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CHARACTERIZATION OF LINEAR AND HARMONIZABLE FRACTIONAL STABLE MOTIONS[1]

by

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CHARACTERIZATION OF LINEAR AND HARMONIZABLE FRACTIONAL STABLE MOTIONS[1)]

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ABSTRACT

We characterize the linear and harmonizable fractional stable motions as the self-similar stable processes with stationary increments whose left-equivalent (or right-equivalent) stationary processes are moving averages and harmonizable respectively.

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1. Introduction

A stochastic process $X = (X(t))_{t\in\mathbb{R}}$ is self-similar with parameter $H > 0$ (H-ss) if $X(c\cdot) \overset{d}{=} c^H X(\cdot)$ for all $c > 0$, and has stationary increments (si) if $X(\cdot + b) - X(b) \overset{d}{=} X(\cdot)$ for all $b \in \mathbb{R}$, where $\overset{d}{=}$ means the equality of all finite-dimensional distributions. Every H-ss si process is stochastically continuous [V1]. A real-valued stochastic process X is α-stable if all finite-dimensional distributions are α-stable where $0 < \alpha \leq 2$, and when $\alpha = 2$, it is Gaussian.

For Gaussian H-ss si processes, $H \leq 1$ and the only H-ss si Gaussian process with $H < 1$ is the fractional Brownian motion (see [MV]). On the other hand, when $0 < \alpha < 2$, many H-ss si α-stable processes have been recognized in the past ten years (see [KM]). Among these, the most important examples of H-ss si α-stable processes for $0 < \alpha < 2$ are the linear fractional stable motion whose properties have been well studied (see, e.g., [CM], [ST], [ALT]), and the harmonizable fractional stable motion (see [CM]).

As discussed in our previous paper [CM], to any stochastically continuous si α-stable process $X = (X(t))_{t\in\mathbb{R}}$ with $1 < \alpha \leq 2$ there correspond two stationary α-stable processes $Y_L = (Y_L(t))_{t\in\mathbb{R}}$ and $Y_R = (Y_R(t))_{t\in\mathbb{R}}$ in the following way:

$$(1.1) \qquad Y_L(t) = \int_{-\infty}^{0} e^{u}[X(t)-X(t+u)]\,du = X(t) - \int_{-\infty}^{t} e^{u-t}\,X(u)\,du, \quad t \in \mathbb{R},$$

$$(1.2) \qquad Y_R(t) = \int_{0}^{\infty} e^{-u}[X(t)-X(t+u)]\,du = X(t) - \int_{t}^{\infty} e^{t-u}\,X(u)\,du, \quad t \in \mathbb{R}.$$

A straightforward calculation gives for all $s < t$,

$$(1.3) \qquad X(t) - X(s) = Y_L(t) - Y_L(s) + \int_s^t Y_L(u)\,du,$$

$$(1.4) \qquad X(t) - X(s) = Y_R(t) - Y_R(s) - \int_s^t Y_R(u)\,du.$$

((1.3) is given in (2.2) on p. 307 in [CM], where the minus sign before the

integral should be replaced by a plus sign as in (1.3)). (1.4) is derived from (1.2) in the same way as (1.3) is derived from (1.1) in [CM].

The existence of the integrals in (1.1) and (1.2), and the derivation of (1.3) and (1.4) from (1.1) and (1.2), respectively, are justified only when $1 < \alpha \leq 2$. When $0 < \alpha < 1$, (1.1) and (1.2) do not always make sense at least when X is the linear fractional stable motion, because of lack of integrability (cf. [S]). Therefore, whenever Y_L or Y_R is considered, we assume $1 < \alpha \leq 2$. The correspondence between the increments of X and Y_L is one-to-one and so is the correspondence between the increments of X and Y_R. Furthermore, it follows from (1.1)-(1.4) that either one of X and Y_L (resp. Y_R) is expressed linearly in terms of past (resp. future) values of the other, so they are left- (resp. right-) equivalent in the sense that if $\mathcal{L}_L(\Delta X,t)$ (resp. $\mathcal{L}_R(\Delta X,t)$) is the closure in probability of all finite linear combinations of the increments of X to the left (resp. right) of t and $\mathcal{L}_L(Y_L,t)$ (resp. $\mathcal{L}_R(Y_R,t)$) is the closure in probability of all finite linear combinations of the values of Y_L (resp. Y_R) to the left (resp. right) of t, then for all $t \in \mathbb{R}$,

$$\mathcal{L}_L(\Delta X,t) = \mathcal{L}_L(Y_L,t), \quad \mathcal{L}_R(\Delta X,t) = \mathcal{L}_R(Y_R,t).$$

We will call Y_L and Y_R the left-equivalent and right-equivalent stationary process of the si process X.

The purpose of this paper is to give characterizations of the linear and the harmonizable fractional stable motions in terms of their left- or right-equivalent stationary processes.

2. The linear fractional stable motion

One of the most important classes of stationary stable processes consists of moving averages:

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\, dM_\alpha(u), \quad t \in \mathbb{R}, \tag{2.1}$$

where $0 < \alpha \leq 2$, $g \in L^{\alpha}(\mathbb{R})$, ($g \log|g| \in L^{1}(\mathbb{R})$ if $\alpha = 1$ and $\beta \neq 0$ in (2.2) below), M_{α} is α-stable motion (Brownian motion when $\alpha = 2$), i.e., has stationary independent α-stable increments with

$$E[\exp\{i\theta[M_{\alpha}(t)-M_{\alpha}(s)]\}] = \begin{cases} \exp\{-|t-s|\theta^2/2\}, & \alpha = 2, \\ \exp\{-|t-s||\theta|^{\alpha}[1 - i\beta\ \mathrm{sgn}(\theta)\ \tan(\pi\alpha/2)]\}, & 0 < \alpha < 2, \ \alpha \neq 1, \\ \exp\{-|t-s||\theta|[1 + i\beta(2/\pi)\mathrm{sgn}(\theta)\ell n|\theta|]\}, & \alpha = 1, \end{cases} \tag{2.2}$$

where $|\beta| \leq 1$. We note that for each $\alpha \in (0,2]$, $\alpha \neq 1$ (or $\alpha = 1$, $\beta = 0$) M_{α} is $1/\alpha$-ss.

We first note that the left- and right-equivalent stationary processes of an si stable process can be moving averages only simultaneously.

<u>Theorem 1.</u> Let $1 < \alpha < 2$ and X be an si α-stable process. The left-equivalent stationary process Y_L of X is a moving average if and only if the right-equivalent stationary process Y_R of X is a moving average.

