NUMERICAL METHOD FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We propose a method for numerical approximation of Backward Stochastic Differential Equations. Our method allows the final condition of the equation to be quite general and simple to implement. It relies on an approximation of Brownian Motion by simple random walk.

1. Introduction

In this paper we propose a new method to approximate solutions of Backward Stochastic Differential Equations (BSDEs). Our method allows the final condition of the equation to be quite general and it is simple to implement. It relies on an approximation of Brownian motion by simple random walk.

This type of equation appears in numerous problems in Finance, in contingent claim valuation when there are constraints on the hedging portfolios (see El Karoui, Peng & Quenez [9]).

Some numerical methods for approximate solutions of BSDEs have already been developed. A four step algorithm developed by J. Ma, P. Protter and J. Yong to solve a class of more general equations called forward-backward SDE's has been proposed in [13]. A numerical scheme was developed based on this method in [7]. Bally [2] presents a random time scheme to approximate BSDEs. The convergence result only needs regularity assumptions. However his scheme requires a further approximation to give an implementation. On the other hand Chevance [5] gives a numerical method for solving BSDEs associated with a forward stochastic differential equation (FSDE). His method requires strong regularity assumptions for its implementation. Finally a new result of Bally and Pagès allows for the numerical treatment of BSDEs and also reflected BSDEs [3].

We should note that there is another type of approximating solution to BSDEs, via the discretization of filtration (see, for example, Antonelli and Kohatsu-Higa [1], and Coquet, Mackevičius, and Mémin [6]). Although our method uses similar ideas to those used in these results, in that we also approximate the Brownian motion by discrete processes, the main feature of our result is that we *do not* assume that the discretized filtrations "converge" to the original

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Brownian filtration in order to guarantee the convergence of the solutions. Such a relaxation reduces the complexity in constructing the approximating solutions.

Let $\Omega = C([0,1], I\!R^d)$ and consider the canonical Wiener space $(\Omega, \mathcal{F}, I\!P, \mathcal{F}_t)$, in which $B_t(\omega) = \omega(t)$ is a standard d-dimensional Brownian Motion. We consider the following BSDE

$$Y_t = \xi + \int_t^1 f(s, Y_s)ds - \int_t^1 Z_s dB_s, \tag{1}$$

where ξ is a $\mathcal{F}_1$ measurable square integrable random variable and f is Lipschitz continuous in the space variable with Lipschitz constant L. The solution of (1) is a pair of adapted processes (Y, Z) which satisfies the equation. Existence and uniqueness for the solutions of such equations are proved in [14], [15], and further results for the extension of uniqueness under more general assumptions for the generator have been developed for example in [12] in the one dimensional case. We note that for technical reasons in this paper we only treat the BSDEs with generator f being independent of the component Z. Nevertheless, our result for f independent of Z, combined with a simple Girsanov argument, yields a result that can include a Z term in the drift in a linear way. We explain this in the Remark at the end of the article. We hope to be able to address the fully general case in our future publications.

2. A Numerical Scheme for BSDEs

The numerical scheme we propose in this paper is based upon a discretization of (1) and replacing B by a simple random walk. To be more precise, let $t_i^{(n)} = \dfrac{i}{n} : i = 0, \dots, n$. For the sake of simplicity we shall denote t_i instead of $t_i^{(n)}$ when the dependence on n is possible; also, we shall consider only the one dimensional case, although the generalization to the d-dimensional case is rather clear. We denote $M^{(n)}$ to be the approximating binomial random walk, whose increments are $1/\sqrt{n}$ and $-1/\sqrt{n}$ with probability $1/2$. Further, we assume that the sequence $\{M^{(n)}\}$ is i.i.d. We note that while most of the results presented in this paper also hold for other approximations of B, we work with $M^{(n)}$ mainly for simplicity.

In what follows we denote $(\mathcal{F}^n)$ to be the natural filtration of $M^{(n)}$. In some of our computations we shall use the linear interpolation associated to the discrete process $M^{(n)}$, which will then become a continuous process and will still be denoted as $M^{(n)}$ itself. Consequently, if F is a functional defined on Ω, then by a slight abuse of notation we shall identify $F(M_0^{(n)}, \dots, M_1^{(n)})$ and $F(M^{(n)})$ as the same, although the latter means F is evaluated over the linear interpolation of $M^{(n)}$, while in the former F is considered as a function on $I\!R^{d(m+1)}$.

Let us now consider the discrete version of the BSDE (1):

$$\tilde{Y}_{t_i}^{(n)} = F(M^{(n)}) + \frac{1}{n}\sum_{j=i}^{n} f(t_j, \tilde{Y}_{t_j}^{(n)}) - \sum_{j=i}^{n-1} \tilde{Z}_{t_j}^{(n)} \Delta M_{t_{j+1}}^{(n)}. \tag{2}$$

This equation has a unique solution $(\tilde{Y}^{(n)}, \tilde{Z}^{(n)})$ since the martingale $M^{(n)}$ has the predictable representation property (see Buckdahn [4]). It can be checked that solving this equation is

equivalent to finding a solution to the following implicit iteration problem:

$$\tilde{Y}_{t_i}^{(n)} = I\!E\left\{\tilde{Y}_{t_{i+1}}^{(n)} + \frac{1}{n}f(t_i, \tilde{Y}_{t_i}^{(n)}) \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\},$$

which, due to the adaptedness condition, is equivalent to

$$\tilde{Y}_{t_i}^{(n)} - \frac{1}{n}f(t_i, \tilde{Y}_{t_i}^{(n)}) = I\!E\left\{\tilde{Y}_{t_{i+1}}^{(n)} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\}. \tag{3}$$

