Scaling, Renormalization, and Universality in Combinatorial Games: the Geometry of Chomp

(with technical appendices)

Eric J. Friedman[1] and Adam Scott Landsberg[2*]

[1]*School of ORIE, Cornell University, Ithaca, NY 14853, USA*

[2]*Joint Science Department, Claremont McKenna, Pitzer and Scripps Colleges, Claremont, California 91711, USA.*

**Current address: School of ORIE, Cornell University, Ithaca NY 14853, USA.*

__Abstract__: **We develop a new approach to combinatorial games (e.g., chess, Go, checkers, Chomp, Nim) that unveils connections between such games and nonlinear phenomena commonly seen in nature: scaling behaviors, complex dynamics and chaos, growth and aggregation processes. Using the game of Chomp (as well as variants of the game of Nim) as prototypes, we discover that the game possesses an underlying geometric structure that "grows" (reminiscent of crystal growth), and show how this growth can be analyzed using a renormalization procedure. This approach not only allows us to answer some open questions about the game of Chomp, but opens a new line of attack for understanding (at least some) combinatorial games more generally through their underlying connection to nonlinear science.**

Combinatorial games, which include chess, Go, checkers, Chomp, dots-and-boxes, and Nim, have both captivated and challenged mathematicians, computer scientists, and players alike (*1-10*). Analysis of these two-player games has generally relied upon a few beautiful analytical results (*1,11-14*) or on numerical algorithms that combine heuristics with look-ahead approaches (α–β pruning) (*15,16*). Using Chomp as a prototype, we report on a new geometrical approach which unveils unexpected parallels between combinatorial games and key ideas from physics and dynamical systems, most notably notions of scaling, renormalization, universality, and chaotic attractors. Our central finding is that underlying the game is a probabilistic geometric structure that encodes essential information about the game, and that this structure exhibits a type of scale invariance: Loosely speaking, the geometry of "small" winning positions and "large" winning positions are the same after rescaling. (This general finding also holds for at least some other combinatorial games, as we explicitly demonstrate with a variant of Nim.) This geometric insight not only provides (probabilistic) answers to some open questions about Chomp, but suggests a natural pathway toward a new class of algorithms for more general combinatorial games, and hints at deeper links between such games and nonlinear science.

Chomp is an ideal candidate for our study, since in certain respects it appears to be among the simplest in the class of "hard" games. Its history is marked by some significant theoretical advances (*19*-*22*), but it has yet to succumb to a complete analysis in the 30 years since its introduction by Gale (*17*) and Schuh (*18*). The rules of Chomp are easily explained. Play begins with an N x M array of counters (Fig. 1a). On each turn a player selects a counter and removes it along with all counters to the north and east of it (Fig. 1b). Play alternates between the two players until one player takes the last counter, thereby losing the game. (An intriguing feature of Chomp, as shown by Gale, is that although it is very easy to prove that the player who moves first can always win (under optimal play), what this opening move should be has been an open question. Our methodology will in fact provide a probabilistic answer to this question.)

For simplicity, we will focus here on the case of three-row (M=3) Chomp, a subject of recent study by Zeilberger (*19*-*20*) and Sun (*21*). Generalizations to four-row and higher Chomp are analogous. To start, we note that the configuration of the counters at any stage of the game can be described (using Zeilberger's coordinates) by the position p=[x,y,z], where x specifies the number of columns of height three, y specifies the number of columns of height two, and z the number with height one (Fig. 1b). Each position p may be classified as either a *winner*, if a player starting from that position can always force a win (under optimal play), or as a *loser* otherwise. (This classification is well defined by Zermelo's theorem.) The set of all losers contains the information for solving the game. One may conveniently group the losing positions according to their x values by defining a "loser sheet" L_x to be an infinite two-dimensional matrix whose $(y,z)^{th}$ component is a 1 if position [x,y,z] is a loser, and a 0 otherwise. (As noted by Zeilberger, one can express L_x in terms of all preceding loser sheets L_{x-1}, L_{x-2}, …, L_0.) Studies by Zeilberger (*19*,*20*) and others (*21*-*23*) have detected several numerical patterns along with a few analytical features about the losing positions, and their interesting but non-obvious properties have even led to a conjecture that Chomp may be "chaotic in a yet-to-be-made-precise sense" (*20*). However, many of the numerical observations to date have remained largely unexplained, and disjoint from one another.

To provide broader insight into the general structure of the game, we depart from the usual analytic/algebraic/algorithmic approaches. We instead show how the analysis of the game can be recast and transformed into a type of "renormalization" problem commonly seen in physics (and later apply this methodology to other combinatorial games besides Chomp). Analysis of the resulting renormalization problem not only explains earlier numerical observations, but provides a unified, global description of the overall structure of the game. We remark that this approach will be distinguished by its decidedly geometric flavor, and by the incorporation of probabilistic elements into the analysis, despite the fact that the combinatorial games we consider are all games of no chance which lack any inherent probabilistic components to them whatsoever.

To proceed, we turn consideration to so-called "instant-winner sheets", defined as follows: A position p=[x,y,z] is called an *instant* winner (in Zeilberger's terminology) if from that position a player can legally move to a losing position with a smaller x-value. We therefore define an instant-winner sheet W_x to be the infinite, two-dimensional matrix consisting of all instant winners with the specified x-value, i.e., the $(y,z)^{th}$ component of matrix W_x is a 1 if position [x,y,z] is an instant winner, and a 0 otherwise. These instant-winner sheets will prove crucial for understanding the geometric structure of the game (and, as will be seen, contain all the information needed to construct the loser sheets).

Our first insight comes from numerical simulations, wherein we numerically construct the instant winner sheets $\{W_x\}$ for various x values (using a recursive algorithm which will be made clear below). Figs. 2a,b show the structure of W_{700} and W_{350}, respectively, and are representative of what is observed at other x values. Each sheet exhibits a nontrivial internal structure characterized by several distinct regions: a solid (filled) triangular region at the lower left, a series of horizontal bands extending to the right (towards infinity), and two other triangular regions of different densities. Most importantly, however, we observe that the set of instant-winner sheets $\{W_x\}$ possess a remarkable scaling property: their overall geometric shape is identical up to a scaling factor! In particular, as x increases, all boundary-line slopes, densities, and shapes of the various regions are preserved from one sheet to the next (although the actual point-by-point locations of the instant winners within each sheet are different). Hence, upon rescaling, the overall geometric structure of these sheets is identical (in a probabilistic sense). We point out that the "growth" (with increasing x) of the instant-winner sheets is strikingly similar to certain crystal-growth and aggregation processes found in physics – in each case, the structures grow through the accumulation of new points along current boundaries, and exhibit geometric invariance during this process. The loser sheets $\{L_x\}$ can be numerically constructed in a similar manner; their characteristic geometry is revealed in Fig. 2c. It is found to consist of three (diffuse) lines: a lower line of slope m_L and density of points λ_L, an upper line of slope m_U and density λ_U, and a flat line extending to infinity. The upper and lower lines originate from a point whose height (i.e., z-value) is αx. The flat line (with density one) is only present with probability γ in randomly selected loser sheets. Like the instant-winner sheets, the loser sheets also exhibit this remarkable geometric scaling property: as x increases, the geometric structure of L_x grows in size, but its overall shape remains unchanged (the only caveat being that, as previously noted, the flat line seen in Fig. 2c is sometimes absent in some of the loser sheets). We emphasize that while these are only numerical findings, they will provide critical intuition for the results that follow.

Our second key finding is that there exists a well-defined, analytical recursion operator that relates one instant winner sheet to its immediate predecessor. Namely, we can write $W_{x+1} = \mathbf{R}\, W_x$, where **R** denotes the recursion operator. (The operator **R** can be decomposed as $\mathbf{R}=\mathbf{L}(\mathbf{I}+\mathbf{DM})$, where **L** is a left-shift operator, **I** is the identity operator, **D** is a diagonal element-adding operator, and **M** is a "sheet-valued" version of the standard mex operator which is often used for combinatorial games.) A detailed derivation of this result is provided in Appendix A. However, for our present purposes it suffices to simply note that a recursion operator relating the instant-winner sheets exists. We point out that once a given instant-winner sheet W_x has been constructed, the corresponding loser sheer L_x can be found via L_x = **M** W_x (Appendix A).