The linear fractional stable motion $(\Delta_{\alpha,H}(a,b;t))_{t\in\mathbb{R}}$ is an H-ss si α-stable process defined for $0 < H < 1$, $0 < \alpha \leq 2$, $\alpha \neq 1$, (or $\alpha = 1$, $\beta = 0$), $H \neq 1/\alpha$, $a,b \in \mathbb{R}$, by

$$\Delta_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \{a[(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}] + b[(t-u)_-^{H-1/\alpha} - (-u)_-^{H-1/\alpha}]\}\ dM_{\alpha}(u), \tag{2.3}$$

with the convention $0^{\gamma} = 0$ even for $\gamma < 0$. When $\alpha = 2$, the processes $\Delta_{2,H}(a,b;\cdot)$ are multiples of the fractional Brownian motion for all a and b. However, when $0 < \alpha < 2$, different lines through the origin of the (a,b) plane give different processes ([CM], [ST]).

We now assume $1 < \alpha < 2$. As shown in [CM], the left-equivalent stationary α-stable process $Y_L(a,b;\cdot)$ of the linear fractional stable motion $\Delta_{\alpha,H}(a,b;\cdot)$ is

a moving average

$$Y_L(a,b;t) = \int_{-\infty}^{\infty} g_{a,b}(t-u)\, dM_\alpha(u), \quad t \in \mathbb{R}, \tag{2.4}$$

where

$$\begin{aligned} g_{a,b}(x) &= a\left[x_+^{H-1/\alpha} - e^{-x} \int_0^{x_+} e^v\, v^{H-1/\alpha}\, dv \right] + b\left[x_-^{H-1/\alpha} - e^{-x} \int_{x_-}^{\infty} e^{-v}\, v^{H-1/\alpha}\, dv \right] \\ &=: a\, g_1(x) + b\, g_2(x). \end{aligned} \tag{2.5}$$

Likewise, the right equivalent stationary α-stable process Y_R of the linear fractional stable motion $\Delta_{\alpha,H}$ is a moving average

$$Y_R(a,b;t) = \int_{-\infty}^{\infty} h_{a,b}(t-u)\, dM_\alpha(u), \quad t \in \mathbb{R} \tag{2.6}$$

where

$$\begin{aligned} h_{a,b}(x) &= a\left[x_+^{H-1/\alpha} - e^{x} \int_{x_+}^{\infty} e^{-v}\, v^{H-1/\alpha}\, dv \right] + b\left[x_-^{H-1/\alpha} - e^{x} \int_0^{x_-} e^{v}\, v^{H-1/\alpha}\, dv \right] \\ &=: a\, h_1(x) + b\, h_2(x). \end{aligned} \tag{2.7}$$

One of the main purposes of this paper is to prove the converse of the above observation characterizing the linear fractional stable motion as follows.

<u>Theorem 2</u>. Fix $1 < \alpha < 2$, $0 < H < 1$, $H \neq 1/\alpha$. Let X be a nondegenerate H-ss si α-stable process. If its left- (or equivalently right-) equivalent stationary process is moving average, then

$$X(\cdot) \overset{d}{=} \Delta_{\alpha,H}(a,b;\cdot) \quad \text{for some } a,\ b \in \mathbb{R}.$$

When $g(x) = 0$ for $x < 0$ in (2.1), Y is a nonanticipating moving average process as it is expressed in terms of the past increments of the stable motion M_α: $Y(t) = \int_{-\infty}^{t} g(t-u)\, dM_\alpha(u)$; and when $g(x) = 0$ for $x > 0$, Y is a fully anticipating moving average as it is expressed in terms of the future increments of the stable motion M_α: $Y(t) = \int_t^{\infty} g(t-u)\, dM_\alpha(u)$. When $Y_L(a,b;\cdot)$ is a nonanticipating moving average of M_α, namely $g_{\alpha,b}(x) = 0$ for $x < 0$ in (2.5), then

$b = 0$, since for $x < 0$, $g_1(x) = 0$ but $g_2(x) \neq 0$. The linear fractional stable motion with $b = 0$, $\Delta_{\alpha,H}(a,0;\cdot)$, is the one introduced in [TW] and is called the left linear fractional stable motion. Hence, given a linear fractional stable motion, if its left-equivalent stationary process Y_L is a nonanticipating moving average, then it is a left linear fractional stable motion. When $Y_R(a,b;\cdot)$ is a fully anticipating moving average of M_α, namely $h_{a,b}(x) = 0$ for $x > 0$ in (2.7), then $a = 0$, since for $x > 0$, $h_2(x) = 0$ but $h_1(x) \neq 0$. Thus, given a linear fractional stable motion, if its right-equivalent stationary process Y_R is a fully anticipating moving average, then it is a right linear fractional stable motion. We generalize this characterization to general H-ss si α-stable process as follows.

<u>Theorem 3.</u> Fix $1 < \alpha < 2$, $0 < H < 1$, $H \neq 1/\alpha$. Let X be a nondegenerate H-ss si α-stable process.

(i) If its left-equivalent stationary process Y_L is a nonanticipating moving average, then

$$X(\cdot) \stackrel{d}{=} \Delta_{\alpha,H}(a,0;\cdot) \quad \text{for some } a \neq 0.$$

(ii) If its right-equivalent stationary process Y_R is a fully anticipating moving average, then

$$X(\cdot) \stackrel{d}{=} \Delta_{\alpha,H}(0,b;\cdot) \quad \text{for some } b \neq 0.$$

It is seen from (2.4) - (2.7) that the left-equivalent stationary process Y_L of a nondegenerate linear fractional stable motion cannot be a fully anticipating moving average, because if $g_{a,b}(x) = 0$ for $x > 0$, then $a = b = 0$ and $g_{a,b} \equiv 0$; and, likewise, the right-equivalent stationary process Y_R cannot be a nonanticipating moving average.

A related characterization of the linear fractional stable motions would be as the only H-ss si α-stable processes X of the form

$$X(t) = \int_{-\infty}^{\infty} [G(t-u) - G(-u)]\, dM_\alpha(u), \quad t \in \mathbb{R}, \tag{2.8}$$

where $G(t - \cdot) - G(-\cdot) \in L^{\alpha}(\mathbb{R})$ for each $t \in \mathbb{R}$. Expressed in this way, this characterization does not involve the corresponding stationary process Y, and hence is stated for all $0 < \alpha < 2$. Vervaat [V2] derives such a characterization under a self-similarity assumption on the kernel of (2.8) of the following form: for all $t, u \in \mathbb{R}$, $c > 0$,

$$G(c(t-u)) - G(-cu) = c^{\beta}[G(t-u) - G(-u)] \tag{2.9}$$

and $G(0) = 0$. Indeed, (2.9) implies immediately that $G(x) = a\, x_+^{\beta} + b\, x_-^{\beta}$. This self-similarity of the kernel G also clearly implies the self-similarity of the process X. Theorem 2 (or more precisely, its proof in Section 4) establishes this characterization when the self-similarity of G is replaced by that of X. We state this characterization in the following

<u>Theorem 4.</u> Fix $0 < \alpha < 2$, $0 < H < 1$, $H \neq 1/\alpha$. If X is a nondegenerate H-ss si α-stable process of the form (2.8), then X is a linear fractional stable motion.