We point out, as we shall prove in Lemma 3.1, that one can in fact assume without loss of generality that the *generator* f is bounded, and henceforth we denote its bound by R. Furthermore, once $\tilde{Y}_{t_{i+1}}^{(n)}$ is determined, $\tilde{Y}_{t_i}^{(n)}$ is solved via (3) by a fixed point technique:

$$\begin{cases} X^0 & = I\!E\left\{\tilde{Y}_{t_{i+1}}^{(n)} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\} \\ X^{k+1} & = X_0 + \frac{1}{n}f(t_i, X^k). \end{cases}$$

It is standard to show that, if f is uniformly Lipschitz in the spatial variable x with Lipschitz constant L, then the iterations of this procedure will converge to the true solution of (3) at a geometric rate L/n. Therefore, in the case when n is large enough, one iteration would already give us the error estimate: $|\tilde{Y}_{t_i}^{(n)} - X^1| \le \frac{LR}{n^2}$, producing a good approximate solution of (3). Consequently, we propose the following explicit numerical scheme for the BSDE (1)

$$\begin{cases} \hat{Y}_1^{(n)} & = F(M^{(n)});\ \hat{Z}_1^{(n)} = 0 \\ \hat{X}_{t_i} & = I\!E\left\{\hat{Y}_{t_{i+1}}^{(n)} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\} \\ \hat{Y}_{t_i}^{(n)} & = \hat{X}_{t_i} + \frac{1}{n}f(t_i, \hat{X}_{t_i}) \\ \hat{Z}_{t_i}^{(n)} & = I\!E\left\{[\hat{Y}_{t_{i+1}}^{(n)} + \frac{1}{n}f(t_i, \hat{Y}_{t_i}^{(n)}) - \hat{Y}_{t_i}^{(n)}](\Delta M_{t_{i+1}}^{(n)})^{-1} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\}. \end{cases}$$

Let us now analyze the error of this scheme. Clearly, the error produced by this method is bounded by

$$|\tilde{Y}_{t_i}^{(n)} - \hat{Y}_{t_i}^{(n)}| \le \left|I\!E\left\{\tilde{Y}_{t_{i+1}}^{(n)} - \hat{Y}_{t_{i+1}}^{(n)} \middle| \mathcal{F}_{t_i}^{(n)}\right\}\right| + \frac{1}{n}|f(t_i, \tilde{Y}_{t_i}^{(n)}) - f(t_i, \hat{Y}_{t_i}^{(n)})| + \frac{1}{n}|f(t_i, \hat{Y}_{t_i}^{(n)}) - f(t_i, \hat{X}_{t_i})|,$$

which yields

$$\sup_\omega |\tilde{Y}_{t_i}^{(n)} - \hat{Y}_{t_i}^{(n)}| \le \gamma(\sup_\omega |\tilde{Y}_{t_{i+1}}^{(n)} - \hat{Y}_{t_{i+1}}^{(n)}| + \frac{LR}{n^2}),\ a.s.,$$

where $\gamma = (1 - L/n)^{-1}$. Iterating this inequality one obtains

$$\sup_\omega |\tilde{Y}_{t_i}^{(n)} - \hat{Y}_{t_i}^{(n)}| \le \gamma^{n-i} \sup_\omega |\tilde{Y}_1^{(n)} - \hat{Y}_1^{(n)}| + \frac{LR\gamma}{n^2}\frac{\gamma^{n-i} - 1}{\gamma - 1}.$$

For large n ($2L \le n$ works) one obtains that $(\frac{1}{1-L/n})^{n-i} \le (1 + \frac{2L}{n})^{n-i} \le e^{2L}$. Therefore we get

$$\sup_\omega |\tilde{Y}_{t_i}^{(n)} - \hat{Y}_{t_i}^{(n)}| \le \frac{R(e^{2L} - 1)}{n}.$$

Using this bound we obtain the corresponding bound for $\tilde{Z}^{(n)} - \hat{Z}^{(n)}$. In fact, we have

$$\begin{aligned} \tilde{Z}_{t_i}^{(n)} - \hat{Z}_{t_i}^{(n)} = \ & [\tilde{Y}_{t_{i+1}}^{(n)} + \tfrac{1}{n}f(t_i, \tilde{Y}_{t_i}^{(n)}) - \tilde{Y}_{t_i}^{(n)}](\Delta M_{t_{i+1}}^{(n)})^{-1} - \\ & I\!E\left\{[\hat{Y}_{t_{i+1}}^{(n)} + \tfrac{1}{n}f(t_i, \hat{Y}_{t_i}^{(n)}) - \hat{Y}_{t_i}^{(n)}](\Delta M_{t_{i+1}}^{(n)})^{-1} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\}, \end{aligned}$$

which by the adaptedness of $\tilde{Z}^{(n)}$ yields

$$\tilde{Z}^{(n)}_{t_i} - \hat{Z}^{(n)}_{t_i} = I\!E\left\{ \left[\tilde{Y}^{(n)}_{t_{i+1}} - \hat{Y}^{(n)}_{t_{i+1}} + \frac{1}{n}(f(t_i, \tilde{Y}^{(n)}_{t_i}) - f(t_i, \hat{Y}^{(n)}_{t_i})) + \tilde{Y}^{(n)}_{t_i} - \hat{Y}^{(n)}_{t_i} \right] (\Delta M^{(n)}_{t_{i+1}})^{-1} \Big| \mathcal{F}^{(n)}_{t_i} \right\}.$$

Finally one obtains,

$$\sup_{\omega} |\tilde{Z}^{(n)}_{t_i} - \hat{Z}^{(n)}_{t_i}| \leq \frac{R(e^{2L}-1)(2+L/n)}{\sqrt{n}}, \ a.s.$$

This means that for the convergence of the numerical method we just need to concentrate on the solution of (2).

