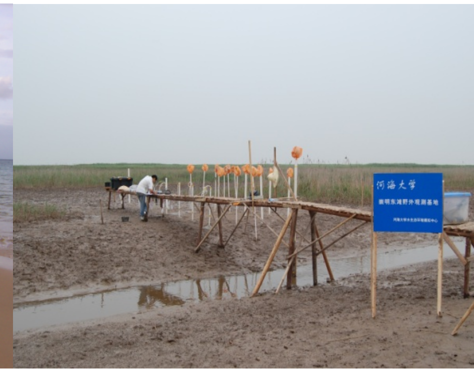





Capillary effects on groundwater waves in unconfined coastal aquifers

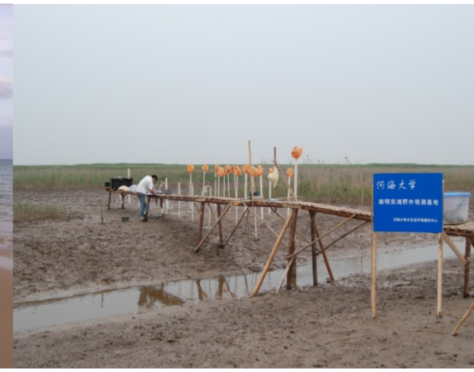
Jun Kong, Cheng-Ji Shen, Pei Xin, Zhi-Yao Song, Ling Li, D.A. Barry, D.-S. Jeng, D. A. Lockington, F. Stagnitti and J.-Y. Parlange

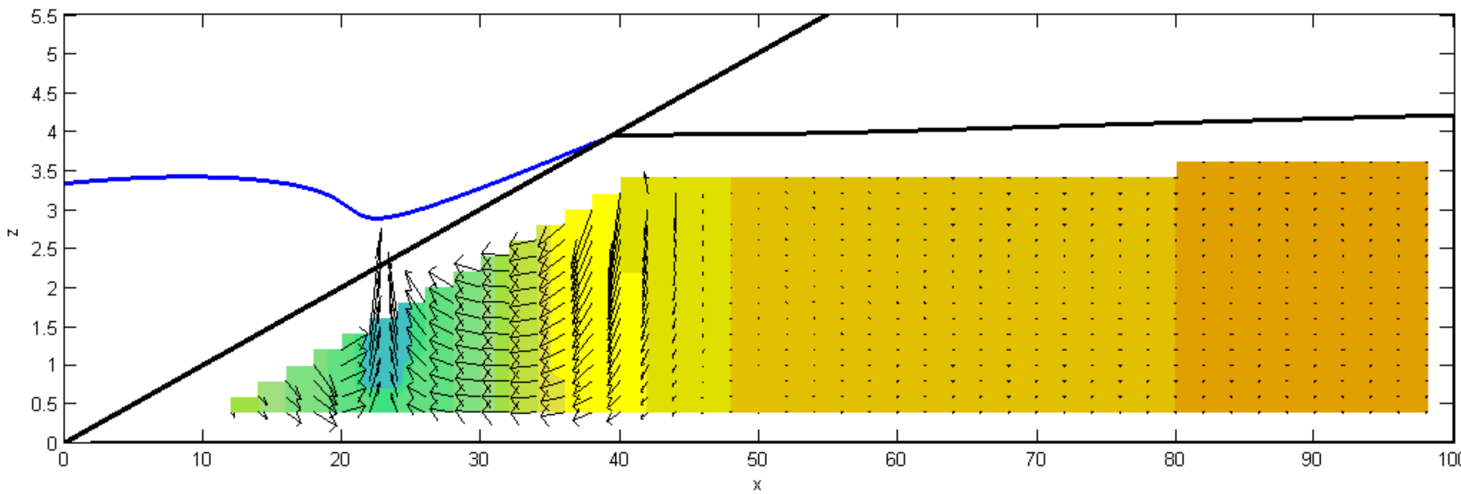



Outline

- Parlange and Brutsaert 1987
- Barry et al. 1996
- Li et al. 1997, 2000
- Liu and Wen 1997
- Nielsen and Perrochet 2000
- Jeng et al. 2003, 2005
- Cartwright et al. 2005, 2006
- Xin et al. 2010
- Kong et al. 2012



