left submatrix in $\otimes_{i=1}^{m} E_i^{-1}A_i$ by S.

Claim: S is strictly upper triangular if K is even, and of the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & 0 & 0 \\ & & 1 \end{bmatrix}$$

if K is odd. Recall $K = \Pi_{i=1}^{m} n_i$, and induct on m. If $m = 1$, the assertion is true, as previously noted. So suppose for all $m < k$ the result holds and let $m = k$.

The induction hypothesis says that $\otimes_{i=2}^{k} E_i^{-1}A_i$ is strictly upper triangular if $\Pi_{i=2}^{k} n_i$ is even, and of the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & 0 & 0 \\ & & 1 \end{bmatrix}$$

if $\Pi_{i=2}^{k} n_i$ is odd. Also $E_1^{-1}A_1$ is strictly upper triangular if n_1 is even, and of the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & 0 & 0 \\ & & 1 \end{bmatrix}$$

if n_1 is odd.

Now suppose K is even; then either n_1 is even, or n_1 is odd and $\Pi_{i=2}^{k} n_i$ is even. In the first instance, S has the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \otimes R = \begin{bmatrix} \boxed{0\cdot R} & & & \\ & \boxed{0\cdot R} & & \\ & & \boxed{0\cdot R} & \\ & & & \boxed{0\cdot R} \end{bmatrix},$$

where R denotes $\otimes_{i=2}^{k} E_i^{-1} A_i$, and the broken lines show the blocks on the diagonal and below the diagonal in S. In the second instance, S has the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \otimes R = \left[\begin{array}{ccc} \boxed{0\cdot R} & & \\ & \boxed{0\cdot R} & \\ & & \boxed{0\cdot R} \\ \hline \multicolumn{2}{c}{P} & Q \end{array}\right],$$

where the blocks in region P consist of $0\cdot$(the upper $\frac{1}{2}(\Pi_{i=1}^{k} n_i)$ rows of R), hence P is all 0's; and Q is $1\cdot$(the upper left corner of R, size $\frac{1}{2}(\Pi_{i=2}^{k} n_i)$). Since $\Pi_{i=2}^{k} n_i$ is even, Q is strictly upper triangular. Thus if K is even, S is strictly upper triangular.

On the other hand, if K is odd, then n_1 must be odd and $\Pi_{i=2}^{k} n_i$ must be odd. S has the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \otimes R = \left[\begin{array}{ccc} \boxed{0\cdot R} & & \\ & \boxed{0\cdot R} & \\ & & \boxed{0\cdot R} \\ \hline \multicolumn{2}{c}{P} & Q \end{array}\right].$$

where the blocks in P consist of $0\cdot$(the upper $\frac{1}{2}(\Pi_{i=2}^{k} n_i + 1)$ rows of R), hence P is all 0's; and Q is $1\cdot$(the upper left corner of R, size $\frac{1}{2}(\Pi_{i=2}^{k} n_i + 1)$). Since $\Pi_{i=2}^{k} n_i$ is odd, Q has the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & 0 & \\ & & 1 \end{bmatrix}.$$

Thus for K odd, S has the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & 0 & \\ & & 1 \end{bmatrix},$$

and this proves the claim.

Finally, let T' be the upper left submatrix of the "correction" matrix T. Recall that

$$T' = \begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix}$$

if K is even, and

$$\begin{bmatrix} -1 & & 0 & \\ & \ddots & & \\ 0 & & -1 & \\ & & & -2 \end{bmatrix}$$

if K is odd. So if K is even, $S+T'$ has the form

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} + \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix},$$

and if K is odd, it has form

$$\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & -2 \end{bmatrix}.$$

In either event $S+T'$ is triangular; therefore its rank is $\lceil \frac{K}{2} \rceil$. ∎

§5 Why Not a Basis?

We close with a discussion of how the results in the previous section illustrate facts about completely balanced complexes established in Bayer and Billera [4]. Also presented here is an examination of the difficulties encountered in trying to determine the polytopes whose extended h-vectors form a basis for $\mathrm{aff}(h_b(C_a^d))$.