Stepping back for a moment, what we now have is a renormalization problem akin to those so often encountered in physics and the nonlinear sciences, such as the famous period-doubling cascade discovered by May (*24*) in a biological mapping and analyzed by Feigenbaum using renormalization techniques (*25*). In particular, we have objects (instant winner matrices) that exhibit similar structure at different size scales (cf., Figs. 2a,b), and a recursion operator relating them. Our task therefore is to determine an invariant geometric structure W such that if we act with the recursion operator followed by an appropriately-defined rescaling operator **S**, we get W back again: W = **S R** W (i.e., we seek a fixed point of the "renormalization-group operator" **S R**.) This can be done, but before doing so we point out a critical feature of the analysis. Even though the recursion operator **R** is exact and the game itself has absolutely no stochastic aspects to it, it is necessary to adopt a probabilistic framework in order to solve this recursion relation. Namely, our renormalization procedure will show that the slopes of all boundary lines and densities of all regions in the W_x's (and L_x's) are preserved – not that there exists a point-by-point equivalence. In essence, we will bypass consideration of the random-looking 'scatter' of points surrounding the various lines and regions of W_x and L_x by effectively averaging over these 'fluctuations'.

The key to implementing the renormalization analysis is to observe that the losers in L_x (Fig. 2c) are constrained to lie along certain boundary lines of the W_x plot (Fig. 2b), and are conspicuously absent from the various interior regions of W_x (for all x). In other words, the interior regions of each W_x remain "forbidden" to the losers. Hence the geometry of W_x's must be very tightly constrained if it is to preserve these forbidden regions under the recursion operator $W_x \xrightarrow{\mathbf{R}} W_{x+1}$ (for otherwise the W_x's would not remain geometrically scale invariant). Each forbidden region in W_x imposes a constraint on the permissable structural form that the W_x's can take, and can be formulated as an algebraic equation relating the hitherto unknown parameters m_L, λ_L, m_U, λ_U, γ, α that define the loser sheets. A detailed calculation, described in Appendix B, shows that there are six independent conditions:

$$\lambda_U + \lambda_L = 1, \quad \frac{\lambda_U}{1+m_U} = 1, \quad \frac{1}{\alpha+1} - \frac{\lambda_L}{m_L+1} = 1, \quad (\gamma-1)\frac{m_L}{\alpha-m_L} + \frac{1}{\alpha+1} = 1,$$
$$\frac{\alpha\lambda_L}{\alpha-m_L}\left(\frac{m_U-m_L}{m_U\alpha - m_L\alpha + m_L\gamma}\right) + \frac{1}{\alpha+1} = 1, \quad \frac{\lambda_L}{\alpha-m_L} - \frac{\alpha}{\alpha+1}\left(1-\frac{\lambda_U}{\alpha-m_U}\right) = 0.$$

Stated differently, these are the necessary conditions for the instant-winner sheets to be fixed points of the renormalization operator **S R**. Solving the above relations yields

$$\alpha = \tfrac{1}{\sqrt{2}}, \quad \lambda_L = 1-\tfrac{1}{\sqrt{2}}, \quad \lambda_U = \tfrac{1}{\sqrt{2}}, \quad m_L = -1-\tfrac{1}{\sqrt{2}}, \quad m_U = -1+\tfrac{1}{\sqrt{2}}, \quad \gamma = \sqrt{2}-1$$

These six key parameters completely characterize the loser sets $\{L_x\}$ of the game, and from these the properties of the associated instant winner sheets $\{W_x\}$ are readily deduced. We thus have a fundamental (probabilistic) description of the global geometric structure of the game. We note that these six (analytically derived) parameter values also provide an explanation for existing numerical observations about Chomp, including key numerical conjectures on the game's loser properties by Brouwer (*23*). We mention that only a single assumption was needed to construct the six preceding parameter relations; namely, that fluctuations associated with the diagonal operator **D** were uncorrelated with the fluctuations surrounding the upper line in L_x.

Several interesting results immediately follow. First, having analytically determined the geometric structure of the loser sheets, we can now show that the winning opening move in Chomp (from the initial position $[x_0, 0, 0]$) must be to a position that lies in the vicinity of one of the two points: $[x_0/\sqrt{2},\ x_0(2-\sqrt{2})/2,\ 0]$ or $[x_0(2-\sqrt{2}),\ 0,\ x_0(\sqrt{2}-1)]$. (Here, "vicinity" is defined by the width of the loser band surrounding the analytical loser lines in loser sheets $\{L_x\}$.) Moreover, this winning opening move can be shown to be unique. (See Appendix C for details.) (Previously, uniqueness was known to hold numerically for $x_0<90{,}000$ (23)). Second, for most winning positions (except those near a boundary), knowing their location within W_x allows us to compute the expected number of winning moves based on which lines in the loser sheets are accessible. Third, knowledge of the geometrical structure of the loser sheets suggests a natural pathway to more efficient algorithms by simply designing the search algorithm to aim directly for the analytically determined loser lines in L_x. This is in fact a general feature of our methodology (not limited to just Chomp): once the geometry of a combinatorial game has been identified by the renormalization procedure, efficient geometrically-based search algorithms can be constructed. Lastly, as seen in Fig. 2c, the co-existence of order (i.e., analytically well-defined loser lines) and disorder (i.e., the scatter of points around these lines) signifies that combinatorial games such as Chomp may be unsolvable yet still informationally compressible, in the language of Chaitin (*26*).

The probabilistic renormalization approach we have employed naturally gives rise to a whole new set of interesting questions about combinatorial games. For instance, we can construct variants of standard games simply by perturbing an instant-winner sheet by the addition of a finite number of new points. (Such additions effectively modify the game by declaring these new positions to be automatic winners.) We can then examine whether or not the same instant-winner-sheet geometry appears in these variants (i.e., is the geometric structure an attractor?). Simulations show that for a sizeable class of variants of Chomp the original geometric structure of Fig. 2 re-emerges. Hence it appears stable (in a probabilistic sense). In the language of renormalization we would say that such game variants fall into the same universality class as the original game. A related issue concerns sensitivity to initial conditions, a hallmark of chaos in dynamical systems theory. Using our recursion operator $W_{x+1} = \mathbf{R}\, W_x$, we can examine how small perturbations to W_x propagate. Although the overall instant-winner geometry of the perturbed and unperturbed systems will be the same if they lie in the same universality class, they will differ on a point-by-point basis. We find (see Fig. 3) that small initial perturbations can in fact significantly alter the actual loser locations quite dramatically, highly reminiscent of chaotic systems. For example, adding just a single point to W_{100} can, after only 25 iterations, alter the locations of nearly half of all losing positions in all subsequent loser sheets $\{L_x \mid x>125\}$.

We can also apply our methodology to other combinatorial games. Consider the simple game of (three-heap) Nim (*1,27*): Play begins with a set of counters stacked into three piles (heaps). The number of counters in the heaps will be described by coordinates [x,y,z]. At each turn, a player selects a heap and removes one or more counters from it. Play alternates between the two players until no counters remain. Under ordinary play whoever takes the last counter(s) wins. It is straightforward to construct the recursion and renormalization operators for this game (Appendix D), and to analyze its properties analogously. Fig. 4a shows the geometry of an instant-winner sheet W_x for three-heap Nim. As in Chomp, this structure exhibits a geometric scaling property (athough the W_x's do depend on their x-values). Unlike Chomp however, ordinary Nim is known to be a completely solvable game, and we find that the geometry of its W_x's is unstable. Indeed, if we add just a few random perturbations to one of the sheets (so that the game is no longer readily solvable), then a very different-looking instant winner structure of the form shown in Fig. 4b emerges. This striking new structure, just as for Chomp, is remarkably stable, generic (i.e., it seems to naturally emerge for most perturbations), and scale invariant. In fact, we speculate that the ordinary game of Nim has an unstable, nongeneric geometry precisely because of its solvable nature, and that the robust geometry of Fig. 4b for variants of Nim is much more typical. It is not unreasonable to conjecture more generally that generic combinatorial games will have robust underlying geometric structures, while those of solvable games will be structurally unstable to perturbations.

Lastly, we remark that the "growth" (with increasing x) of the geometric structures W_x (Figs. 4b and 2a) for games such as Nim and Chomp is suggestive of certain crystal growth and aggregation processes in physics (*28*) and activation-inhibition cellular automata models in biology (*29*). This semblance arises because the recursion operators governing the game evolution ($W_x \xrightarrow{\mathbf{R}} W_{x+1}$) typically act by attaching new points to the boundaries of the existing (instant-winner sheet) structures. Although the details vary, this type of attachment-to-boundaries process is a common feature of many physical growth models. Viewed in this way, then, our procedure offers a means of transforming the study of a combinatorial game into that of a shape-preserving growth process – and with it the hope that some of the tools which physicists have developed for analyzing such growth models may now be brought directly to bear on combinatorial games.

We must conclude by emphasizing that though we have applied our methodology successfully to a few games – Chomp, Nim, and their variants – and it has yielded some interesting insights, these games represent just a small handful in the set of established combinatorial games, and consequently the limitations of this methodology and its scope of applicability are not known at present. However, we hope that this novel (renormalization-based) approach to combinatorial games and the tantalizing connections it raises to key ideas from the nonlinear sciences will stimulate further research along these lines.