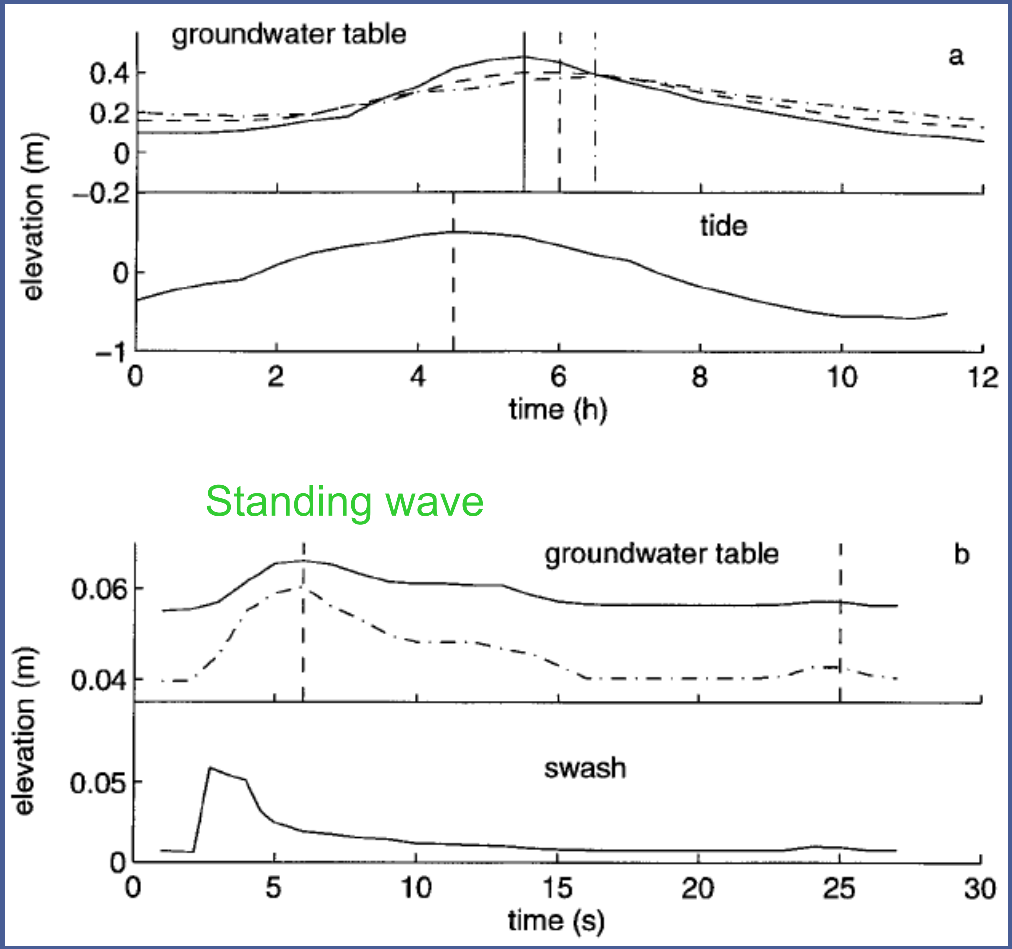


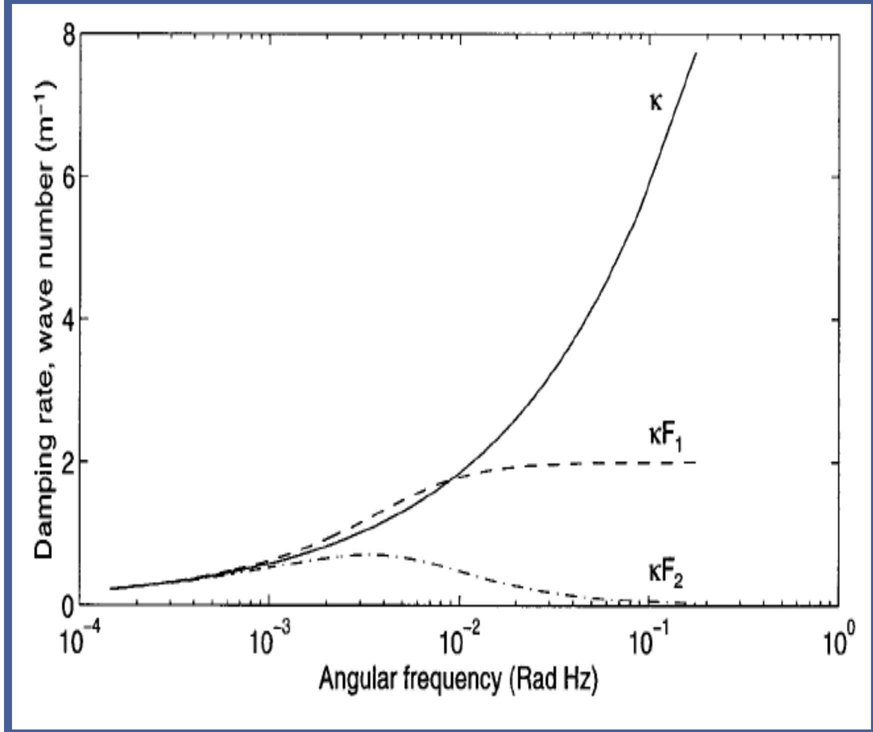

5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
z
0
10
20
30
40
50
60
70
80
90
100
x