3. <u>The harmonizable fractional stable motion</u>

Another important class of stationary complex valued stable processes consists of harmonizable stable processes:

$$Y(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda), \quad t \in \mathbb{R}, \tag{3.1}$$

where Z has complex, independent, rotationally invariant α-stable increments and finite control measure μ, i.e.

$$E[\exp\{i\, \mathrm{Re}(\bar{z} \int f dZ)\}] = \exp\{-|z|^{\alpha} \int |f|^{\alpha} d\mu\}$$

for all complex numbers z and functions f in $L^{\alpha}(\mu)$. The measure μ is called the spectral measure of the harmonizable process Y.

We first note that (as for moving averages in Theorem 1) the left- and right-equivalent processes of an si stable process can be harmonizable only simultaneously.

<u>Theorem 5.</u> Let $1 < \alpha < 2$ and X be an si stable process. The left-equivalent stationary process Y_L of X is harmonizable if and only if the right-equivalent stationary process Y_L of X is harmonizable.

The harmonizable fractional stable motion $(\theta_{\alpha,H}(a,b;t))_{t\in\mathbb{R}}$ is an H-ss si complex valued α-stable process defined for $0 < H < 1$, $0 < \alpha \leq 2$, $a,b \in \mathbb{R}$, by

$$\theta_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \frac{e^{it\lambda}-1}{i\lambda} \left[a\, \lambda_+^{1-H-1/\alpha} + b\, \lambda_-^{1-H-1/\alpha}\right] d\tilde{M}_\alpha(\lambda),$$

where $\tilde{M}_\alpha$ is a complex α-stable motion, i.e. has complex, independent, stationary, rotationally invariant increments (and Lebesgue control measure). When $\alpha = 2$, the processes $\theta_{2,H}(a,b;\cdot)$ are multiples of the fractional Brownian motion for all a and b. However, when $1 < \alpha < 2$ different rays from the origin of the (a,b) plane give different processes [CM]. Assume $1 < \alpha < 2$. As shown in [CM], the left- and right-equivalent stationary α-stable processes $Y_L(a,b;\cdot)$ and $Y_R(a,b;\cdot)$ of the harmonizable fractional stable motion $\theta_{\alpha,H}(a,b;\cdot)$ are harmonizable stable process

$$Y_L(a,b;t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ_L(\lambda), \qquad Y_R(a,b;t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ_R(\lambda),$$

where

$$dZ_L(\lambda) = \frac{1}{-i\lambda-1}\,(a\, \lambda_+^{1-H-1/\alpha} + b\, \lambda_-^{1-H-1/\alpha})\, d\tilde{M}_\alpha(\lambda),$$

$$dZ_R(\lambda) = \frac{1}{i\lambda-1}\,(a\, \lambda_+^{1-H-1/\alpha} + b\, \lambda_-^{1-H-1/\alpha})\, d\tilde{M}_\alpha(\lambda).$$

We now characterize the harmonizable fractional stable motion.

<u>Theorem 6.</u> Fix $1 < \alpha < 2$, $0 < H < 1$. Let X be a nondegenerate H-ss si complex-valued α-stable process. If its left- (or equivalently right-) equivalent stationary process is harmonizable, then

$$X(\cdot) \stackrel{d}{=} \theta_{\alpha,H}(a,b;\cdot) \quad \text{for some } a,b \in \mathbb{R}.$$

One-sided harmonizable fractional stable motions can be characterized in the following way.

<u>Theorem 7.</u> Fix $1 < \alpha < 2$, $0 < H < 1$. Let X be a nondegenerate H-ss si complex valued α-stable process.

(i) If its left- or right-equivalent stationary process Y_L or Y_R is harmonizable with spectral measure vanishing on some interval of positive numbers then

$$X(\cdot) \stackrel{d}{=} \theta_{\alpha,H}(0,b;\cdot) = b \int_{-\infty}^{0} \frac{e^{i\cdot\lambda}-1}{i\lambda} |\lambda|^{1-H-1/\alpha} \, d\tilde{M}_\alpha(\lambda) \quad \text{for some } b \neq 0.$$

(ii) If its left- or right-equivalent stationary process Y_L or Y_R is harmonizable with spectral measure vanishing on some interval of negative numbers then

$$X(\cdot) \stackrel{d}{=} \theta_{\alpha,H}(a,0;\cdot) = a \int_{0}^{\infty} \frac{e^{i\cdot\lambda}-1}{i\lambda} \lambda^{1-H-1/\alpha} \, d\tilde{M}_\alpha(\lambda) \quad \text{for some } a \neq 0.$$

A real harmonizable stable process is of the form $\mathcal{R}e Y(t)$, where $Y(t)$ is as in (3.1). It is shown in [CM] that the real harmonizable fractional stable processes $\mathcal{R}e\ \theta_{\alpha,H}(a,b;\cdot)$ are all multiples of each other in distribution. The following characterization of the real harmonizable fractional stable process $\mathcal{R}e\ \theta_{\alpha,H}(1,1;\cdot)$ follows from Theorem 6.

<u>Corollary</u>. Fix $1 < \alpha < 2$, $0 < H < 1$. Let X be a nondegenerate H-ss si α-stable process. If its left- (or equivalently right-) equivalent stationary process is real harmonizable, then for some constant c,

$$X(\cdot) \stackrel{d}{=} c\ \mathcal{R}e\ \theta_{\alpha,H}(1,1;\cdot).$$

4. Proofs for linear fractional stable motion (Theorems 1, 2 and 3)

In what follows, we write M for M_α for notational simplicity.

Proof of Theorem 1. Suppose Y_L is a moving average:

$$Y_L(t) = \int_{-\infty}^{\infty} g_L(t-u)\, dM(u), \quad t \in \mathbb{R},$$

for some $g_L \in L^\alpha(\mathbb{R})$. Then by (1.3), for $s < t$,

$$X(t) - X(s) = \int_{-\infty}^{\infty} [G_L(t-u) - G_L(s-u)]\, dM(u),$$

where

$$G_L(x) = g_L(x) + \int_0^x g_L(v)dv, \quad x \in \mathbb{R}.$$

Thus by (1.2) for all $t \in \mathbb{R}$,

$$Y_R(t) = \int_0^\infty e^{-u} \left\{ \int_{-\infty}^{\infty} [G_L(t-v) - G_L(t+u-v)]\, dM(v) \right\} du$$

$$= \int_{-\infty}^{\infty} \left\{ \int_0^\infty e^{-u}\, [G_L(t-v) - G_L(t-v+u)]\, du \right\} dM(v) =: \int_{-\infty}^{\infty} g_R(t-v)\, dM(v),$$

where

$$g_R(x) = \int_0^\infty e^{-u}\, [G_L(x) - G_L(x+u)]\, du, \quad x \in \mathbb{R}.$$