Fig. 1: The game of Chomp. **(a)** play begins with an MxN rectangular array of counters (three-row Chomp is illustrated). At each turn, a player selects a counter and removes it along with all counters lying in the northeast quadrant extending from the selected counter. Play alternates between the two players until one player is forced to take the 'poison' counter (shown in red), thereby losing the game. **(b)** a sample game configuration after player 1 selects counter A, followed by player 2 selecting counter B. More generally, an arbitrary game configuration can be specified by coordinates [x,y,z], as shown.

Fig. 2: The geometry of Chomp. **(a)** the instant-winner sheet geometry for three-row Chomp, shown for x=700. Instant winner locations in the y-z plane are shown in blue. **(b)** the instant-winner sheet for x=350. Comparison of W_{350} to W_{700} highlights the central scaling property of the instant winner sheets $\{W_x\}$: as they "grow" in size with increasing x, they remain geometrically identical up to a scale factor (i.e., shapes, densities, and boundary-line slopes remain fixed). (Note: This geometric invariance is not especially apparent at very small values of x, but rapidly emerges as x increases.) **(c)** the loser-sheet geometry L_x, shown for x=350. Observe that losers in each sheet all lie near one of three lines: a lower line of slope m_L, density λ_L; an upper (tilted) line of slope m_U, density λ_U; and an upper flat line (of density one) which only exists for some x-

values. The probability that a flat line in L_x exists for a randomly chosen x is γ. The lower and upper tilted lines both emanate from a point near (y,z)=(0, α x). Note that these loser lines are located along boundaries of the associated instant-winner sheet (compare L_{350} with W_{350}) The geometrical structure of the L_x's, like that of the W_x's, remains invariant (up to a scale factor) as one goes to progressively larger x values (not shown). As described in the text, the analysis of this invariance property allows for a complete geometrical/probabilistic characterization of the structures shown in these figures (including an analytical determination of the parameters m_L, λ_L, m_U, λ_U, γ, α).

Fig. 3: Dependence on initial conditions. The figure illustrates how perturbing an instant winner matrix by a single point subsequently spreads and "infects" the loser sheets at higher x values (i.e., altering the precise locations of the losing positions). The red data points show the fraction of losers along the upper tilted and lower lines (e.g., Fig. 2c) that are affected when one adds a single point to W_{400} and then iterates. The blue data points show the corresponding effect when the initial perturbation is to W_{100}. The green data shows (a rolling average of) the spread of the infection to losers lying along the flat tail of Fig. 2c (for an initial perturbation to W_{400}). Note that the effects can be pronounced in spite of the linear appearance of the initial growth for small iteration numbers (see blue, red data). For example, the blue data shows that changing just a single point (out of the approximately 4,000 points making up the relevant region of W_{100}), will, after only about 25 iterations of the recursion operator, shift the location of nearly half of all losing positions!

Fig. 4: The geometries of ordinary and variant Nim. **(a)** the instant winner structure W_x at x=256 for ordinary 3-heap Nim. As in Chomp, this geometrical structure is preserved (up to an overall scale factor) with increasing x values;

i.e., W_x and W_{2x} look identical (not shown). However, unlike Chomp, the geometry is highly unstable to perturbations, and also exhibits an internal periodicity such that W_x and W_{x+1} are similar but not wholly identical in structure. In the figure the instant winners have been color coded based on their "age" (as defined by the recursion algorithm which generated them); blue is oldest, red youngest. **(b)** the instant winner structure W_x at x=256 for a generic Nim variant. Nim variants are similar to ordinary Nim, except that one or more heap configurations are arbitrarily declared to be automatic winners. The striking geometrical structure shown in the figure is both stable and reproducible, i.e., it typically emerges whenever one or more random heap configurations are declared automatic winners. As in Chomp, this attracting structure is preserved (up to scale factors) as one goes to increasingly large x-values. (We note, however, that the scaling behavior appears more pronounced for $W_x \rightarrow W_{2x}$ than it is for $W_x \rightarrow W_{x+1}$, a remnant, we believe, of the underlying solvable structure of ordinary Nim upon which these Nim variants are based.)

References

1. Berlekamp, E. R., Conway, J. H., & Guy, R. K., *Winning Ways for Your Mathematical Plays*. Academic Press, London (1982).

2. Nowakowski, R. J., *Games of No Chance*. Cambridge University Press, Cambridge (1996).

3. Nowakowski, R. J., *More Games of No Chance*. Cambridge University Press, Cambridge (2002).

4. Fleischer, R. H. & Nowakowski, R. J., *Algorithmic combinatorial game theory*. Elsevier, Amsterdam (2004) .

5. Berlekamp, E., *The Game of Dots and Boxes - sophisticated child's play*. by A K Peters Ltd, Natick, MA (2000).

6. Berlekamp, E. & Wolfe, D., Mathematical Go: Chilling Gets the Last Point. A K Peters Ltd, Natick, MA (1994).

7. Demaine, E. D., Playing games with algorithms: Algorithmic combinatorial game theory, *Lecture Notes in Computer Science*, **2136** 1832 (2003).

8. Fraenkel, A.S., Complexity, appeal and challenges of combinatorial games. *Theoretical Computer Science* **313** 393-415 (2004).

9. Demaine, E.D., Fleischer, R., Fraenkel, A.S. & Nowakowski, R.J., Open problems at the 2002 Dagstuhl Seminar on Algorithmic combinatorial game theory. *Theoretical Computer Science* 313 539-543,(2004).

10. Guy, R., & Nowakowski, R., *Unsolved Problems in Combinatorial Games*. In Nowakowski, R. J., ed., *More Games of No Chance*. Cambridge University Press, Cambridge (2002).

11. Sprague, R., Uber mathematische Kampfspiele. *Tohoku Mathematical Journal*, **41** 438–444, (1936).

12. Grundy, P. M., Mathematics and games. Eureka 2 6–8, (1939).

13. Smith, C., Graphs and composite games. *Journal of Combinatorial Theory* **1** 51–81 (1966).

14. Conway, J. H., *On Numbers and Games*. AK Peters, Ltd., Natick, Mass (2000).

15. Newborn, M., *Kasparov versus Deep Blue: Computer Chess Comes of Ag*e. Springer-Verlag, New York, (1997).

16. Schaeffer, J., *One Jump Ahead: Challenging Human Supremacy in Checkers.* Springer-Verlag, New York (1997)

17. Gale, D., A Curious Nim-type game. *Amer. Math. Monthly* **81,** 876-879 (1974).

18. Schuh, F., Spel van delers. *Nieuw Tijdschrift voor Wiskunde* **39,** 299-304 (1952).

19. Zeilberger, D., Three-rowed Chomp. *Adv. in Appl. Math*.**26**, 168-179 (2001).

20. Zeilberger, D., Chomp, Recurrences, and Chaos. *J. Difference Equations and its Applications* 10, 1281-1293 (2004).

21. Sun, X., Improvements on Chomp. *Integers* **2**, G1, 8 pp http://www.integers-ejcnt.org/ (2002).

22. Byrnes, S., Poset Games Periodicity. *Integers* **3,** G3, 8 pages. http://www.integers-ejcnt.org/ (2003).

23. Brouwer, A.E., The game of Chomp. *On-line document*, http://www.win.tue.nl/~aeb/games/chomp.html (2004)

24. May, R.M., Biological populations obeying difference equations: Stable points, stable cycles, and chaos. *Journal of Theoretical Biology,* **51**(2) 511-524 (1975).

25. Feigenbaum, M., Universal Behavior in Nonlinear Systems. *Los Alamos Science*, **1**, 4-27 (1980).

26. Chaitin, G., *Information, Randomness & Incompleteness.* 2nd edition, World Scientific, Singapore (1990).

27. Bouton, C.L., Nim, a game with a complete mathematical theory. *Annals of Mathematics*, Series 2, **3** 35–39, (1902).