Let C^d be the set of all completely balanced homology (d-1)-spheres, that is $C^d = C_a^d$ where $a = (a_1,\ldots,a_d)$ and $a_i = 1$ for $i = 1,\ldots,d$. In [4], $\mathrm{aff}(h_b(C^d))$ was shown to have dimension $2^{d-1}-1$, which is in agreement with our computation $\lfloor \frac{K-1}{2} \rfloor$, where $K = \Pi_{i=1}^{d} (a_i+1) = 2^d$. Furthermore, a set of polytopes was exhibited whose extended h-vectors are a basis for $\mathrm{aff}(h_b(C^d))$; we now describe briefly the method by which this was accomplished.

Note that if b_i is an entry in $b \leq a$, b_i is either 0 or 1. Define a lexicographic order on $\{b: b \leq a\}$ as follows. For

$b = (b_1,\ldots,b_d)$ and $b' = (b'_1,\ldots,b'_d)$, $b < b'$ if $\sum_{i=1}^{d} b_i < \sum_{i=1}^{d} b'_i$ or if $\sum_{i=1}^{d} b_i = \sum_{i=1}^{d} b'_i$ and for some $j \leq d$, $b_j < b'_j$ while for $i < j$ $b_i = b'_i$. Set up a $2^{d-1} \times 2^{d-1}$ matrix, columns indexed by the first 2^{d-1} vectors $b \leq a$ in increasing order, and rows indexed by the last 2^{d-1} vectors $x \leq a$, in decreasing order. The (x,b)-entry is defined to be $2-h_b(P^x)$, and the matrix is shown to be lower triangular with 1's along the diagonal. The conclusion is that the h-vectors of the boundary complexes of the polytopes P^x where x ranges over the last half of the vectors in the ordering, are a basis for $\mathrm{aff}(h_b(C^d))$.

For example, if $a = (1,1)$, the lexicographic order on b and x used by Bayer and Billera agrees with that used here in the previous section, except that they take the x in decreasing order. Denote the matrix whose (x,b)-entry is $h_b(P^x)$ by L. Then

$L =$

x \ b	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1	1	1	1
(0,1)	1	1	1	1
(1,0)	1	1	1	1
(1,1)	1	2	2	1

Bayer and Billera take the lower left 2×2 block, written with the rows switched, and subtract it from a matrix of 2's to get

x \ b	(0,0)	(0,1)
(1,1)	1	0
(1,0)	1	1

,

which has rank 2.

In our method, we subtract the first row of L from all others, and then multiply it by -1 to get

$(A_1 \otimes A_2) - (C_1 \otimes C_2) - (E_1 \otimes E_2) =$

x \ b	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1	1	1	1
(0,1)	1	1	1	1
(1,0)	1	1	1	1
(1,1)	1	1	1	1

-

x \ b	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1	1	1	1
(0,1)	1	0	1	0
(1,0)	1	1	0	0
(1,1)	1	0	0	0

-

x \ b	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1	1	1	1
(0,1)	0	1	0	1
(1,0)	0	0	1	1
(1,1)	0	0	0	1

where

$$A_i = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad E_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_i = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{for} \quad i = 1,2.$$

Using

$$E_i^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

multiply the equation by $E_1^{-1} \otimes E_2^{-1}$ to get

$$(E_1^{-1}A_1 \otimes E_2^{-1}A_2) + T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

$$= \left[\begin{array}{cc:cc} -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ \hdashline 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right],$$

whose upper left corner has rank 2.

Unfortunately, this method gives no specific information about which subset $S \subseteq \{x: x \leq a\}$ should be chosen in order for $h(P^x)_{x \in S}$ to be a basis. Even identifying the elementary row operations which transform L to $E_1^{-1}A_1 \otimes E_2^{-1}A_2 + T$, and specifying which rows of the final matrix form independent sets does not determine which rows of L

form independent sets. So working backwards to find a basis among the rows of L is not feasible.

We close with two examples which show the apparent unpredictability of the balanced, but not completely balanced, case.

Examples

1) a = (3,1). The h-vector is symmetric so only half the b's are listed. The x's are arranged lexicographically as specified by Bayer and Billera. Arrows indicate identical rows.

L =

x \ b	(0,0)	(1,0)	(0,1)	(2,0)
→ (3,1)	1	2	2	2
(2,1)	1	2	2	3
→ (3,0)	1	2	1	2
→ (1,1)	1	2	2	2
→ (2,0)	1	2	1	2
→ (0,1)	1	1	1	1
→ (1,0)	1	1	1	1
→ (0,0)	1	1	1	1

L has rank $\lfloor \frac{K-1}{2} \rfloor + 1 = \lfloor \frac{8-1}{2} \rfloor + 1 = 4$, which is the exact number of distinct rows. Notice that neither the upper half nor the lower half of L has rank 4, so the simplicity of the completely balanced case does not extend. Numerous examples of this kind were considered, and no pattern of independent rows emerged.