High frequency watertable fluctuations

- Tide- and wave-induced groundwater table fluctuations





Starting point and early work

- Parlange and Brutsaert 1987 – modified Boussinesq equation

$$n_e \frac{\partial \eta}{\partial t} = K \frac{\partial}{\partial x}\left(\eta \frac{\partial \eta}{\partial x}\right) + \boxed{B \frac{\partial^2}{\partial t \partial x}\left(\eta \frac{\partial \eta}{\partial x}\right)}$$

$$B = \int_{-\infty}^{0} (\theta - \theta_{res}) d\psi$$

- Barry et al. 1996: solution for periodic forcing

$$\eta(0, t) = D[1 + \varepsilon \cos(\omega t)]$$

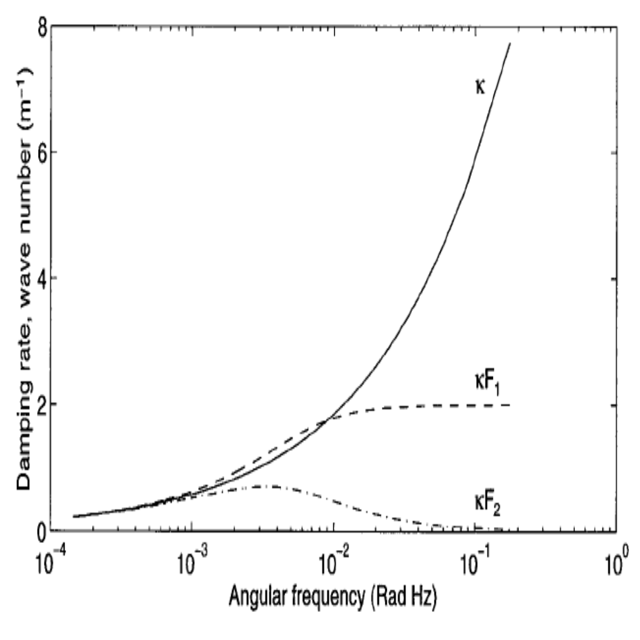
$$\eta(x, t) = D[1 + \varepsilon \exp(-x\kappa F_1) \cos(\omega t - x\kappa F_2)]$$

$$\kappa = \sqrt{\frac{n_e \omega}{2KD}},$$

$$F_1 = \sqrt{\frac{N_{\mathrm{CAR}}}{\sqrt{1 + N_{\mathrm{CAR}}^2}} + \frac{N_{\mathrm{CAR}}}{1 + N_{\mathrm{CAR}}^2}},$$

$$F_2 = \sqrt{\frac{N_{\mathrm{CAR}}}{\sqrt{1 + N_{\mathrm{CAR}}^2}} - \frac{N_{\mathrm{CAR}}}{1 + N_{\mathrm{CAR}}^2}}.$$