28. Gouyet, J.-F., *Physics of Fractal Structures.* Springer-Verlag, New York (1995).

29. Bar-Yam, Y., *Dynamics of Complex Systems*, Addison-Wesley, Massachusetts. (1997).

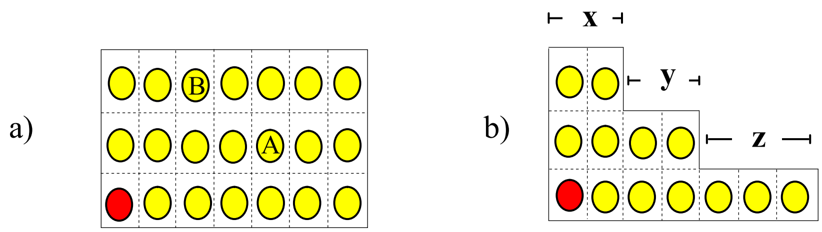
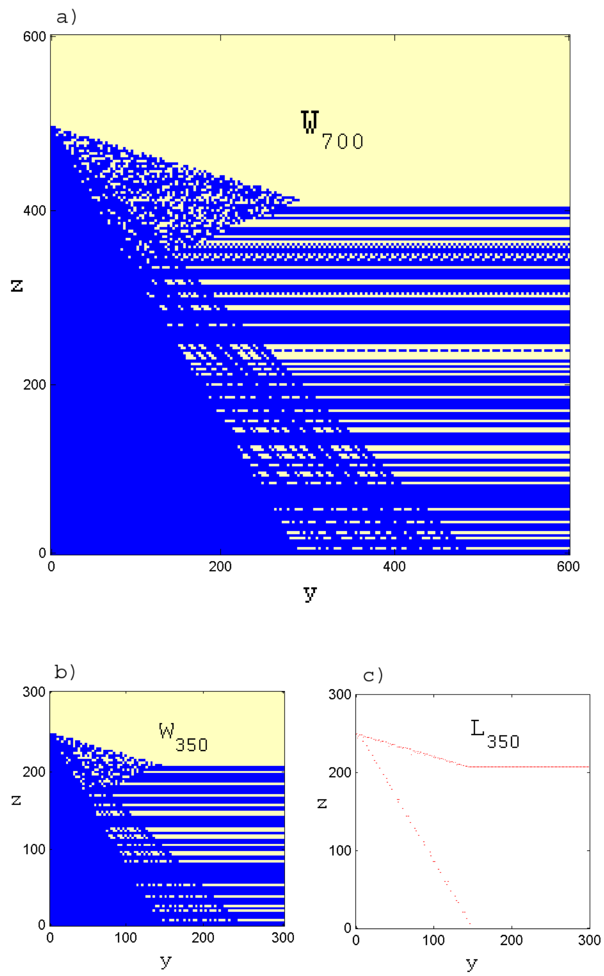


Figure 1



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0
z
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y
b)
W350
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c)
L350
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y

Figure 2

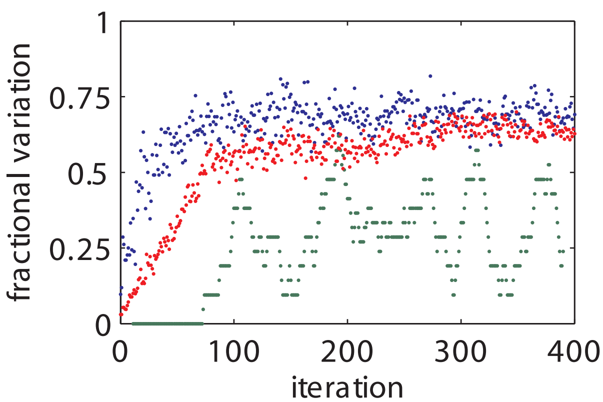
fractional variation
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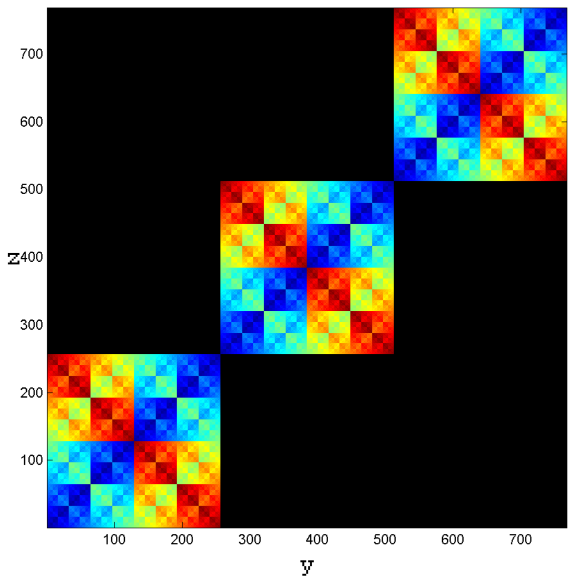
Figure 3

a)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b)

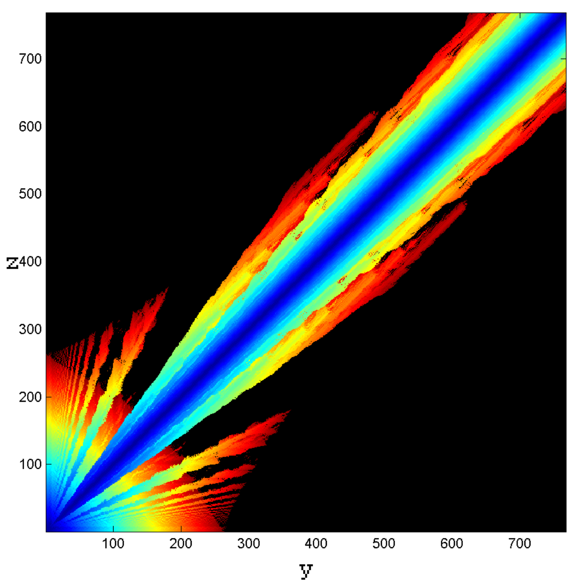
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Figure 4

Technical Appendices:

<u>Appendix A</u>: Derivation of recursion relation $W_{x+1}= R\ W_x$

Recall the following definitions:

- Position p=[x,y,z] defines the current game configuration, with x specifying the number of columns of height three, y the number of columns of height two, and z the number of height one (see Fig. 1 of main text). Each position p is classified as a *winner* (if a player starting from that position can always force a win under optimal play), or as a *loser* otherwise.

- A *loser sheet* L_x is the infinite, two-dimensional matrix whose $(y,z)^{th}$ component is a 1 if position [x,y,z] is a loser, and a 0 otherwise. The set of loser sheets $\{L_0, L_1, L_2, \ldots\}$ provides a convenient way of organizing the game's losing positions in a hierarchical fashion according to their x values; each loser is uniquely assigned to a particular loser sheet.

- A winning position p=[x,y,z] called an *instant* winner if from that position a player can reach a losing position with a smaller x-value in exactly one move. (The set of instant winners is a <u>subset</u> of the winners.) An *instant-winner sheet* W_x is defined to be an infinite, two-dimensional matrix whose $(y,z)^{th}$ component is a 1 if position [x,y,z] is an instant winner, and a 0 otherwise (i.e., the instant-winner sheets group the instant winners according to their x values).

Consider an arbitrary position p=[x,y,z]. Under the game rules, the set of positions which are accessible from p in a single move (i.e., the "children" of p) are:

$[x, y-t, z+t]$	$0 < t \le y$	M1
$[x, y-t, 0]$	$0 < t \le y$	M2
$[x, y, z-t]$	$0 < t \le z$	M3
$[x-t, y+t, z]$	$0 < t \le x$	M4
$[x-t, 0, z+y+t]$	$0 < t \le x$	M5
$[x-t, 0, 0]$	$0 < t \le x$	M6

By inverting these relations, the set of positions from which it is possible to reach p in a single move (i.e., the "parents" of p) can be readily determined. Note that the first three moves (M1-M3) are to positions with the same x value as p, while the last three are to positions with lower x values. We also note three elementary though useful facts:

(*) The children of any winning position must contain at least one loser.

(**) The parents of any losing position must all be winners.

(***) If all children of a position are winners, then the position must be a loser.

We next define a set of operators which can act on sheets (here, "sheets" can refer to the loser sheets, the instant-winner sheets, or, more generally, to any infinite, two-dimensional matrix consisting of 0's and 1's). In what follows we denote a general sheet by A, and its $(y,z)^{th}$ component by A(y,z), where y specifies the column of matrix A, and z its row. The labeling begins with zero (i.e., $y,z \in \{0,1,2,\dots\}$); y=0 refers to the leftmost column, and z=0 to the bottom row:

- Define **L** to be the *left-shift* operator which shifts all elements of a sheet to the left: **L**A(y,z)=A(y-1,z). (In other words, **L** effectively eliminates the leftmost column of A.)
- Define an *addition* operator + on sheets by the logical OR: Given two sheets A,B, (A+B)(y,z)=1 if either A(y,z)=1 or B(y,z)=1.
- Define **I** to be the identity operator: **I**A=A for any sheet A.
- Define the *diagonal element-adding* operator **D,** which acts on loser sheets, as follows: Observe first that for each loser sheet L_x there is a unique losing position with y=0, i.e., there is a unique z-value (denoted z*(x)) such that $L_x(0,z^*(x))=1$. (Such a loser must exist since the set of instant winners is bounded in z; uniqueness holds by (**) in conjunction with M3). Now define operator **d** which acts on loser sheets as follows: **d**L_x is a matrix which is zero everywhere except at the following points: **d**$L_x(t,z^*(x)-t)=1$ for all $0 \le t \le z^*(x)$ (Geometrically, the non-zero components of matrix **d**L_x form a diagonal line of 1's extending downwards and to the right from the losing entry at (0,z*(x)) at a 45° angle.) Finally, the desired diagonal element-adding operator **D** is defined by **D**=**I**+**d**. Geometrically, **D**L_x is thus identical to L_x except for the addition of the diagonal row of 1's. Henceforth, we will call the diagonal elements associated with **D** the "deadly diagonals" (since, as will become clear, they tend to complicate the analysis considerably).