We remark that the conditional expectations with respect to the discrete filtration $(\mathcal{F}^{(n)})$ can be computed explicitly as follows. We assume Γ is an $\mathcal{F}^{(n)}_{t_{k+1}}$-measurable random variable, and we take the 2^k atoms corresponding to the trajectories of the martingale $M^{(n)}$ in $\mathcal{F}^{(n)}_{t_k}$. Each atom in $\mathcal{F}^n_{t_k}$ splits into 2 atoms of $\mathcal{F}^n_{t_{k+1}}$. Then we have

$$I\!E(\Gamma|\mathcal{F}^{(n)}_{t_k})(\omega) = \frac{1}{2}(a+b)$$

where a, b are the values of Γ in the two atoms of $\mathcal{F}^{(n)}_{t_{k+1}}$ coming from the corresponding atom in $\mathcal{F}^{(n)}_{t_k}$ containing ω.

Our main result is the following:

Theorem 1. *Assume that in the BSDE (1) the following conditions hold:*

- $\xi = F(B)$, *where* $F : \Omega \to I\!R^d$ *is a bounded Lipschitz function with respect to the uniform topology on* Ω*; that is, there exists a constant* κ *such that for all* $\omega, \omega' \in \Omega$ *it holds that*
$$|F(\omega) - F(\omega')| \leq \kappa \sup_{0 \leq t \leq 1} |\omega(t) - \omega'(t)|;$$
- $f : [0,1] \times I\!R^d \to I\!R^d$ *is a continuous function and it is uniformly Lipschitz, that is, there exists a constant* L *such that for all* $x, y \in I\!R^d$
$$\sup_{0 \leq t \leq 1} |f(t,x) - f(t,y)| \leq L|x-y|.$$

Let $(\tilde{Y}^{(n)}, \tilde{Z}^{(n)})$ *be the solution of (2), and let*

$$U^{(n)}_{t_i} = \sum_{j=0}^{i-1} \tilde{Z}^{(n)}_{t_j} \Delta M^{(n)}_{t_{j+1}} = \tilde{Y}^{(n)} - F(M^{(n)}) - \frac{1}{n}\sum_{j=1}^{i} f(t_j, \tilde{Y}^{(n)}_{t_j}).$$

Then the sequence $(\tilde{Y}^{(n)}, U^{(n)})$ *converges weakly in the Skorohod topology to* $(Y, \int Z dB)$*, where* (Y, Z) *is the unique solution of (1).*

3. Proof of the main result

We first show that under the assumptions of Theorem 1 we can in fact assume without loss of generality that the function f is bounded. To this end, notice that f is continuous, it is bounded on any compact set $[0,1] \times [-A, A]$, $A > 0$. Thus it suffices to show that there exists a constant

$A > 0$ such that the solution of (1) satisfies $|Y_t| \leq A$, $\forall t$, a.s. Denoting $R_0 = \sup_{s\in[0,1]} |f(s,0)|$, we have the following lemma.

Lemma 3.1. *Under the assumptions on F and f as in Theorem 1, the solution Y of (1) is bounded by $(D+R_0)e^L$, where D is a bound for F.*

Proof. In the one dimensional case this can be proved by the comparison Theorem. However, we shall provide an argument that does not involve any comparison results, therefore it works for the general case.

For any $A > 0$ we consider the following BSDE:

$$X_t^A = F(B) + \int_t^1 f(s, \phi_A(X_s^A))ds - \int_t^1 \Lambda_s dB_s, \tag{4}$$

where ϕ_A is any bounded Lipschitz function with constant 1, such that $\phi_A(x) = x$ for $|x| \leq A$, and we shall determine the constant A later. Since f is continuous, it is bounded on $[0,1] \times [-2A, 2A]$ and we let C_A denote this bound. Note that X^A also solves the problem

$$X_t^A = \mathbb{E}\left\{ F(B) + \int_t^1 f(s, \phi_A(X_s^A))ds \;\middle|\; \mathcal{F}_t \right\},$$

we see that X^A is bounded by $D + C_A$ and the following estimate holds:

$$|X_t^A| \leq D + \mathbb{E}\Big\{ \int_t^1 |f(s, \phi_A(X_s^A))|ds \Big| \mathcal{F}_t \Big\} \leq D + R_0 + L\mathbb{E}\Big\{ \int_t^1 |X_s^A|ds \Big| \mathcal{F}_t \Big\}.$$

Denote $h^A(t) := |X_t|_\infty$, the $L^\infty(\Omega)$ norm of the random variable X_t. Then h^A is a deterministic, measurable function and satisfies the inequality

$$h^A(t) \leq D + R_0 + L\int_t^1 h^A(s)ds. \tag{5}$$

Since $h^A(t) \leq D + C_A$ for all t, iterating (5) we derive that, for any $N > 0$,

$$h^A(t) \leq (D+R_0)\sum_{k=0}^{N} \frac{L^k(1-t)^k}{k!} + (D+C_A)\frac{L^{N+1}(1-t)^{N+1}}{(N+1)!}.$$

Letting $N \to \infty$ we obtain a *generalized* Gronwall inequality:

$$|X_t^A| \leq h^A(t) \leq (D+R_0)e^{L(1-t)} \leq (D+R_0)e^L.$$

Note that the right side of this inequality does not depend on A, therefore if we choose $A = (D+R_0)e^L$ in (4), then X^A actually solves the equation

$$X_t^A = F(B) + \int_t^1 f(s, X_s^A)ds - \int_t^1 \Lambda_s dB_s.$$

The uniqueness of the solution of the BSDE (1) then implies that $(X^A, \Lambda) \equiv (Y, Z)$, proving the lemma. ■

The next two lemmas give some fine properties of the discretized solution $\{\tilde{Y}^{(n)}\}$.