Thus, Y_R is also a moving average. Similarly, if Y_R is a moving average, then so is Y_L. □

Remark. A simple calculation yields

$$g_L(x) = g_R(x) - 2\, e^{-x} \int_{-\infty}^{x} e^{v}\, g_R(v)\, dv.$$

Similarly, we have

$$g_R(x) = g_L(x) - 2\, e^{x} \int_{x}^{\infty} e^{-v}\, g_L(v)\, dv.$$

<u>Proof of Theorem 2.</u> From (1.3) and (2.1), we have for s, t $\in \mathbb{R}$,

$$X(t) - X(s) = \int_{-\infty}^{\infty} [g(t-u) - g(s-u) + \int_{s-u}^{t-u} g(v)\, dv]\, dM(u) \tag{4.1}$$

$$= \int_{-\infty}^{\infty} [G(t-u) - G(s-u)]\, dM(u),$$

where

$$G(x) = g(x) + \int_0^x g(v)\, dv, \quad x \in \mathbb{R}.$$

Fix $t \neq 0$. Then we have for any $a_1, \ldots, a_N$, $x_1, \ldots, x_N \in \mathbb{R}$ and $c > 0$,

$$\sum_{n=1}^{N} a_n[X(c(t + x_n)) - X(cx_n)]$$

$$\stackrel{d}{=} \int_{-\infty}^{\infty} \left\{ \sum_{n=1}^{N} a_n\, [G(c(t + x_n)-u) - G(cx_n - u)] \right\} dM(u)$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_{n=1}^{N} a_n\, [G(c(t + x_n - v)) - G(c(x_n - v))] \right\} dM(cv)$$

$$\stackrel{d}{=} c^{1/\alpha} \int_{-\infty}^{\infty} \left\{ \sum_{n=1}^{N} a_n\, [G(c(t + x_n - v)) - G(c(x_n - v))] \right\} dM(v)$$

(where we have used the $1/\alpha$-self-similarity of M),

$$\sum_{n=1}^{N} a_n\, [X(t + x_n) - X(x_n)] \stackrel{d}{=} \int_{-\infty}^{\infty} \left\{ \sum_{n=1}^{N} a_n\, [G(t + x_n - v) - G(x_n - v)] \right\} dM(v).$$

Hence by the self-similarity of X and

$$|E[\exp\{i \int_{-\infty}^{\infty} f(v)\, dM(v)\}]| = \exp\{-\int_{-\infty}^{\infty} |f(v)|^{\alpha}\, dv\},$$

we have for each $c > 0$,

$$c \int_{-\infty}^{\infty} \left| \sum_{n=1}^{N} a_n\, [G(c(t + x_n - v)) - G(c(x_n - v))] \right|^{\alpha} dv$$

$$= c^{\alpha H} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{N} a_n \, [G(t + x_n - v) - G(x_n - v)] \right|^{\alpha} dv,$$

namely, the functions $c^{1/\alpha - H}[G(c(t-\cdot)) - G(c(-\cdot))]$ and $G(t-\cdot) - G(-\cdot)$ have equal L_α-norms of all linear combinations of their translates. Then, by Kanter's result [K], for every $t \neq 0$ and every $c > 0$,

$$(4.2) \quad c^{1/\alpha - H}[G(c(t-u)) - G(-cu)] = \epsilon(t,c) \, [G(t-u-\tau(t,c)) - G(-u-\tau(t,c))]$$

for almost all u, where $\epsilon(t,c) \in \{-1,+1\}$, $\tau(t,c) \in \mathbb{R}$. For simplicity, we write p for $1/\alpha - H$ below.

We first show that for some $\theta \in \mathbb{R}$,

$$(4.3) \qquad \tau(t,c) = \theta(1 - \frac{1}{c}), \; t > 0, \; c > 0,$$

where θ is independent of t and c.

We have by (4.2), for c_1, $c_2 > 0$,

$$(4.4) \quad (c_1c_2)^p \, [G(c_1c_2(t-u)) - G(-c_1c_2u)] = c_1^p \, c_2^p \, [G(c_2(c_1t - c_1u)) - G(-c_2c_1u)]$$

$$= c_1^p \, \epsilon(c_1t,c_2) \, [G(c_1t - c_1u - \tau(c_1t,c_2)) - G(-c_1u - \tau(c_1t,c_2))]$$

$$= \epsilon(c_1t,c_2) \, c_1^p \, [G(c_1(t-u-c_1^{-1}\tau(c_1t,c_2))) - G(-c_1(u + c_1^{-1}\tau(c_1t,c_2)))]$$

$$= \epsilon(c_1t,c_2) \, \epsilon(t,c_1) \, [G(t-u-c_1^{-1}\tau(c_1t,c_2)-\tau(t,c_1))$$

$$- G(-u-c_1^{-1}\tau(c_1t,c_2)-\tau(t,c_1))]$$

for almost all u. On the other hand, applying (4.2) for $c = c_1c_2$ we have

$$(4.5) \quad (c_1c_2)^p \, [G(c_1c_2(t-u)) - G(-c_1c_2u)]$$

$$= \epsilon(t_1,c_1c_2) \, [G(t-u-\tau(t,c_1c_2)) - G(-u-\tau(t,c_1c_2))]$$

for almost all u.

We now claim that (4.4) and (4.5) imply that for every $t > 0$ and c_1, $c_2 > 0$,

(4.6) $$\tau(t,c_1c_2) = \tau(t,c_1) + c_1^{-1}\,\tau(c_1t,c_2),$$

(4.7) $$\epsilon(t,c_1c_2) = \epsilon(c_1t,c_2)\,\epsilon(t,c_1).$$

To see this we observe that, if we put

$$H_t(u) = G(t-u) - G(-u), \quad u \in \mathbb{R},$$

and if for some $t > 0$,

(4.8) $$\epsilon\, H_t(u + \delta) = \epsilon'\, H_t(u + \delta') \quad \text{a.a. } u,$$

where ϵ, $\epsilon' \in \{-1,1\}$ and δ, $\delta' \in \mathbb{R}$, then

(4.9) $$\delta = \delta' \quad \text{and} \quad \epsilon = \epsilon'.$$

Indeed, (4.8) is equivalent to

(4.10) $$\frac{\epsilon}{\epsilon'}\, H_t(u+\delta-\delta') = H_t(u) \quad \text{a.a. } u.$$

Since $\int_{-\infty}^{\infty} |H_t(u)|^{\alpha}\, du < \infty$, if $\delta - \delta' \neq 0$, then (4.10) would imply that $H_t(u) = 0$ for almost all u. But, then

$$X(t) = \int_{-\infty}^{\infty} [G(t-u) - G(-u)]\, dM(u) = 0.$$

Note that for a nondegenerate ss process, $P\{X(t) = 0\} < 1$ for every $t > 0$. Thus the obtained contradiction proves $\delta-\delta' = 0$; and (4.10) implies $\epsilon/\epsilon' = 1$. Now (4.4) and (4.5) imply

$$\epsilon(t,c_1c_2)\, H_t(u + \tau(t,c_1c_2))$$

$$= \epsilon(c_1t,c_2)\,\epsilon(t,c_1)\, H_t(u + c_1^{-1}\tau(c_1t,c_2) + \tau(t,c_1)) \quad \text{a.a. } u.,$$

and by (4.8) and (4.9), we obtain (4.6) and (4.7).