Using the above operators, we next derive a relationship between a given instant winner sheet W_x and the set of loser sheets $\{L_{x-1}, L_{x-2}, \ldots, L_0\}$. First recall that, by definition, position p=[x,y,z] is an instant winner in W_x (i.e., $W_x(y,z)=1$) iff there is a loser reachable from p with a lower x value. So to construct W_x we need only determine which positions [x,y,z] are capable of reaching a loser in $\{L_{x-1},L_{x-2},\ldots,L_0\}$ (i.e., we seek the parents of the losers). For this purpose only moves M4-M6 are relevant (since moves M1-M3 maintain the same x-value.) From M4 (or its inverse), it is easily verified that the terms $\mathbf{L}L_{x-1}$, $\mathbf{L}^2L_{x-2}, \ldots,\mathbf{L}^xL_0$ will all contribute to W_x. From M5 (and its inverse), terms $\mathbf{Ld}L_{x-1}$, $\mathbf{L}^2\mathbf{d}L_{x-2}, \ldots, \mathbf{L}^x\mathbf{d}L_0$ will contribute. There are no contributions from rule M6 because, by Gale's argument, positions of the form [x,0,0] are never losers (for any x>0). Thus, combining all contributions we have

$$W_x = \sum_{t=1}^{t=x} \mathbf{L}^t(\mathbf{I}+\mathbf{d})\, L_{x-t} = \sum_{t=1}^{t=x} \mathbf{L}^t\mathbf{D}\, L_{x-t} \qquad (1)$$

We next show that W_x contains all the necessary information for constructing L_x. Specifically, we develop a sheet-valued mex operator **M** (which we dub "supermex") and show that

$$L_x = \mathbf{M}\, W_x \qquad (2)$$

To understand the construction of L_x from W_x via the supermex operator, we begin with a preliminary observation: if we rank all positions in the game by size using the standard "dictionary" ordering (i.e., define [x,y,z] > [x',y',z'] if either (x>x') or (x=x' and y>y') or (x=x', y=y', and z>z')), then as play progresses the size of successive positions will strictly decrease, as may be verified by considering the game rules M1-M6 (i.e., children are always smaller than their parents). Now the basic intuition behind (2) is as follows: Start with the instant-winner sheet W_x, and locate the smallest position which is not an instant winner (i.e., the smallest (y,z) such that $W_x(y,z)=0$). Call this position q. q must be a loser, as shown by the following argument: Since q is not an instant winner, then by definition none of its children can be losers with a lower x-value. Moreover, none of its children can be losers with the same x-value either, since q is (by construction) the *smallest* non-instant-winner in $W_{x,}$ and so its children in W_x (which under the dictionary ordering must be smaller in size) can only be instant winners. Hence, all the children of q are winners, and by (***) it follows that q is a loser. Having thus identified the first loser in L_x, we can then find the next loser as follows: First find all parents of the first loser q which have the same x-value as q (using rules M1-M3) and mark these as "winners" in the W_x matrix by setting the appropriate entries equal to 1. (Note that both these new winners and the original instant winners in W_x will be labeled with a 1.) We point out that in the formal algorithm presented below, we do no actually want to alter W_x itself (since it is well-defined via (1)), so instead we define a scratch sheet T_x given by $T_x=W_x$, and make the alterations to T_x. Having marked the parents of q, we again find the smallest remaining position in T_x (apart from q) which is neither a winner or instant winner. By the same argument used above, this position must be a loser. Through this iterative process (of marking parents of losers and finding the smallest remaining non-winner), we can generate all losers in L_x.

More formally, we can algorithmically define the action of the supermex operator **M** on an instant-winner sheet W_x to generate L_x (equation (2)) as follows:

Supermex algorithm:

1. Set $L_x=0$ (i.e., $L_x(y,z)=0$ for all $y,z \in \{0,1,2,\ldots\}$)
2. Set $T_x=W_x$

3. Set y=0 (i.e., we will start with the first column of T_x)
4. Let $z_{small}(y)$ =mex({z|$T_x(y,z)=1$}) (i.e., find the z-value of the smallest non-winner)
5. Set $L_x(y, z_{small}(y))=1$ (i.e., mark the point as a loser)
6. Set $T_x(y,t)=1$ for all $t \geq z_{small}(y)$ and set $T_x(y+t, z_{small}(y)-t)=1$ for all $t \leq z_{small}(y)$
7. Set $y \rightarrow y+1$
8. If $z_{small}(y)=0$ stop; else go to step 4.

Several remarks are in order. (i) The mex operator in step 4 is the standard minimal excluded value operator defined on sets of non-negative integers (e.g., mex({0,1,2,5,7})=3; mex({1,4})=0). (ii) The program finds the smallest non-winners by searching T_x column by column, starting with column y=0 (steps 2 and 7). (iii) step 6 corresponds to finding the parents of the current smallest losing position via rules M3 and M1 and marks them as winners. (Any winners which are not instant winners are called "implied" winners in Zeilberger's terminology.) Geometrically, the implied winners arising from rule M3 fill up the entire column directly above the loser, while the implied winners from rule M1 form a diagonal line running down and to the right from the loser at a 45° angle. This diagonal line will prove important for later considerations, and in support of Zeilberger's terminology, we will refer to this diagonal as an "implied" diagonal. (Implied diagonals are distinct from the "deadly diagonals" associated with the operator **D** discussed previously.) (iv) if $z_{small}(y)=0$ (step 8), then by rule M2 no other losers in L_x exist and the search terminates. (v) if $z_{small}(y) \neq 0$ for any y, then the computation will not terminate. (As we show later, the probability of it terminating is

$\gamma = \sqrt{2} - 1.$) However, Byrnes [22] has shown that even when it does not terminate it does eventually becomes periodic and hence predictable.

Now, combining our expressions for W_x and L_x (equations (1) and (2)), we have

$$W_x = \sum_{t=1}^{t=x} \mathbf{L}^t \mathbf{D}\,\mathbf{M} W_{x-t}$$

Finally, we make the critical observation that if we create W_{x+1} by substituting $x \rightarrow x+1$ above and then compare it to the original expression for W_x, it becomes possible to re-express W_{x+1} in terms of W_x as follows:

$$W_{x+1} = \mathbf{L}\,(\mathbf{I} + \mathbf{D}\,\mathbf{M})\, W_x \equiv \mathbf{R}\, W_x \qquad (3)$$

where

$$\mathbf{R} \equiv \mathbf{L}\,(\mathbf{I} + \mathbf{D}\,\mathbf{M}) \qquad (4)$$

Appendix B: Fixed point of the renormalization operator $\mathbf{W} \rightarrow \mathbf{S}\,\mathbf{R}\,\mathbf{W}$

In this appendix we derive analytical values for the six key parameters characterizing the overall geometric structure of the loser and instant-winner sheets. Recall that the geometry of the loser sheets L_x (Fig. 2c of main text) consists of three diffuse lines: a lower line of slope m_L and density of points (per unit y) λ_L, an upper line of slope m_U and density λ_U (per unit y), and a flat line extending to infinity. The upper and lower lines originate from a point whose height (i.e., z-value) is αx. The flat line (with density one) is altogether absent in some loser sheets; we define γ to be the probability that a flat loser line is present in a randomly selected loser sheet. We will show here that these parameters must have the following values:

$$\begin{aligned}
\alpha &= \tfrac{1}{\sqrt{2}}, \\
\lambda_L &= 1 - \tfrac{1}{\sqrt{2}}, \\
\lambda_U &= \tfrac{1}{\sqrt{2}}, \\
m_L &= -1 - \tfrac{1}{\sqrt{2}}, \\
m_U &= -1 + \tfrac{1}{\sqrt{2}}, \\
\gamma &= \sqrt{2} - 1
\end{aligned} \tag{5}$$

These are the (unique) parameter values for which the W_x's and L_x's can exhibit the observed scaling invariance. Said differently, these parameter values define an invariant geometry of the instant-winner sheets that is a fixed point of the renormalization-group operator **SR**.   As we show below, this invariance property can be expressed via a set of algebraic equalities, the solution of which leads to the above parameter values.