Lemma 3.2. *Let $\tilde{Y}^{(n)}$ be the solution of (2). Then the jumps of $\tilde{Y}^{(n)}$ converge uniformly to zero. Moreover*

$$\sup_{\omega} |\tilde{Y}^{(n)}_{t_{i+1}} - \tilde{Y}^{(n)}_{t_i}| \leq \frac{\kappa e^{2L}}{\sqrt{n}} + \frac{R}{n}.$$

Proof. We use induction. Let us start with $\tilde{Y}^{(n)}_1 - \tilde{Y}^{(n)}_{t_{n-1}}$ which is given by

$$\tilde{Y}^{(n)}_1 - \tilde{Y}^{(n)}_{t_{n-1}} = F(M^{(n)}) - I\!E(F(M^{(n)}) \mid \mathcal{F}^n_{t_{n-1}}) - \frac{1}{n} f(t_{n-1}, \tilde{Y}^{(n)}_{t_{n-1}}).$$

Computing the conditional expectation, and recalling our convention on the process $M^{(n)}$ and its linear interpolation (see section 2) we see that

$$\begin{aligned} \tilde{Y}^{(n)}_1 - \tilde{Y}^{(n)}_{t_{n-1}} = \; & F(M^{(n)}_0, \dots, M^{(n)}_{t_{n-1}}, M^{(n)}_1) - \tfrac{1}{2} F(M^{(n)}_0, \dots, M^{(n)}_{t_{n-1}}, M^{(n)}_{t_{n-1}} + \tfrac{1}{\sqrt{n}}) \\ & - \tfrac{1}{2} F(M^{(n)}_0, \dots, M^{(n)}_{t_{n-1}}, M^{(n)}_{t_{n-1}} - \tfrac{1}{\sqrt{n}}) - \tfrac{1}{n} f(t_{n-1}, \tilde{Y}^{(n)}_{t_{n-1}}), \end{aligned}$$

from which we deduce, using that F is Lipschitz, the upper bound

$$|\tilde{Y}^{(n)}_1 - \tilde{Y}^{(n)}_{t_{n-1}}| \leq \frac{\kappa}{\sqrt{n}} + \frac{R}{n}.$$

On the other hand, since

$$\tilde{Y}^{(n)}_{t_{n-1}} - \frac{1}{n} f(t_{n-1}, \tilde{Y}^{(n)}_{t_{n-1}}) = I\!E\{F(M^{(n)}) \mid \mathcal{F}^n_{t_{n-1}}\},$$

we can denote $G_{n-1}(\cdot)$ to be the inverse function of the mapping $y \mapsto y - \frac{1}{n} f(t_{n-1}, y)$, which exists if $L < n$. (Recall here that L is the Lipschitz constant for f in the spatial variable y.) Observe that G_{n-1} is a Lipschitz function with Lipschitz constant $\gamma = (1 - L/n)^{-1}$. Furthermore, it holds that

$$\tilde{Y}^{(n)}_{t_{n-1}} = G_{n-1}\left(\frac{1}{2} F(M^{(n)}_0, \dots, M^{(n)}_{t_{n-1}}, M^{(n)}_{t_{n-1}} + \frac{1}{\sqrt{n}}) + \frac{1}{2} F(M^{(n)}_0, \dots, M^{(n)}_{t_{n-1}}, M^{(n)}_{t_{n-1}} - \frac{1}{\sqrt{n}})\right).$$

Clearly, the right side above is a Lipschitz function of $(M^{(n)}_0, \dots, M^{(n)}_{t_{n-1}})$ with Lipschitz constant $\kappa\gamma$. If we take $\tilde{Y}^{(n)}_{t_{n-1}}$ as the terminal value for a discrete BSDE with generator f on $\{0, \dots, t_{n-1}\}$ we can apply the same argument as above to deduce that

$$|\tilde{Y}^{(n)}_{t_{n-1}} - \tilde{Y}^{(n)}_{t_{n-2}}| \leq \frac{\kappa\gamma}{\sqrt{n}} + \frac{R}{n},$$

and by an inductive argument

$$|\tilde{Y}^{(n)}_{t_i} - \tilde{Y}^{(n)}_{t_{i-1}}| \leq \frac{\kappa\gamma^{n-i}}{\sqrt{n}} + \frac{R}{n} \leq \frac{\kappa e^{2L}}{\sqrt{n}} + \frac{R}{n},$$

completing the proof. ■

Lemma 3.3. *The sequence $\tilde{Y}^{(n)}$ is tight in the Skorohod topology.*

Proof. We use the criterium given in [8], Theorem 2.3, for locally square integrable semimartingales. First notice that $\tilde{Y}^{(n)}$ has the following decomposition: $\tilde{Y}^{(n)} = U^{(n)} + A^{(n)}$, where $A^{(n)}_{t_i} = \tilde{Y}^{(n)}_0 + \frac{1}{n} \sum_{j=0}^{i-1} f(t_j, \tilde{Y}^{(n)}_{t_j})$ is a predictable process with finite variation, and $U^{(n)}_{t_i} = \sum_{j=0}^{i-1} \tilde{Z}^{(n)}_{t_j} \Delta M^{(n)}_{t_{j+1}}$ is a locally square integrable martingale. Define $G^{(n)} = [U^{(n)}, U^{(n)}] + V(A^{(n)})$, where $V(A^{(n)})$