Now, let m be a positive integer. We have from (4.2),

$$c^p\,[G(cm-cu) - G(-cu)] = \Sigma_{k=1}^{m}\, c^p\,[G(c\cdot 1-c(u-(m-k))) - G(-c(u-(m-k)))]$$

(4.11) $$= \Sigma_{k=1}^{m}\, \epsilon(1,c)\,[G(m-k-u+1-\tau(1,c)) - G(m-k-u-\tau(1,c))]$$

$$= \epsilon(1,c)\ [G(m-u-\tau(1,c)) - G(-u-\tau(1,c))]$$

for almost all u. Comparing (4.11) with (4.2) for t=m, we obtain for a.a. u,

$$\epsilon(1,c)\ [G(m-u-\tau(1,c)) - G(-u-\tau(1,c))] = \epsilon(m,c)\ [G(m-u-\tau(m,c)) - G(-u-\tau(m,c))].$$

It then follows from (4.8) and (4.9) that

$$\tau(m,c) = \tau(1,c), \quad \epsilon(m,c) = \epsilon(1,c), \quad \text{for every } c > 0, \quad m = 1,2,\ldots .$$

Likewise, if we start with $\tau(t,c)$ instead of $\tau(1,c)$, then the same procedure will give

$$\tau(mt,c) = \tau(t,c), \quad \epsilon(mt,c) = \epsilon(t,c), \quad \text{for all } t > 0,\ c > 0,\ m = 1,2,\ldots .$$

It follows that for any m,n = 1,2,...,

$$\tau(\tfrac{m}{n}t,c) = \tau((\tfrac{m}{n}t)n,c) = \tau(mt,c) = \tau(t,c).$$

Thus

$$\tau(qt,c) = \tau(t,c) \tag{4.12}$$

for every $t > 0$, $c > 0$, and rational $q > 0$, and similarly

$$\epsilon(qt,c) = \epsilon(t,c). \tag{4.13}$$

Comparing (4.12) with (4.6), we conclude that

$$\tau(t,qc) = \tau(t,q) + \frac{1}{q}\tau(t,c) \tag{4.14}$$

for every $t > 0$, $c > 0$, rational $q > 0$. Now letting $c > 0$ be rational as well, we obtain from (4.6) and (4.12)

$$\tau(t,qc) = \tau(t,c) + \frac{1}{c}\tau(t,q), \tag{4.15}$$

and so by (4.14) and (4.15),

$$\tau(t,q) + \frac{1}{q}\tau(t,c) = \tau(t,c) + \frac{1}{c}\tau(t,q)$$

for every $t > 0$ and rationals q, $c > 0$, and thus

$$\tau(t,q)\ (1 - \tfrac{1}{c}) = \tau(t,c)(1 - \tfrac{1}{q}).$$

It follows that for $t > 0$, $\tau(t,1) = 0$ and for rationals $0 < q$, $c \neq 1$,

$$\tau(t,q)/(1 - \frac{1}{q}) = \tau(t,c)/(1 - \frac{1}{c}),$$

implying that for some $a(t) \in \mathbb{R}$,

$$\tau(t,q) = a(t)(1 - \frac{1}{q}), \quad t > 0, \tag{4.16}$$

for all rationals $q > 0$.

It follows from (4.12) and (4.16) that

$$a(t) = a(qt) \quad \text{for every } t > 0 \text{ and rational } q > 0. \tag{4.17}$$

The next step is to prove that $\tau(t,c)$ is continuous in $t \neq 0$. Set

$$J(y) = \int_0^y G(-u)\, du, \quad y \in \mathbb{R}.$$

Note that $J(\cdot)$ is continuous. We have

$$\int_0^y c^p\, [G(c(t-u)) - G(-cu)]\, du = c^{p-1} \left[\int_{-ct}^{c(y-t)} G(-v)\, dv - \int_0^{cy} G(-v)\, dv \right]$$

$$= c^{p-1}[J(c(y-t)) - J(-ct) - J(cy)]. \tag{4.18}$$

Also

$$\int_0^y [G(t-u-\tau(t,c)) - G(-u-\tau(t,c))]\, du \tag{4.19}$$

$$= \int_{-t+\tau(t,c)}^{y-t+\tau(t,c)} G(-v)\, dv - \int_{\tau(t,c)}^{y+\tau(t,c)} G(-v)\, dv$$

$$= J(y-t+\tau(t,c)) - J(-t+\tau(t,c)) - J(y+\tau(t,c)) + J(\tau(t,c)).$$

Thus, by (4.2), (4.18) and (4.19),

$$c^{p-1}[J(c(y-t)) - J(-ct) - J(cy)] \tag{4.20}$$

$$= \epsilon(t,c)\, [J(y-t+\tau(t,c)) - J(-t+\tau(t,c)) - J(y+\tau(t,c)) + J(\tau(t,c))]$$

for all $y \in \mathbb{R}$, $t > 0$, $c > 0$.

Keep now $c > 0$ fixed and let $\tilde{\tau}(t) := \tau(t,c)$. Let $t_0 > 0$ and suppose that $\tilde{\tau}(t_n)$ does not converge to $\tilde{\tau}(t_0)$ for some $\{t_n\}$ tending to t_0. Then there is a number $\ell(t_0) \in [-\infty,\infty]$ such that $\ell(t_0) \neq \tilde{\tau}(t_0)$ and $\tilde{\tau}(t_{n_k}) \to \ell(t_0)$ as $k \to \infty$ along some subsequence $\{n_k\}$.

We now show that for every $y_1, y_2 \in \mathbb{R}$,

$$\lim_{x\to \pm\infty} [J(y_1+x) - J(y_1-y_2+x) - J(y_2+x) + J(x)] = 0. \tag{4.21}$$

Take for example $0 < y_2 < y_1$. Then for $x \to \infty$,

$$|J(y_1+x) - J(y_1-y_2+x) - J(y_2+x) + J(x)| = \left|\int_{y_1-y_2+x}^{y_1+x} G(-u)\,du - \int_x^{y_2+x} G(-u)\,du\right|$$

$$= \left|\int_x^{y_2+x} [G(-u+y_2-y_1) - G(-u)]\,du\right|$$

$$\leq y_2^{1-1/\alpha} \left[\int_x^{y_2+x} |G(-u+y_2-y_1) - G(-u)|^\alpha\,du\right]^{1/\alpha}$$

$$\to 0 \quad \text{as} \quad x \to \infty$$

by the integrability of $|H_t(u)|^\alpha$. The case $x \to -\infty$ is similar and (4.21) is thus proved.