To begin, we express the recursion relation (3) as

$$W_{x+1} = \mathbf{L}\,(W_x + \mathbf{DM}W_x) \tag{6}$$

and observe that the new instant-winner sheet W_{x+1} is generated from the old W_x by the following sequence of steps: First, the mex operator acts to create the loser sheet L_x (i.e., $\mathbf{M}W_x = L_x$).   This loser sheet is then modified by the operator **D**, which adds a diagonal line of 1's into the matrix. We then add this modified loser sheet ($\mathbf{D}L_x$) to the original W_x sheet, and then simply left-shift the whole sheet, yielding W_{x+1}.

The most critical stage of this "growth" of W_x into W_{x+1} is when the original instant-winner sheet W_x is altered by the addition of the modified loser sheet $\mathbf{D}L_x$. To best understand this change, we decompose $\mathbf{D}L_x$ into its four basic components: the lower loser line in L_x, the upper (tilted) loser line in L_x, the flat loser line in L_x (when it exists), and the deadly diagonal line created by **D**. (See Fig. A1 of the appendix for a geometric depiction of these lines.) This in turn allows us to similarly decompose any W_x: The basic idea is that as we recursively construct W_x via $W_0 \xrightarrow{\mathbf{R}} W_1 \xrightarrow{\mathbf{R}} \ldots \xrightarrow{\mathbf{R}} W_{x-1}$, at each step we are adding a modified loser sheet to the current instant-winner sheet. Hence, once W_x is built up, we can separate out the contributions that came from lower loser lines, upper loser lines, flat lines, and deadly diagonals. Figure A2 of the appendix shows a typical W_x (for x=100); Figure A3 shows its decomposition into the four sets (defined by which component of the modified loser sheets contributed). We label these sets LL (for lower loser lines), U (for upper loser lines), F (for flat loser lines), and DD (for deadly diagonals). Note that DD overlaps with U and with F, but that no other sets overlap. In particular, LL, U, and F cannot overlap with one another for the simple reason that as the three loser lines (in a modified loser sheet) are being laid down, they never intersect the current instant-winner sheet to which they are being added (since they are losers). DD and LL do not overlap for a simple geometrical reason: each time a new DD line is laid down, it is located above the existing LL region (since the LL lines have slope $m_L<-1$, whereas the DD lines have slope -1). Note that while we have ignored the left-shift operator **L** in eqn. (6) in the above discussion, the action of this shift operator is in a certain sense trivial, and does not alter any of the preceding conclusions.

Each of the regions LL, U, F, and DD in Fig. A3 are constructed from a series of lines. (The series of deadly diagonal lines and the flat lines making up DD and F respectively are clearly visible in the figure; the lower loser lines in LL and upper loser lines in U are less apparent primarily because the lower and upper loser lines (Fig. A1) which are being laid down are not solid (i.e., their density of points is less than one).) We now calculate the density of lines in LL, DD, U, and F. We will start with LL, DD, and U, since they all follow from the same argument: Consider a given W_x. The lower loser line, upper loser line, and deadly diagonal line in $\mathbf{D}L_x$ (that will eventually be added to W_x to create W_{x+1}) all originate from the same point (y,z)=(0,αx). Thus their initial "height" in the y-z plane is αx, and all have the same general form z=my+ αx. In the construction of W_{x+1} from W_x, these loser lines are added to W_x and then the resulting sheet is left-shifted (as described by the equation (6)). In this process, these former loser lines become the leading edges of the new instant-winner sheet W_{x+1}. Their height (in W_{x+1}) becomes αx+m (since the operator **L** shifted these lines leftward by one unit, thereby dropping their height by |m|). Now consider the new loser lines for W_{x+1} (i.e., $\mathbf{D}L_{x+1}$), which will be used in the construction of W_{x+2}. These new loser lines will come in at height α(x+1). When these are added to W_{x+1} and left-shifted, they become the leading edge of W_{x+2} at height α(x+1)+m. Meanwhile, the old lines (which had been the leading edge of W_{x+1}) will drop in height again by |m| owing to the second left-shift (to height αx+2m), and will now form the next-to-leading edge of W_{x+2}. Comparing the heights of the lines forming the leading edge and next-to-leading edge of W_{x+2} yields a difference of α-m. This is the vertical spacing between successive lines comprising the LL, U, and DD regions of an instant-winner sheet. Hence, the density of lines (per unit z) is

$$\frac{1}{\alpha - m} \qquad \text{(vertical line density)} \qquad (7)$$

where m = m_L, m_U, -1 for lower losers, upper losers, and deadly diagonals, respectively. It follows from elementary geometry that the density of lines per unit y is

$$\frac{-m}{\alpha - m} \qquad \text{(horizontal line density)} \qquad (8)$$

where, as before, m = m_L, m_U, -1 for lower losers, upper losers, and deadly diagonals, respectively. We emphasize that when applied to LL and DD, results (7), (8) give the average density of the lower and upper *lines* in an instant winner sheet – they do not represent the actual density of *points* making up those lines (since the lower and upper loser lines are not in fact solid, but are comprised of points with densities λ_L,λ_U, respectively). Lastly, the density (per unit z) of the flat lines in F is also readily calculated. Recall that the probability that a flat loser line exists in a randomly selected L_x is γ. So, for a given W_x, the total number of flat lines that have been generated during the recursive construction procedure is simply γx. From Fig. A1 we see that the vertical extent of these lines is simply $\alpha x(1 - m_U / m_L)$. Thus, we have

$$\frac{\gamma}{\alpha(1 - \frac{m_U}{m_L})} \qquad \text{(vertical density of flat loser lines)} \qquad (9)$$

We now derive six algebraic constraints on the geometric structure of instant-winner and loser sheets which, when solved, yield the desired parameter values given in (5).

Constraint 1: *Existence and uniqueness of losers*

Consider the region of a loser sheet L_x where the upper and lower loser lines exist. Then there exists a unique loser in each column of L_x in this region. This result follows directly from the supermex algorithm described above. (The coordinates of these unique losers are $(y,z_{small}(y))$). Accordingly, since λ_U, λ_L denote the density of losers (per unit y) along the upper and lower loser lines, we have

$$\lambda_U + \lambda_L = 1$$

<u>Constraint 2</u>: *Region III is forbidden*

Notice that the upper triangular region of the W_x's (labeled region III of Fig. A2) is devoid of losers for all x. In particular, when the losers L_x are constructed from the instant-winner sheet W_x via the supermex operator (L_x=**M**W_x), they are forbidden from appearing in any of the existing "holes" in region III (i.e., locations (y,z) where $W_x(y,z)=0$). The mechanism preventing their appearance there is the implied diagonals generated during the supermex operator. Specifically, each time a new loser along the upper loser line is created, it casts down an implied diagonal (see step 6 of the supermex algorithm) and thereby fills in some of remaining holes in region III. These implied diagonals cannot overlap with one another, and as a set must effectively fill up all holes in region III (which is necessary for the scaling property of the W_x's to be preserved). The condition on the implied diagonals filling the gaps is actually stronger than it might appear at first glance: since the locations of the gaps are not well correlated with the locations of the upper losers, in fact the implied diagonals cast down not only just fill the gaps, but in fact densely fill region III. Now, since the slope of the upper loser line is m_U, and the losers are scattered with density λ_U (per unit y) along this line, we can calculate the density (per unit vertical z) of the implied diagonals they cast off via a simple geometric argument: The average horizontal separation between successive losers on the upper loser line is $1/\lambda_U$, and since the slope of the line is m_U, their average vertical is $-m_U/\lambda_U$. (See Fig. A4). The average vertical spacing between implied diagonals is thus $1/\lambda_U + m_U/\lambda_U$, and so their density per unit vertical is simply the reciprocal of this, namely $\lambda_U/(1+m_U)$. Hence, demanding that the implied diagonals fill the region, we set their density equal to one, which yields:

$$\frac{\lambda_U}{1+m_U} = 1$$

(An entirely analogous argument applied to region F shows that the density of losers along the flat loser line equals unity.)

Constraint 3: *Region II is forbidden*

Region II is made up of contributions from DD (along with F in the lower part of the region and U in the upper part), with DD playing the key role. In particular, losers are prevented from appearing inside region II because the implied diagonals which are cast down by losers on the lower loser line during the supermex process mesh perfectly with the existing DD, thereby completely filling the region and making it forbidden to losers. (The underlying reason behind the perfect meshing of the implied diagonals and DD is that every loser created on the lower loser line must necessarily be filling in an existing hole, and so there could not have been a DD line already there. Thus, the implied diagonal emanating from the loser will not overlap any DD. Moreover, as we move across the columns of W_x and lay down lower losers (via the supermex operation), the lowest remaining hole in the current column must necessarily get filled by a loser.