is the total variation of $A^{(n)}$. We have $G^{(n)}$ is bounded by an increasing function g that only depends on t. In fact $V(A^{(n)}) \leq C$ where C is a constant not depending on n. On the other hand $[U^{(n)}, U^{(n)}] = \sum(\Delta U^{(n)})^2 = \sum(\tilde{Z}^{(n)}_{t_i})^2(M^{(n)}_{t_{i+1}} - M^{(n)}_{t_i})^2$. Since $\tilde{Y}^{(n)}$ satisfies equation (2) we have

$$\tilde{Y}^{(n)}_{t_{i+1}} - \tilde{Y}^{(n)}_{t_i} = -\frac{1}{n} f(t_i, \tilde{Y}^{(n)}_{t_i}) + \tilde{Z}^{(n)}_{t_i} \Delta M^{(n)}_{t_{i+1}}.$$

Using Lemma 3.2 and the fact $|\Delta M^{(n)}| = \frac{1}{\sqrt{n}}$, we obtain

$$|\tilde{Z}^{(n)}_{t_i}| \leq \kappa e^{2L} + \frac{2R}{\sqrt{n}},$$

and

$$[U^{(n)}, U^{(n)}] \leq (\kappa e^{2L} + \frac{2R}{\sqrt{n}})^2$$

By choosing g equal to a constant, we have that $G^{(n)}$ satisfies conditions $C1$ and $C2$ of theorem 2.3 given in [8] and the conclusions of this theorem implies that $(\tilde{Y}^{(n)})$ is relatively compact under the Skorohod topology. ■

We point out that the sequence of predictable finite variation processes $\{\tilde{Y}^{(n)}_0 + \frac{1}{n}\sum f(t_j, \tilde{Y}^{(n)}_{t_j})\}$ is bounded in total variation, and therefore is relatively compact under the Skorohod topology. From the previous proof we also know that the sequence $\{U^{(n)}\}$ is relatively compact.

The following lemma is a standard result in the theory of BSDEs, and will give us the basic estimates for the proof of our main result. To simplify presentation let us introduce some notation. For a process Y we denote the $L^{2,\infty}$ norm of Y as

$$\|Y\|_{L^{2,\infty}} := \left\{ I\!E \sup_{0 \leq t \leq 1} |Y_t|^2 \right\}^{1/2}.$$

Further, we call the pair of the functions F and f in (1) the *generator* of this BSDE; and we denote the adapted solution of (1) by $Y(F, f)$ (or $(Y(F, f)), Z(F, f))$ if necessary), when the generators are to be specified. We have the following Lemma.

Lemma 3.4. *Let (G, g) and (F, f) be continuous and bounded generators. If $Y(G, g)$ and $Y(F, f)$ are the corresponding solutions for the BSDEs, then there exists a constant $C < \infty$ which depends only on the Lipschitz constant L of f such that*

$$\|Y(G, g) - Y(F, f)\|_{L^{2,\infty}} \leq C \left(|G(B) - F(B)|_{L^2(\Omega)} + \|f - g\|_\infty \right).$$

In particular if we have a sequence (G_k, g_k) converging to (F, f) in $L^2(\Omega) \times L^\infty([0, 1] \times I\!R^d)$, then the corresponding solutions converges in the $L^{2,\infty}$ norm.

Proof For notational convenience we denote by $Y = Y(F, f)$ and $\bar{Y} = Y(G, g)$. Since $Y - \bar{Y}$ satisfies the following equation

$$Y_t - \bar{Y}_t = I\!E(F(B) - G(B) + \int_t^1 f(Y_s) - g(\bar{Y}_s)\, ds | \mathcal{F}_t), \tag{6}$$

we deduce that $|Y_t - \bar{Y}_t| \leq N_t = I\!E(|F(B) - G(B)| + \int_0^1 |f(Y_s) - g(\bar{Y}_s)|\ ds|\mathcal{F}_t)$, and therefore from Doob's maximal inequality

$$I\!E(\sup_{0\leq t\leq 1} |Y_t - \bar{Y}_t|^2) \leq 12 \left(I\!E(|F(B) - G(B)|^2 + L^2 \int_0^1 |Y_s - \bar{Y}_s|^2 ds) + \|f - g\|_\infty^2 \right).$$

On the other hand using (6) we obtain

$$I\!E(|Y_t - \bar{Y}_t|^2) \leq 3 \left(I\!E(|F(B) - G(B)|^2 + L^2 \int_t^1 |Y_s - \bar{Y}_s|^2 ds) + \|f - g\|_\infty^2 \right),$$

which by Gronwall's inequality gives

$$I\!E(|Y_t - \bar{Y}_t|^2) \leq C_1 \left(I\!E(|F(B) - G(B)|^2 + \|f - g\|_\infty^2 \right),$$

for some finite constant C_1 which depends only on L, and the result follows. ■

We note that if in the previous result the sequence $Y(G_k, g_k)$ is uniformly bounded by A, then we can replace $\|f - g\|_\infty$ by $\sup_{|x|\leq A} |f(x) - g(x)|$. Using that $M^{(n)}$ converges weakly to a Brownian motion we obtain the following result whose proof is in the same spirit as the previous Lemma.