Suppose now that $\ell(t_0) = \pm\infty$. It follows from (4.20) and (4.21) that

$$c^{p-1}[J(c(y-t_0)) - J(-ct_0) - J(cy)] = 0 \quad \text{for any } y \in \mathbb{R}. \tag{4.22}$$

Since J is absolutely continuous and $J'(y) = G(-y)$ for a.a. y, we conclude from (4.22) that

$$G(ct_0-y) - G(-y) = 0 \quad \text{for a.a. } y,$$

which implies $X(ct_0) = 0$, contradicting as before $P\{X(ct_0) = 0\} < 1$.

Let us now consider the case $\ell(t_0) \in (-\infty,\infty)$. It follows from (4.20) that

$$\limsup_{k\to\infty} \epsilon(t_{n_k},c)\,[J(y-t_0+\ell(t_0)) - J(-t_0+\ell(t_0)) - J(y+\ell(t_0)) + J(\ell(t_0))]$$

$$= \epsilon(t_0,c)\ [J(y-t_0+\tilde{\tau}(t_0)) - J(-t_0+\tilde{\tau}(t_0)) - J(y+\tilde{\tau}(t_0)) + J(\tilde{\tau}(t_0))]$$

and thus

$$G(t_0-y-\ell(t_0)) - G(-y-\ell(t_0)) = \pm\ \epsilon(t_0,c)\ [G(t_0-y-\tilde{\tau}(t_0)) - G(-y-\tilde{\tau}(t_0))]$$

for almost all $y \in \mathbb{R}$, which is again impossible if $\ell(t_0) \neq \tilde{\tau}(t_0)$, because of (4.8)-(4.9).

This proves that $\tau(t,c)$ is continuous in $t > 0$. It follows now from (4.16) that $a(t)$ is continuous in $t > 0$ so that by (4.17), $a(t) = \theta$. Thus, from (4.16), we conclude (4.3).

Since $J(\cdot)$ is continuous and $\tau(t,c)$ is continuous in $t > 0$, (4.20) implies that $\epsilon(t,c)$ is also continuous in $t > 0$, which together with (4.13) and $\epsilon(t,c) \in \{-1,+1\}$ means that for any $t > 0$,

$$\epsilon(t,c) \equiv +1 \quad \text{or} \quad \epsilon(t,c) \equiv -1. \tag{4.23}$$

Comparing (4.7) and (4.23), we obtain

$$\epsilon(t,c) \equiv 1 \quad \text{for all } t > 0, \quad c > 0.$$

Therefore we conclude from (4.2) that for every $t > 0$ and $c > 0$,

$$c^p\ [G(c(t-u)) - G(-u)] = G(t-u-\ \theta(1-\tfrac{1}{c})) - G(-u-\ \theta(1-\tfrac{1}{c}))$$

for almost all $u \in \mathbb{R}$, so that for any $t > 0$, $c > 0$,

$$c^p\ G(c(t-u)) - G(t-u-\ \theta(1-\tfrac{1}{c})) = c^p\ G(-cu) - G(-u-\ \theta(1-\tfrac{1}{c})) \tag{4.24}$$

for almost all $u \in \mathbb{R}$. By Fubini's theorem, for any $c > 0$ and for almost all $u \in \mathbb{R}$, (4.24) holds for almost all $t > 0$. Therefore, the right hand side of (4.24) is independent of u (almost everywhere). Thus, there exists a function $k(c)$ such that

$$c^p\ G(cu) - G(u-\theta(1-\tfrac{1}{c})) = k(c), \quad \text{a.a. } u. \tag{4.25}$$

Then for any c_1, $c_2 > 0$, for almost all $u \in \mathbb{R}$,

$$k(c_1c_2) = (c_1c_2)^p\, G(c_1c_2u) - G(u-\theta(1-\frac{1}{c_1c_2}))$$

$$= c_1^p\, [c_2^p\, G(c_1c_2u)] - G(u-\theta(1-\frac{1}{c_1c_2}))$$

$$= c_1^p\, [k(c_2) + G(c_1u-\theta(1-\frac{1}{c_2}))] - G(u-\theta(1-\frac{1}{c_1c_2}))$$

$$= c_1^p\, k(c_2) + c_1^p\, G(c_1(u-\frac{\theta}{c_1}(1-\frac{1}{c_2}))) - G(u-\frac{\theta}{c_1}(1-\frac{1}{c_2})-\theta(1-\frac{1}{c_1}))$$

$$= c_1^p\, k(c_2) + k(c_1).$$

Thus for every $c_1, c_2 > 0$,

$$k(c_1c_2) = c_1^p\, k(c_2) + k(c_1),$$

and likewise

$$k(c_1c_2) = c_2^p\, k(c_1) + k(c_2).$$

Hence $k(1) = 0$ and for $0 < c_1, c_2 \neq 1$ we have

$$\frac{k(c_1)}{1-c_1^p} = \frac{k(c_2)}{1-c_2^p},$$

namely for some $\gamma \in \mathbb{R}$,

(4.26) $$k(c) = \gamma(1 - c^p), \quad c > 0.$$

Then, from (4.25) and (4.26) we have that for any $c > 0$,

(4.27) $$c^p\, G(cu) - G(u-\theta(1-\frac{1}{c})) = \gamma(1 - c^p), \quad \text{a.a. } u.$$

Let

(4.28) $$\tilde{G}(u) := G(u) + \gamma.$$

Then (4.27) is equivalent to

$$c^p\, \tilde{G}(cu) = \tilde{G}(u-\theta(1-\frac{1}{c})), \quad \text{a.a. } u.$$

and putting $y = u+\theta_+/c$, we obtain

(4.29) $$c^p\, \tilde{G}(cy-\theta) = \tilde{G}(y-\theta), \quad \text{a.a. } y.$$

Take $y_1, y_2 > 0$. It follows from (4.29) that

$$\tilde{G}(y_1y_2-\theta) = c^p\,\tilde{G}(cy_1y_2-\theta) = (cy_1)^p\,y_1^{-p}\,\tilde{G}((cy_1)y_2-\theta) = y_1^{-p}\,\tilde{G}(y_2-\theta)$$

for almost all pairs $(y_1,y_2) \in (0,\infty) \times (0,\infty)$. Similarly

$$\tilde{G}(y_1y_2-\theta) = y_2^{-p}\,\tilde{G}(y_1-\theta), \quad \text{a.a.} \quad y_1 > 0, \quad y_2 > 0.$$