Hence, the implied diagonals and the DD's must mesh completely, entirely filling every column of region II.) So the constraint is that the densities (per unit vertical) of the DD's and the implied diagonals must sum to unity. The density of the DD's is given by eqn. (7) above with m=-1. The density of the implied diagonals associated with the lower losers can be found via the same geometric-type argument used to find the density of implied diagonals for the upper losers in Constraint 2 (see the construction in Fig. A4): $-\lambda_L/(1+m_L)$. Thus, the constraint for region II is

$$\frac{1}{\alpha+1}-\frac{\lambda_L}{1+m_L}=1$$

Constraint 4: *Bottom row of region I is forbidden*

Observe from Fig. A2 that the bottom-most row (z=0) of region I is completely filled, and hence forbidden to losers. (The same is true for all other rows of region I, but those will be handled separately.) We now derive the constraint that must be satisfied in order that the bottom row of every W_x (for all x-values) always remains filled. To begin, note that only LL and DD contribute to the bottom row of region I, and that they do not overlap. (F would potentially contribute too were it not for the fact that the bottom row of F happens to be completely empty.) Since the bottom row of region I is completely filled, it follows that the densities of the DD's and LL (in the bottom row) must sum to unity. The density of the DD's has been calculated previously (eqn. (7) with m=-1). The density of LL along the bottom row follows from a critical observation: Whenever a flat loser line is absent in a loser sheet (which occurs with probability 1-γ), it is because a loser with height zero has been generated in the lower loser line (recall step 8 of the supermex algorithm). Hence, the average density (per unit horizontal) of losers along the bottom row of a given W_x must equal the average density (per unit horizontal) of the LL's (eqn (8)) times the probability that a height-zero loser is generated (1-γ). This yields $(\gamma-1)m_L/(\alpha-m_L)$. Setting the sum of the densities of the DD's and LL in the bottom row equal to unity yields the desired constraint:

$$\frac{(\gamma-1)m_L}{\alpha - m_L}+\frac{1}{\alpha+1}=1$$

<u>Constraint 5</u>: *Lower Region I is forbidden*

All rows of region I (not only the bottom row as discussed above) are forbidden to losers. Consider now a row in the lower part of region I, where LL, DD, and F all contribute (but not U). The sum of these three contributions fills up the entire row. To simplify the analysis we select a row in which the horizontal band of F is entirely empty. Hence LL and DD alone must fill this row, and since they do not overlap, it follows that the sum of the densities (per unit y) of DD and LL must equal unity for that row. Now, the density of the DD's is already known (eqn. (7) with m=-1), so we need only compute the density of points contributed from LL. At first glance, it might seem that we could do so simply by multiplying the horizontal density of the LL lines (eqn. (8) with $m=m_L$) by the expected number of points contributed to a given row by each LL line ($-\lambda_L/m_L$). However, this naïve argument misses the fact that we have restricted consideration to a row in region I in which the horizontal band coming from F happens to be empty, but have ignored the fact that the location of points along a LL line are strongly correlated with empty/filled rows of F. Indeed, points in LL can only exist in empty bands of F. The proper calculation goes as follows: Consider a segment of a LL line. Let its vertical extent be H, so its horizontal extent is $-H/m_L$ (since it has slope m_L). The expected total number of points distributed along this LL line segment is $(-H/m_L)\lambda_L$. Now, from eqn. (9), the number of empty bands of F that will 'hit' the line segment is just $(1-\gamma/\alpha/(1-m_U/m_L))H$ (i.e., the vertical density of empty bands times the vertical height of the line segment). So the number of points on the segment per empty band is $-\lambda_L\alpha(m_L - m_U)/m_L/(\alpha m_L - \alpha m_U - \gamma m_L)$. Multiplying this by the density of LL lines (eqn. (8)) yields the desired horizontal density of points contributed by LL along the row. Finally, setting the sum of this density and the density of the DD's to unity, we find

$$\frac{\lambda_L\alpha(m_U - m_L)}{(\alpha - m_L)(\alpha m_U - \alpha m_L + \gamma m_L)} + \frac{1}{\alpha+1} = 1$$

<u>Constraint 6</u>: *Upper Region I is forbidden*

Consider a row in the upper portion of region I, where U, DD, and LL all contribute (but not F). Note that U and DD overlap with one another, though not with LL. Since the row is completely filled, we have density(LL)+density(U+DD) =1. Now, the density of points contributed to a row by the LL lines is easily found: The density of the LL lines themselves is given by eqn. (8), while the average number of points contributed to a given row by each LL line is just $-\lambda_L/m_L$. So the (horizontal) density of points in a row from LL is simply $\lambda_L/(\alpha- m_L)$. (Note: the intermediate expression ($-\lambda_L/m_L$) used in the above calculation is easily obtained: Consider a segment of a LL line. Let H denote its vertical extent H and $-H/m_L$ its horizontal extent. So the total number of points along this segment is $(-H/m_L)\lambda_L$. Dividing by H yields the expected number of points per row contributed by the line.) Using a similar argument, the (horizontal) density of points contributed in a row contributed from U is $\lambda_U/(\alpha- m_U)$. The density from DD is $1/(\alpha+1)$. Assuming no correlations between the contributions from U and DD, the expected density of their combined contributions is 1 - (1- $\lambda_U/(\alpha- m_U)$)(1-1/(α+1)). Adding this to the density from LL, the requirement that the row is completely filled becomes

$$\frac{\lambda_L}{\alpha - m_L} - \frac{\alpha}{\alpha+1}\left(1-\frac{\lambda_U}{\alpha - m_U}\right) = 0$$

Solving these six algebraic constraint relations yields exact values for the key parameters characterizing the geometric structure of the game. These values are given in (5) above.

Appendix C: Location and uniqueness of winning opening move

Consider a starting position of Chomp $p_0=[x_0,0,0]$. We wish to identify losing positions which are reachable from p_0. Fig. A1 provides a general geometric depiction of all losers in a generic loser sheet L_x, and rules M1-M6 of Appendix A specify all possible moves. A quick inspection reveals that only moves of the type M4 and M5 are viable candidates for reaching a loser from p_0. We start with M4. The reachable positions from p_0 are of the form $[x_0-t, t,0]$. Hence we seek a loser in sheet L_{x_0-t} which lies on the lower loser line at a height z=0. Since the lower lose line has the functional form $z=m_L y+\alpha(x_0-t)$, this requires $y=-\alpha(x_0-t)/m_L$. Hence, we set $t=y=-\alpha(x_0-t)/m_L$. Solving for t, the desired loser location is:

$$\left[\frac{-m_L x_0}{\alpha - m_L},\ \frac{\alpha x_0}{\alpha - m_L},\ 0\right] \qquad \text{(L1)}$$

Consider now M5. The reachable positions from p_0 are of the form $[x_0-t', 0, t']$. From fig. A1 we see that we seek a loser in sheet $L_{x_0-t'}$ located at height $z=\alpha(x_0-t')$ in the y=0 column (i.e., where the upper and lower loser lines meet). Setting $t'= \alpha(x_0-t')$ and solving, the desired loser location is

$$\left[\frac{x_0}{\alpha+1},\ 0,\ \frac{\alpha x_0}{\alpha+1}\right] \qquad \text{(L2)}$$

Hence we conclude that the only possible losing positions which are accessible from p_0 <u>must</u> lie *in the vicinity* of points (L1) or (L2). (Note that (L1), (L2) are not themselves actual losing positions (since they are not integers) – recall from Fig. 2 of main text that losers tend to be scattered within a narrow band surrounding the analytical loser lines, and are usually not directly on these lines). We point out here that the "vicinity" of (L1), (L2) is determined by the width of the loser band surrounding the analytical loser lines in loser sheets $\{L_x\}$. Numerical simulations indicate that this width appears to have a global maximum bound (i.e., for all x) of less than 2.5. Though this bound has yet to be proven analytically, it is in agreement with the general heuristic argument that the supermex operator **M**, by its very construction, tends to place new losers as close as possible to existing boundaries in the instant-winner sheets (i.e., the losers "hug" the analytical loser lines).

From the above considerations, it might appear that there could exist multiple losers in the vicinity of either (L1) or (L2) which are all accessible from starting position p_0, in which case there would be more than one winning opening move. However, as we show now, this is not the case – there is but a single losing position accessible from p_0, and hence the winning opening move is unique. To see this, start by considering the set of possible losing positions in the vicinity of (L1). They will all have the general form $[x_0\text{-}t, t, 0]$, where t now represents some integer close to the original (non-integer) value of t used above to determine (L1). Suppose more than one of these positions is actually a loser. Let L1* denote the largest of these losers (in the dictionary ordering). By rule M4, however, it is easy to see that the other losers in the neighborhood will be reachable from L1*. Hence, this must mean that these other supposed losers cannot in fact be losers – since a losing position is never accessible from another losing position. So we conclude that there can be at most one losing position in the neighborhood of (L1). Likewise, by noting that the possible losing positions in the vicinity of (L2) must all have the general form $[x_0\text{-}t', 0, t']$, and invoking rule M5, we use a similar argument to show that there can be at most one loser in the vicinity of (L2). Thus far, we see that we can have, at most, one loser near (L1) and one loser near (L2). Suppose both exist. Observe, however, that the loser near (L2) would be accessible from the loser near (L1) by rule M5 (since $t' > t$), leading to a contradiction (see (*) of Appendix A). Hence, there is a unique loser which is accessible from initial position p_0, and so the winning opening move is unique.