Lemma 3.5. *Let (G, g) and (F, f) be continuous and bounded generators. Consider $\tilde{Y}^{(n)}(G, g)$ the solution of the discrete BSDE (2) associated to (G, g) and similarly for $\tilde{Y}^{(n)}(F, f)$. Then, there exists a constant $C < \infty$ which depends only on the Lipschitz constant L of f such that*

$$\limsup_{n\to\infty} \|\tilde{Y}^{(n)}(G, g) - \tilde{Y}^{(n)}(F, f)\|_{L^{2,\infty}} \leq C \left(|G(B) - F(B)|_{L^2(\Omega)} + \|f - g\|_\infty \right).$$

In particular if we have a sequence (G_k, g_k) converging to (F, f) then

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\tilde{Y}^{(n)}(G_k, g_k) - \tilde{Y}^{(n)}(F, f)\|_{L^{2,\infty}} = 0.$$

■

Our next step is to prove that our numerical method converges when the terminal functional F is of discrete type; that is, it depends only on a finite number of points of a continuous path. We also assume that (F, f) are in C_0^∞. For the sake of simplicity we assume that F depends only on two variables, leaving the obvious generalization for the reader.

We assume now the process Y satisfies the following Backward Stochastic Differential Equation

$$Y_t = F(B_{\tau_0}, B_1) + \int_t^1 f(Y_s) ds + \int_t^1 Z_s dB_s \tag{7}$$

where $\tau_0 \in (0, 1)$. Using Itô's formula the process Y is obtained by solving the following system of partial differential equations:

$$\begin{array}{rcll} \frac{\partial u}{\partial t}(t, x, y) + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(t, x, y) & = & -f(u(t, x, y)) & t \in [\tau_0, 1] \\ u(1, x, y) & = & F(x, y) & \\ \frac{\partial v}{\partial t}(t, z) + \frac{1}{2}\frac{\partial^2 v}{\partial z^2}(t, z) & = & -f(v(t, z)) & t \in [0, \tau_0] \\ v(\tau_0, z) & = & u(\tau_0, z, z) & \end{array} \tag{8}$$

In fact the solution is given by

$$Y_t = \begin{cases} u(t, B_{\tau_0}, B_t) & t \in [\tau_0, 1] \\ v(t, B_t) & t \in [0, \tau_0] \end{cases} . \tag{9}$$

In the previous system we can assume that u is defined on the whole interval $[0, 1]$.

In order to compute a discretization of u, we consider (x_i, y_j) such that $x_i = i\delta, y_j = j\delta$, where $\delta = \delta(n) = 1/\sqrt{n}$. We also denote $\Delta t = 1/n$. We fix x_i and consider the following difference equation associated to the first part of (8) which comes from a Taylor expansion of $u(t + \Delta t, x_i, y_j + \delta) + u(t + \Delta t, x_i, y_j - \delta)$ (the idea is that $U^{(n)}(k, i, j) \approx u(k\Delta t, i\delta, j\delta)$). We denote by $k_0 = [\tau_0 n]$

$$\begin{aligned} \tfrac{1}{2}(U^{(n)}(k+1, i, j+1) + U^{(n)}(k+1, i, j-1)) &= U^{(n)}(k, i, j) - f(U^{(n)}(k, i, j))\Delta t \\ U^{(n)}(n, i, j) &= F(x_i, y_j) \end{aligned} \tag{10}$$

Using the Lipschitz condition on f one proves that the previous difference equation has a unique solution, for $k = k_0, \ldots, n-1$ $i, j \in \mathbb{Z}$. With this solution we approximate the second part of (8) by solving

$$\begin{aligned} \tfrac{1}{2}(V^{(n)}(k+1, j+1) + V^{(n)}(k+1, j-1)) &= V^{(n)}(k, j) - f(V^{(n)}(k, j))\Delta t \\ V^{(n)}(k_0, j) &= U^{(n)}(k_0, j, j) \end{aligned} \tag{11}$$

where $k = 0, \ldots, k_0 - 1$, and $j \in \mathbb{Z}$.

Using the results of [11] (Chapter V, pp. 353–381), we know that if (F, f) is C_0^4 the solution of (8) is regular in (t, x, y) and it has bounded derivatives of order 2 in t and order 4 in (x, y). Let $C > 0$ be the common bound for these derivatives. Using this fact and a Taylor expansion for u on the (t, y) variables one obtains an upper bound for $\theta^{(n)}(k) = \sup_{i,j} |u(k\Delta t, x_i, y_j) - U^{(n)}(k, i, j)|$ for $k = k_0, \ldots, n$. In fact if $\gamma = \frac{1}{1 - L\Delta t}$ we get

$$\theta^{(n)}(k) \leq \gamma \left[\theta^{(n)}(k+1) + \frac{4C}{n^{3/2}} \right],$$

which yields the inequality

$$\theta^{(n)}(k) \leq \gamma^{n-k}\theta^{(n)}(n) + \frac{4C}{n^{3/2}} \sum_{p=1}^{n-k} \gamma^p = \gamma^{n-k}\theta^{(n)}(n) + \frac{4C}{n^{3/2}}\gamma\frac{\gamma^{n-k} - 1}{\gamma - 1}.$$

From $\Delta t = 1/n$ one obtains, as before, for large n that $(\frac{1}{1 - L/n})^{n-k} \leq (1 + \frac{2L}{n})^{n-k} \leq e^{2L}$ and therefore

$$\max_{k_0 \leq k \leq n} \theta^{(n)}(k) \leq e^{2L}\theta^{(n)}(n) + \frac{4C(e^{2L} - 1)}{L\sqrt{n}}.$$

Since $\theta^{(n)}(n) = 0$ we get $\max_{k_0 \leq k \leq n} \theta^{(n)}(k) \leq \frac{4C(e^{2L} - 1)}{L\sqrt{n}}$.