2) a = (2,3)

L =

x \ b	(0,0)	(1,0)	(0,1)	(2,0)	(1,1)	(0,2)	b \ x
(2,3)	1	2	2	2	2	2	(2,3)
(1,3)	1	2	2	2	3	2	(1,3)
(2,2)	1	2	2	2	3	3	(2,2)
(0,3)	1	1	2	1	2	2	(0,3)
(1,2)	1	2	2	2	4	3	(1,2)
(2,1)	1	2	2	2	3	2	(2,1)
(0,2)	1	1	2	1	2	2	(0,2)
(1,1)	1	2	2	2	3	2	(1,1)
(2,0)	1	2	1	1	2	1	(2,0)
(0,1)	1	1	1	1	1	1	(0,1)
(1,0)	1	1	1	1	1	1	(1,0)
(0,0)	1	1	1	1	1	1	(0,0)

Here L has rank $\lfloor \frac{K-1}{2} \rfloor + 1 = \lfloor \frac{12-1}{2} \rfloor + 1 = 6$, and the number of distinct rows is 7.

§6 Some Habitats of Balanced Complexes

Here we look at two settings in which balanced and completely balanced complexes arise naturally. One involves stellar subdivision of geometric cellular (not necessarily simplicial) complexes, so we offer a brief description of this based on discussions in Ewald and Shephard [9], and Bayer [3]. A few definitions are called for.

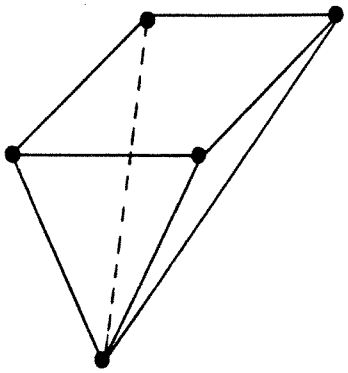
The cell P is a d-pyramid if $P = \mathrm{conv}(B \cup \{v\})$ where B (the base of P) is a $(d-1)$-cell and v (the apex of P) is a point not contained in aff B. For r a positive integer, P is an r-fold d-pyramid with base B of dimension $d-r$ if P is a d-pyramid with base P' where P' is an $(r-1)$-fold $(d-1)$-pyramid with base B. Ewald and Shephard's generalization differs from the subdivision of simplicial polytopes mainly in its requirement that if the cell F to be subdivided in a complex P is contained in a cell H of higher dimension, H must be a $(\dim H - \dim F)$-fold pyramid with base F. The notion of link of F in P carries over intact so if a cell F meets the criterion above, $\ell k_P F$ is simplicial. Since the definition of the join $*$ of simplicial complexes applies also to cellular complexes whose affine hulls are independent affine subspaces, F may be subdivided as $F'*\{t\}*G$ where F' is a proper cell of F, G is a face in $\ell k_P F$, and t is a point in relint F. The d-cells in a d-complex can always be subdivided, and the following fact from [9] sets the stage for subdivisions of lower-dimensional cells. Say F is an $(r-1)$-cell in a complex P for some r, $1 \leq r \leq d-1$; if all r-cells in P which contain F have been subdivided, yielding a

complex P', then all the cells of P' containing F are pyramids with base F. Thus F itself may be subdivided. So in some sense, the order of subdivision in a non-simplicial complex proceeds by decreasing dimension.

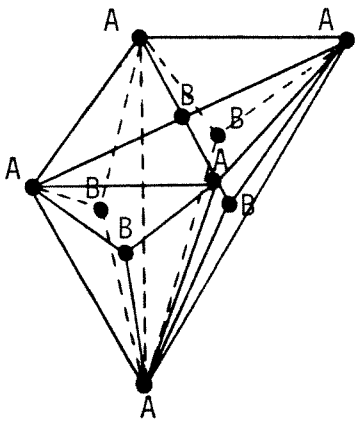
We now go on to the examples. Grünbaum [11] defines a d-polytope to be k-simplicial if every k-face of P is a simplex. When k = d-2, P is also called quasi-simplicial; in this case the facets of P are simplicial. For example, every 3-polytope is quasi-simplicial. If a stellar subdivision is performed on each facet of a quasi-simplicial d-polytope P, the resulting d-polytope P' is simplicial. This suggests a natural way to label the vertices of P' so that it is balanced of type (d-1,1). Give the vertices of P the label A, and assign each vertex introduced by a subdivision of a facet the label B. Then each facet of P' will have d-1 vertices labeled A, and 1 vertex labeled B.