Consequently,

$$y^p\,\tilde{G}(y-\theta) = \text{const.} = a, \quad \text{say}, \quad \text{a.a. } y > 0,$$

and hence

$$\tilde{G}(y) = a(y+\theta)^{-p}; \quad \text{a.a.} \quad y > -\theta_+. \tag{4.30}$$

Now let $y_1 < 0$, $y_2 > 0$. Then by (4.29)

$$\begin{aligned} \tilde{G}(y_1y_2-\theta) &= c^p\,\tilde{G}(cy_1y_2-\theta) = (-cy_1)^p(-y_1)^{-p}\,\tilde{G}((-cy_1)(-y_2)-\theta) \\ &= (-y_1)^{-p}\,\tilde{G}(-y_2-\theta) \end{aligned} \tag{4.31}$$

for almost all pairs $(y_1,y_2) \in (-\infty,0) \times (0,\infty)$. Similarly

$$\tilde{G}(y_1y_2-\theta) = y_2^{-p}\,\tilde{G}(y_1-\theta) \quad \text{a.a.} \quad y_1 < 0, \quad y_2 > 0. \tag{4.32}$$

Hence by (4.31) and (4.32) we conclude

$$(-y_1)^p\,\tilde{G}(y_1-\theta) = y_2^p\,\tilde{G}(-y_2-\theta) \quad \text{a.a.} \quad y_1 < 0, \quad y_2 > 0.$$

Therefore

$$(-y)^p\,\tilde{G}(y-\theta) = \text{const.} = b, \quad \text{say}, \quad \text{a.a.} \quad y < 0$$

and hence

$$\tilde{G}(y) = b\,(-(y+\theta))^{-p}, \quad \text{a.a.} \quad y < -\theta. \tag{4.33}$$

We can combine (4.30) and (4.33) as

$$\tilde{G}(y) = a(y+\theta)_+^{-p} + b(y+\theta)_-^{-p} \quad \text{a.a.} \quad y \in \mathbb{R}.$$

By (4.28), we obtain

$$G(u) = a(u+\theta)_+^{-p} + b(u+\theta)_-^{p} - \gamma \quad \text{a.a.} \quad u \in \mathbb{R}.$$

Therefore

$$X(t) = \int_{-\infty}^{\infty} [G(t-u) - G(-u)]\, dM(u)$$

$$= \int_{-\infty}^{\infty} \{ a[(t-u+\theta)_+^{-p} - (-u+\theta)_+^{-p}] + b[(t-u+\theta)_-^{-p} - (-u+\theta)_-^{-p}] \}\, dM(u)$$

$$\overset{d}{=} \int_{-\infty}^{\infty} \{a[(t-u)_+^{-p} - (-u)_+^{-p}] + b[(t-u)_-^{-p} - (-u)_-^{-p}]\}\, dM(u),$$

$$= \Lambda_{\alpha,H}(a,b;t),$$

since $dM(u-\theta) \overset{d}{=} dM(u)$, where $\overset{d}{=}$ means equality of all finite-dimensional distributions. This completes the proof of Theorem 2. □

<u>Proof of Theorem 3</u>. Since the proofs of (i) and (ii) are similar, we only prove (i). Since $Y_L(t) = \int_{-\infty}^{\infty} g(t-u)\, dM(u)$ is a nonanticipating moving average, $g(x) = 0$ for $x < 0$, and thus $G(x) = 0$, $x < 0$. Then, in (4.25), $k(c) = 0$, and hence $\gamma = 0$ in (4.26). Thus in (4.28), $\tilde{G}(u) = G(u) = 0$, $u < 0$. Consequently, we conclude that $b = 0$ in (4.33), and $X(\cdot) \overset{d}{=} \Lambda_{\alpha,H}(a,0;\cdot)$. □

5. <u>Proofs for harmonizable fractional stable motion (Theorems 5, 6 and 7)</u>

<u>Proof of Theorem 5</u>. Suppose Y_L is harmonizable: $Y_L(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ_L(\lambda)$, $t \in \mathbb{R}$, where Z_L is as in (3.1). Then by (1.3), for $s < t$,

$$X(t) - X(s) = \int_{-\infty}^{\infty} [e^{it\lambda} - e^{is\lambda} + \int_s^t e^{iu\lambda}\, du]\, dZ_L(\lambda) \tag{5.1}$$

$$= \int_{-\infty}^{\infty} (e^{it\lambda} - e^{is\lambda})(1 + \frac{1}{i\lambda})\, dZ_L(\lambda).$$

Thus by (1.2) for all $t \in \mathbb{R}$,

$$Y_R(t) = -\int_0^\infty e^{-u} \left\{ \int_{-\infty}^\infty (e^{i(t+u)\lambda} - e^{it\lambda})(1 + \frac{1}{i\lambda})\, dZ_L(\lambda) \right\} du$$

$$= \int_{-\infty}^\infty e^{it\lambda} \left\{ \int_0^\infty e^{-u}(1 - e^{iu\lambda})\, du \right\} (1 + \frac{1}{i\lambda})\, dZ_L(\lambda)$$

$$= \int_{-\infty}^\infty e^{it\lambda} \frac{i\lambda+1}{i\lambda-1}\, dZ_L(\lambda) =: \int_{-\infty}^\infty e^{it\lambda}\, dZ_R(\lambda),$$

where

$$dZ_R(\lambda) = \frac{i\lambda+1}{i\lambda-1}\, dZ_L(\lambda), \quad \lambda \in \mathbb{R}.$$

Note that therefore Y_L and Y_R have the same spectral measure. Similarly, if Y_R is harmonizable, then so is Y_L. □

<u>Proof of Theorem 6</u>. Suppose Y_L is harmonizable:

$$Y_L(t) = \int_{-\infty}^\infty e^{it\lambda}\, dZ(\lambda), \quad t \in \mathbb{R},$$

where Z is as in (3.1) with finite spectral measure μ. Recall from (5.1) that for all $s < t$,

$$X(t) - X(s) = \int_{-\infty}^\infty (e^{it\lambda} - e^{its})(1 + \frac{1}{i\lambda})\, dZ(\lambda). \tag{5.2}$$

Fix $t > 0$. Then we have for any $a_0,\ldots,a_N$, $x_0,\ldots,x_N \in \mathbb{R}$ and $c > 0$,

$$\sum_{n=0}^N a_n\, [X(c(t+x_n)) - X(cx_n)] = \int_{-\infty}^\infty \sum_{n=0}^N a_n \left[e^{ic(t+x_n)\lambda} - e^{icx_n\lambda}\right](1 + \frac{1}{i\lambda})\, dZ(\lambda)$$

$$= \int_{-\infty}^\infty (1 + \frac{1}{i\lambda}) \sum_{n=0}^N a_n\, e^{-icx_n\lambda}(e^{ict\lambda} - 1)\, dZ(\lambda)$$

$$= \int_{-\infty}^\infty (1 + \frac{c}{i\lambda}) \sum_{n=0}^N a_n\, e^{ix_n\lambda}(e^{it\lambda} - 1)\, Z(\frac{d\lambda}{c})$$

and

$$c^H \sum_{n=0}^{N} a_n [X(t+x_n) - X(x_n)] = c^H \int_{-\infty}^{\infty} (1 + \frac{1}{i\lambda}) \sum_{n=0}^{N} a_n e^{ix_n\lambda} (e^{it\lambda} - 1) \, dZ(\lambda).$$