For the other terms we proceed in the same way. Consider

$$\phi^{(n)}(k) = \sup_{i,j} |v(k\Delta t, y_j) - V^{(n)}(k, j)|,$$

for $0 \le k \le k_0$. Then we have $\phi^{(n)}(k_0) \le C(\tau_0 - k_0/n) + \theta^{(n)}(k_0) \le C(1 + 4(e^{2L} - 1)/L)/\sqrt{n}$ and for $0 \le k \le k_0 - 1$

$$\phi^{(n)}(k) \le \gamma \left[\phi^{(n)}(k+1) + \frac{4C}{n^{3/2}} \right],$$

which yields the upper bound

$$\max_{0 \le k \le k_0} \phi^{(n)}(k) \le e^{2L} \phi^{(n)}(k_0) + \frac{4C(e^{2L} - 1)}{L\sqrt{n}} \le \left(e^{2L} + 4e^{2L} \frac{e^{2L} - 1}{L} + 4 \frac{e^{2L} - 1}{L} \right) \frac{C}{\sqrt{n}}.$$

Therefore, using a suitable constant A, we obtain the following estimate

$$\sup_{i,j \in \mathbb{Z}} \{ \sup_{k_0 \le k \le n} |u(k\Delta t, x_i, y_j) - U^{(n)}(k,i,j)|, \sup_{0 \le k \le k_0} |v(k\Delta t, y_j) - V^{(n)}(k,j)| \} \le \frac{A}{\sqrt{n}}. \tag{12}$$

With this estimate in hand we can prove that $\tilde{Y}^{(n)}$ converges to Y. Using Skorohod's embedding theorem we can assume that $M^{(n)}, B$ are defined in the same space and $M^{(n)}$ converges a.s. uniformly on $[0,1]$ to B. On the other hand it is not hard to see that the unique solution of

$$\tilde{Y}^{(n)}_{t_k} = F(M^{(n)}_{t_{k_0}}, M^{(n)}_1) + \sum_{p=k}^{n-1} f(\tilde{Y}^{(n)}_{t_p}) \Delta t + \sum_{p=k}^{n} Z^{(n)}_{t_p} (M^{(n)}_{t_{p+1}} - M^{(n)}_{t_p}), \tag{13}$$

is given by

$$\tilde{Y}^{(n)}_{t_k} = \begin{cases} U^{(n)}(t_k, M^{(n)}_{t_{k_0}}, M^{(n)}_{t_k}) & t_k \in [t_{k_0}, 1] \\ V^{(n)}(t_k, M^{(n)}_{t_k}) & t_k \in [0, t_{k_0}] \end{cases}.$$

Finally from (12) one obtains that $\tilde{Y}^{(n)}$ converges a.s. uniformly on compact sets to the continuous process (9), which is exactly the solution of (7). This means that Theorem 1 is proven when (F, f) is C_b^∞, and F depends on a finite number of coordinates.

Now, we are ready to complete the proof of Theorem 1 in the general case. First, we take a convergent subsequence of $\{\tilde{Y}^{(n)}\}$, which by simplicity we denote by $\{\tilde{Y}^{(n)}\}$ itself. By Skorohod's embedding theorem we can assume that all are defined on the same space and moreover they converge a.s. uniformly on $[0,1]$, to a continuous process X. We consider the approximation F_m which corresponds to the linear interpolation at $t_0, \dots, t_m = 1$. The processes $\tilde{Y}^{(n,m)} = \tilde{Y}^{(n)}(F_m, f), \tilde{Y}^{(n)} = \tilde{Y}^{(n)}(F, f), Y(F_m, f)$ are uniformly bounded by some constant A and therefore we can approximate (F_m, f) with a sequence of C_0^∞ functions $((F_{m,p}, f_p))_p$, such that

$$\lim_{p \to \infty} \mathbb{E} \left(|F_{m,p}(B_{t_0}, \dots, B_{t_m}) - F_m(B_{t_0}, \dots, B_{t_m})|^2 \right) = 0,$$
$$\text{and} \lim_{p \to \infty} \|f_p - f\|_\infty = 0.$$

Actually we just need $\lim_{p \to \infty} \sup_{|x| \le A} |f_p - f| = 0$, for an appropriate constant A. We denote by $\tilde{Y}^{(n,m,p)} = \tilde{Y}^{(n)}(F_{m,p}, f_p)$ and $Y^{(m,p)} = Y(F_{m,p}, f_p)$. Using the triangle inequality we obtain

$$\begin{aligned} \|Y - X\| \le \; & \|Y - Y^{(m)}\| + \|Y^{(m)} - Y^{(m,p)}\| + \|Y^{(m,p)} - \tilde{Y}^{(n,m,p)}\| + \\ & \|\tilde{Y}^{(n,m,p)} - \tilde{Y}^{(n,m)}\| + \|\tilde{Y}^{(n,m)} - \tilde{Y}^{(n)}\| + \|\tilde{Y}^{(n)} - X\|. \end{aligned} \tag{14}$$

Taking limits first in n, then in p and finally in m we see that the right side of (14) converges to 0. In fact, the first and the second terms converges to zero due to Lemma 3.4. The fourth and fifth converge to zero by Lemma 3.5. The last one tends to zero thanks to the a.s. uniform convergence of $\tilde{Y}^{(n)}$ to X plus the dominated convergence theorem. Finally the third term converges to 0 because $(F_{m,p}, f_p)$ is a smooth function, and $F_{m,p}$ depends on a finite number of coordinates. Therefore the only limit point for the sequence $\tilde{Y}^{(n)}$ is exactly Y the solution of (1), proving the result. ■

Remark.