Example

P =



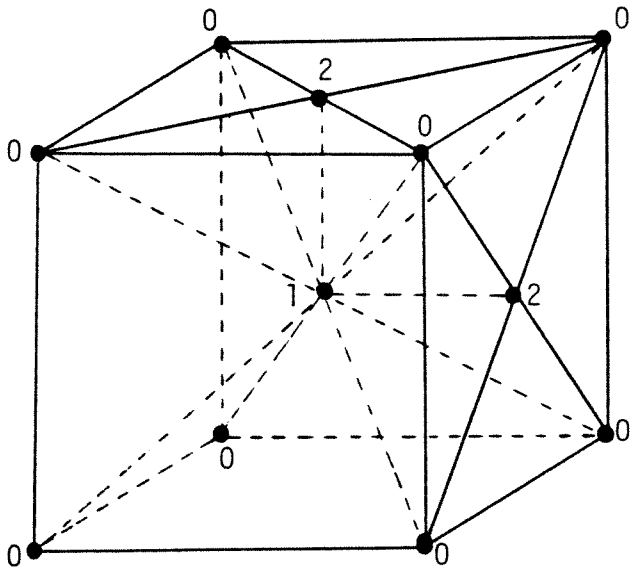
P' =



This idea generalizes to the k-simplicial case. If P is k-simplicial, its non-simplicial faces can be subdivided to yield a simplicial polytope P'. Give the vertices of P the label 0. First subdivide <u>all</u> facets (simplices or not) using vertices labeled 1. Continue to subdivide <u>all</u> faces of a given dimension d-i, $1 \leq i \leq d-k-1$, proceeding in order of decreasing dimension. The new vertices introduced in the subdivision of faces of dimension d-i are labeled i. The polytope P' then has type $(a_0,a_1,\ldots,a_{d-k-1})$ where $a_0 = k+1$ and $a_i = 1$ for $1 \leq i \leq d-k-1$. The label on each vertex conveniently keeps track of the stage in the subdivisions at which the vertex appeared in the polytope.

<u>Example</u>

Consider the 3-cube as a 1-simplicial facet of a 4-cube. The illustration shows the polytope obtained after the subdivision of the 3-face and two of the 2-faces. Note that the facets emerging are type (2,1,1).



Since every d-polytope is 0-simplicial, if all faces of dimension 1 or more in P are subdivided as described above, the resulting simplicial polytope is completely balanced. In this way we can achieve a simplicial polytope P' which is balanced of any type $a = (a_1,\ldots,a_r)$, $r \leq d$, where a_i is a positive integer and $\sum_{i=1}^{r} a_i = d$, just by identifying labels.

The question of whether certain complexes can be completely balanced may be couched in the language of graph colorability. A simplicial complex is n-colorable if each vertex can be labeled with some integer i, $1 \leq i \leq n$, so that the vertices of any simplex have distinct labels. Thus if Δ is a pure simplicial d-complex, saying Δ is (d+1)-colorable is the same as saying Δ is completely balanced.

It is well known (see [2], for example) that a graph which can be embedded in the 2-sphere S^2 is 3-colorable if and only if it is a subcomplex of the 1-skeleton of a triangulation of S^2 in which every vertex has even degree. Goodman and Onishi [10] prove this fact and also its analog of one dimension higher: a triangulation T of a simply-connected closed region in S^3 is 4-colorable if and only if every interior edge in T has an even number of vertices in its link. R.D. Edwards also proved this fact independently, based on the following observation of P. Deligne, R. MacPherson, and J. Morgan [8]: a closed, simply-connected PL triangulated n-manifold is (n+1)-colorable if and only if each (n-2)-simplex is a face of an even number of (n-1)-simplices.

Thus a criterion is provided for when a pure simplicial d-complex which is embedded in S^d can be completely balanced, in terms of

whether or not each interior (d-2)-face has an even number of vertices in its link. There are, no doubt, other questions which can be posed relating (not necessarily completely) balanced complexes to the colorability of graphs subject to certain constraints.

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