Hence by the self-similarity of X, we have for each $c > 0$,

$$\int_{-\infty}^{\infty} |1 + \frac{c}{i\lambda}|^{\alpha} \, |(e^{it\lambda}-1) \sum_{n=0}^{N} a_n e^{ix_n\lambda}|^{\alpha} \, \mu(\frac{d\lambda}{c})$$

$$= c^{H\alpha} \int_{-\infty}^{\infty} |1 + \frac{1}{i\lambda}|^{\alpha} \, |(e^{it\lambda}-1) \sum_{n=0}^{N} a_n e^{ix_n\lambda}|^{\alpha} \, \mu(d\lambda).$$

Take $a_0=1$, $x_0=0$, $N=1$. Then for any $a, x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \left[\frac{\lambda^2+c^2}{\lambda^2}\right]^{a/2} |e^{it\lambda}-1|^{\alpha} \, |1 + a\, e^{ix\lambda}|^{\alpha} \, \mu(\frac{d\lambda}{c}) \tag{5.3}$$

$$= c^{H\alpha} \int_{-\infty}^{\infty} \left[\frac{\lambda^2+1}{\lambda^2}\right]^{\alpha/2} |e^{it\lambda}-1|^{\alpha} \, |1 + a\, e^{ix\lambda}|^{\alpha} \, \mu(d\lambda).$$

Differentiating (5.3) with respect to a and putting a=0, we obtain for any $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{ix\lambda} \left[\frac{\lambda^2+c^2}{\lambda^2}\right]^{\alpha/2} |e^{it\lambda}-1|^{\alpha} \, \mu(\frac{d\lambda}{c}) = c^{H\alpha} \int_{-\infty}^{\infty} e^{ix\lambda} \left[\frac{\lambda^2+1}{\lambda^2}\right]^{\alpha/2} |e^{it\lambda}-1|^{\alpha} \, \mu(d\lambda),$$

which implies

$$\left[\frac{\lambda^2+c^2}{\lambda^2}\right]^{\alpha/2} |e^{it\lambda}-1|^{\alpha} \, \mu(\frac{d\lambda}{c}) = c^{H\alpha} \left[\frac{\lambda^2+1}{\lambda^2}\right]^{\alpha/2} |e^{it\lambda}-1|^{\alpha} \, \mu(d\lambda). \tag{5.5}$$

Therefore

$$(\lambda^2+c^2)^{\alpha/2} \, \mu(\frac{d\lambda}{c}) = c^{H\alpha} \, (\lambda^2+1)^{\alpha/2} \, \mu(d\lambda). \tag{5.6}$$

Integrating (5.6) from 0 to any $a > 0$, we have

$$\int_0^a (\lambda^2+c^2)^{\alpha/2} \, \mu(\frac{d\lambda}{c}) = c^{H\alpha} \int_0^a (\lambda^2+1)^{\alpha/2} \, \mu(d\lambda). \tag{5.7}$$

If we define

$$f(y) := \int_0^y (\lambda^2+1)^{\alpha/2}\, \mu(d\lambda),$$

then the right-hand side of (5.7) is $c^{H\alpha} f(y)$ and the left-hand side is

$$\int_0^y (\lambda^2+c^2)^{\alpha/2}\, \mu(\tfrac{d\lambda}{c}) = \int_0^{y/c} (c^2v^2+c^2)^{\alpha/2}\, \mu(dv) = c^\alpha \int_0^{y/c} (v^2+1)^{\alpha/2}\, \mu(dv) = c^\alpha\, f(\tfrac{y}{c}).$$

We thus have for any $c > 0$, $y > 0$,

$$c^\alpha\, f(\tfrac{y}{c}) = c^{H\alpha}\, f(y).$$

If we put $1/c = x$, $y=1$, we have

$$f(x) = f(1)\, x^{\alpha(1-H)},\ x > 0,$$

namely,

$$\int_0^x (\lambda^2+1)^{\alpha/2}\, \mu(d\lambda) = f(1)\, x^{\alpha(1-H)}.$$

This implies $\mu(d\lambda)$ is absolutely continuous on $(0,\infty)$ with a Radon-Nikodym derivative $\varphi(\lambda)$ such that $\mu(d\lambda) = \varphi(\lambda)d\lambda$, and

$$(\lambda^2+1)^{\alpha/2}\, \varphi(\lambda) = f(1)\, \alpha(1-H)\, \lambda^{\alpha(1-H)-1},$$

namely, for some $a \in \mathbb{R}$,

$$\mu(d\lambda) = \frac{a}{(\lambda^2+1)^{\alpha/2}}\, \lambda^{\alpha(1-H)-1}\, d\lambda, \quad \text{for} \quad \lambda > 0. \tag{5.8}$$

Hence

$$dZ(\lambda) \overset{d}{=} \frac{a}{-i\lambda-1}\, \lambda^{1-H-1/\alpha}\, d\tilde{M}_\alpha(\lambda), \quad \text{for } \lambda > 0.$$

Similarly, we have for some $b \in \mathbb{R}$,

$$dZ(\lambda) \overset{d}{=} \frac{b}{-i\lambda-1}\, |\lambda|^{1-H-1/\alpha}\, d\tilde{M}_\alpha(\lambda), \quad \text{for } \lambda < 0. \tag{5.9}$$

Finally (5.5) implies that any atom of μ at 0 satisfies $c^\alpha|t|^\alpha\mu(\{0\}) = c^{H\alpha}|t|^\alpha\mu(\{0\})$ and thus $\mu(\{0\}) = 0$ (since $t > 0$, $0 < H < 1$).

Therefore

$$dZ(\lambda) \overset{d}{=} \frac{1}{-i\lambda-1}\, (\, a\, \lambda_+^{1-H-1/\alpha} + b\, \lambda_-^{1-H-1/\alpha})\, d\tilde{M}_\alpha(\lambda).$$

Combining this with (5.2), we have

$$X(\cdot) = \int_{-\infty}^{\infty} (e^{i\cdot\lambda}-1)(\frac{i\lambda+1}{i\lambda})\, dZ(\lambda) \overset{d}{=} \theta_{\alpha,H}(a,b;\cdot).$$

This completes the proof of Theorem 6, for Y_L, an its proof for Y_R is similar. □

<u>Proof of Theorem 7</u>. For the left-equivalent process Y_L, (i) follows from (5.8) and (ii) from (5.9) (and likewise for Y_R). □

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