We note that our results in this paper can easily be used to treat a slightly more general case, where there is a term Z in the drift of the BSDE. Indeed, suppose that $(H_s)_{s>0}$ is a predictable process in L^2 for $dPdt$. Also suppose it is known to us. Then we can numerically solve the BSDE

$$Y_t = \xi + \int_t^1 (f(s, Y_s) + H_s Z_s) ds - \int_t^1 Z_s dB_s, \tag{15}$$

by first solving

$$Y_t = \xi + \int_t^1 f(s, Y_s) ds - \int_t^1 Z_s dB_s. \tag{16}$$

We do this as follows. Given equation (15) on $(\Omega, \mathcal{F}, I\!P, \mathcal{F}_t, P)$, we form the equation

$$M_t = 1 + \int_0^t M_s H_s dB_s \tag{17}$$

and then set $\beta = B_t - \int_0^t H_s ds$, which is a Brownian motion under the new probability measure Q defined by $dQ = M_T dP$. We then solve the equation (16) under Q with β replacing B. The solution Y of (16) is then also a solution of (15) under P and using the original Brownian motion B. Last, we wish to note that equations of the type (15) are of interest in Financial Asset Pricing Theory; see, for example, Section 1 of [10].

Summary.

We summarize here the proposed algorithm to solve the BSDE

$$Y_t = F(B) + \int_t^1 f(s, Y_s) ds - \int_t^1 Z_s dB_s,$$

by means of a random walk approximation $M^{(n)}$, in the following scheme

$$\begin{cases} \hat{Y}_1^{(n)} & = F(M^{(n)});\ \hat{Z}_1^{(n)} = 0 \\ \hat{X}_{t_i} & = I\!E\left\{\hat{Y}_{t_{i+1}}^{(n)} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\} \\ \hat{Y}_{t_i}^{(n)} & = \hat{X}_{t_i} + \frac{1}{n} f(t_i, \hat{X}_{t_i}) \\ \hat{Z}_{t_i}^{(n)} & = I\!E\left\{[\hat{Y}_{t_{i+1}}^{(n)} + \frac{1}{n} f(t_i, \hat{Y}_{t_i}^{(n)}) - \hat{Y}_{t_i}^{(n)}](\Delta M_{t_{i+1}}^{(n)})^{-1} \;\middle|\; \mathcal{F}_{t_i}^{(n)}\right\}. \end{cases}$$

where the conditional expectations with respect to the discrete σ-field $\mathcal{F}_{t_i}^{(n)}$ are computed using a tree structure. For example we obtain

$$\hat{X}_{t_{n-1}} = \frac{1}{2}\left\{F\left(M_{t_{n-1}}^{(n)} + \frac{1}{\sqrt{n}}\right) + F\left(M_{t_{n-1}}^{(n)} - \frac{1}{\sqrt{n}}\right)\right\}.$$

References

[1] Antonelli, F. and Kohatsu-Hida, A. *Filtration stability of backward SDE's*. Stochastic Anal. Appl. **18**(1) (2000), 11–37.

[2] Bally, V., *Approximation scheme for solutions of Backward stochastic differential equations* (Paris, 1995-1996). Pitman Res. Notes Math. Ser. **364** (1997), 177–191.

[3] Bally, V., Pagès *A quantization method for the discretization of BSDE's and Reflected BSDE's*. Preprint, (2000).

[4] Buckdahn, R., *Backward Stochastic Differential Equations Driven by a Martingale*, Unpublished preprint. Université de Provence 93-05.

[5] Chevance D., *Discrétisation des Équations Différentieles Stochastiques Rétrogrades*, Numerical Methods in Finance, L.C.G. Rogers & D. Talay (eds), Cambridge UP, (1997), 232–244.

[6] Coquet F., Mackevičius V., and Mémin J. *Stability in D of Martingales and Backward Equations under Discretization of Filtration* Stochastic Processes Appl. **75** (1998), no. 2, 235–248.

[7] Douglas J., Ma J., Protter P. *Numerical methods for forward-backward stochastic differential equations*. Annals of Applied Probability. **Vol. 6** (1996), 940–968.

[8] Jacod J., Mémin J., Metivier M., *On Tightness and Stopping Times*, Stochastic Processes and their Applications. **14** (1983), 109–146.

[9] El Karoui N., Peng S., Quenez M.C., *Backward Stochastic Differential Equations in Finance* , Math. Finance **7** (1997), 1–71.

[10] El Karoui N., Quenez M.C., *Nonlinear pricing theory and backward stochastic differential equations*, in *Financial Mathematics*, Springer Lecture Notes in Math. **1656** (1997), 191–246.

[11] Freidlin M., *Functional integration and partial differential equations*, Princeton New Jersey (1985).

[12] Lepeltier J.P., San Martin J., *Backward Stochastic Differential Equations with continuous generator*, Statist. Probab. Lett. **32** (1997) 425–430.

[13] Ma. J., Protter P., Yong.J., *Solving forward-backward stochastic differential equations explicitly -a four step scheme*, Proba. Theory and Relat. Fields. **98** (1994), 339-359.

[14] Pardoux E. Peng.S., *Adapted Solution of a Backward Stochastic Differential Equation*, System and Control Letters. **14** (1990), 55–61.

[15] Pardoux E. Peng.S., *Backward stochastic differential equation and quasilinear parabolic partial differential equations*. Lectures Notes in CSI. **176** (1992), 200–217.

[16] Protter P., *Stochastic Integration And Differential Equations*, Springer-Verlag (1990).