<u>Appendix D</u>: Recursion operator for three-heap Nim

Let position p=[x,y,z] specify the current configuration of the game, with x denoting the number of counters in the first heap, y the number in the second, and z the third. The positions which are reachable from position p in a single move are given by

$$[x - t, y, z] \quad 0 < t \le x \quad M1$$
$$[x, y - t, z] \quad 0 < t \le y \quad M2$$
$$[x, y, z - t] \quad 0 < t \le z \quad M3$$

As in Chomp, let L_x represent a loser sheet ($L_x(y,z)$=1 if [x,y,z] is a loser and 0 otherwise), and W_x an instant-winner sheet. From M1 it is clear that the parents of an arbitrary position p=[x,y,z] at a higher x-value will have the same (y,z) as p. Thus, using the parents of losers in the loser sheets to construct the instant-winner sheet yields

$$W_x = L_0 + L_1 + \ \dots \ + L_{x-1}$$

The loser sheets are constructed from the instant-winner sheets via a supermex operator **M** (*not* the same supermex operator as in Chomp)

$$L_x = \mathbf{M} \, W_x$$

where the action of **M** on an instant-winner sheet W_x is defined algorithmically via

<u>Supermex algorithm:</u>

1. Set L_x=0 (i.e., $L_x(y,z)$=0 for all $y,z \in \{0,1,2,\dots\}$)
2. Set T_x=W_x (T_x serves as a scratch sheet)
3. Set y=0 (i.e., we will start with the first column of T_x)

4. Let $z_{small}(y)$ =mex({z|$T_x(y,z)=1$}) (i.e., find z-value of smallest non-winner)
5. Set $L_x(y, z_{small}(y))=1$ (i.e., mark the point as a loser)
6. Set $T_x(y, z_{small}(y)+t)=1$ and $T_x(y+t, z_{small}(y))=1$ for all $t \geq 0$ (see M3, M2)
7. Set $y \rightarrow y+1$
8. Go to step 4.

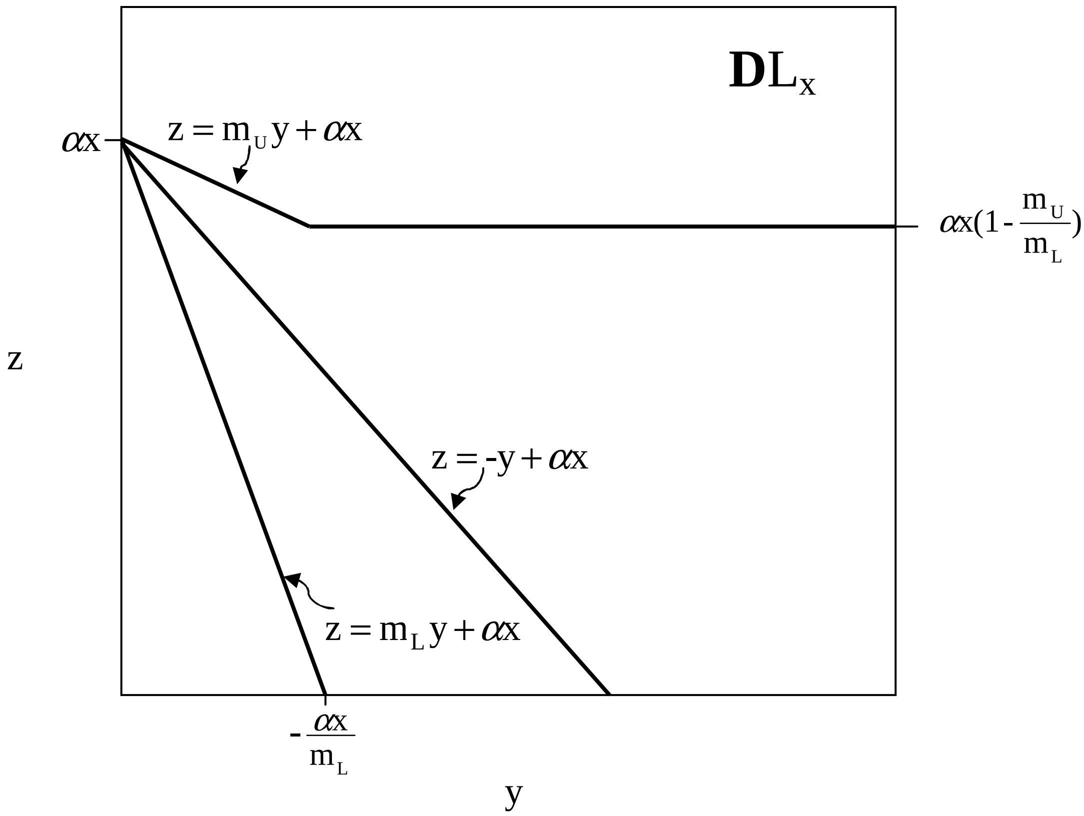
Combining the expressions for W_x and L_x above yields

$$W_x = \sum_{n=0}^{n=x-1} \mathbf{M} W_n$$

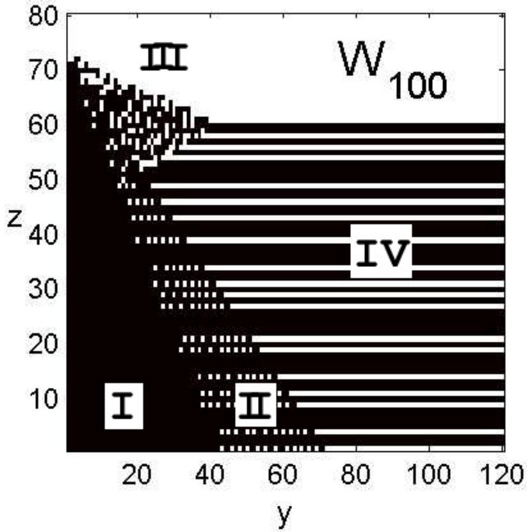
Using this expression to construct W_{x+1}, and then comparing it to W_x, leads to the desired recursion relation for Nim:

$$W_{x+1} = (\mathbf{I}+\mathbf{M})\, W_x$$

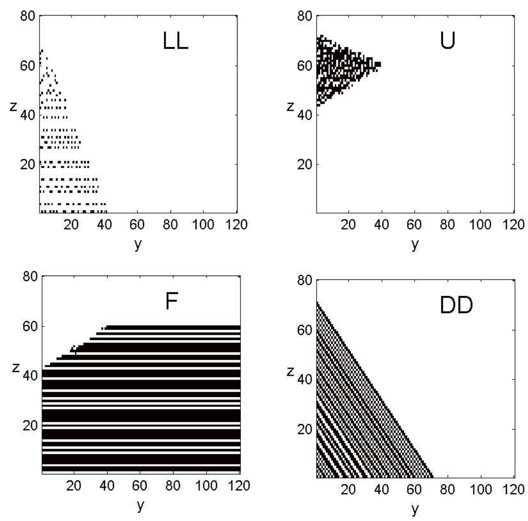
where **I** denotes the identity operator.

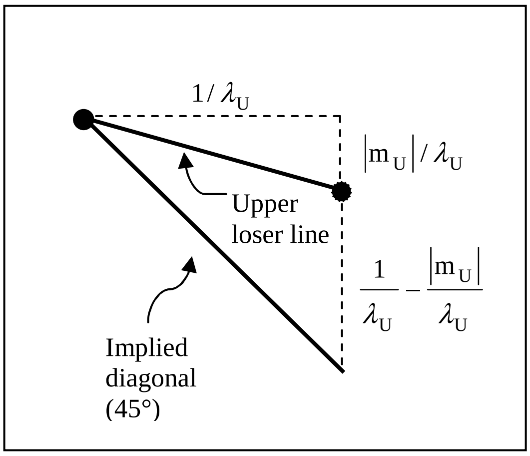


DLx
αx
z = m_U y + αx
αx(1 - m_U/m_L)
z
z = -y + αx
z = m_L y + αx
- αx/m_L
y



W100
III
IV
I
II
z
y


LL
U
F
DD
z
y
0
20
40
60
80
100
120



$1/\lambda_U$
$|m_U|/\lambda_U$
Upper loser line
$\frac{1}{\lambda_U} - \frac{|m_U|}{\lambda_U}$
Implied diagonal (45°)