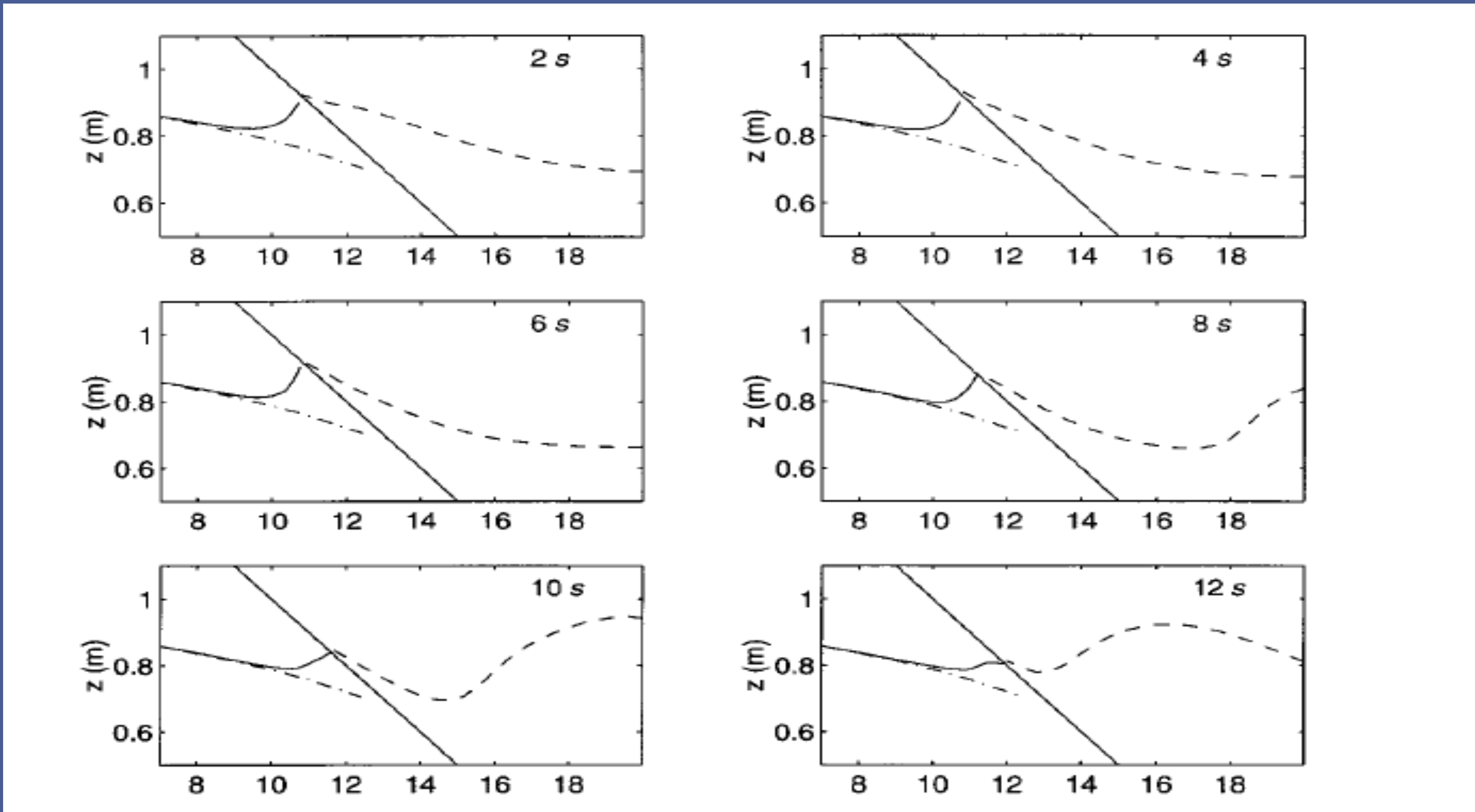
$$N_{\mathrm{CAR}} = \frac{K}{B\omega},$$



Starting point and early work

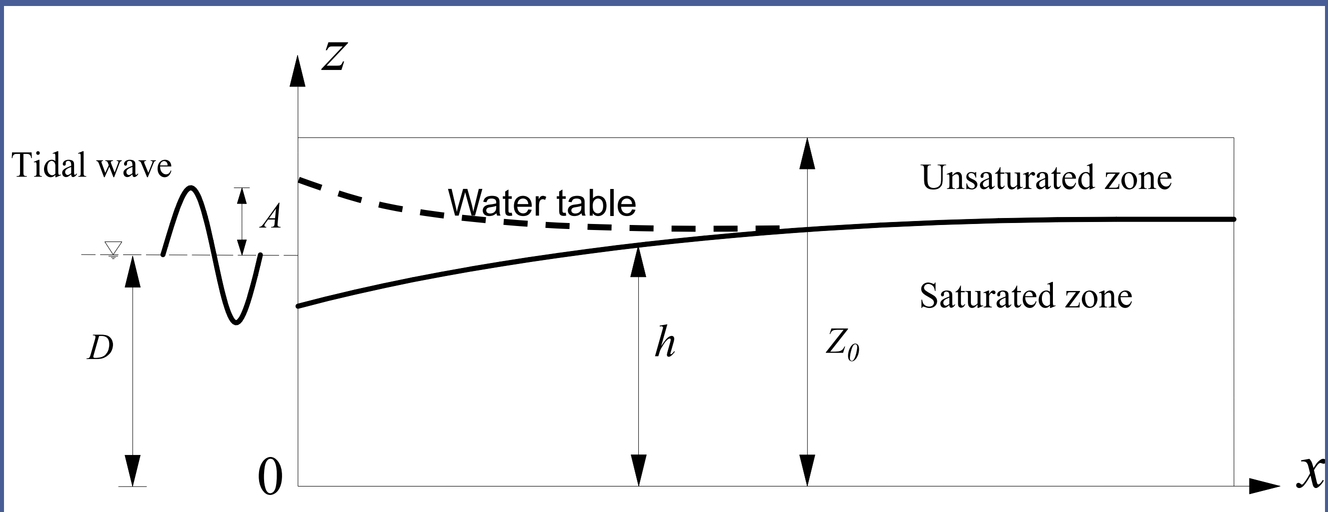
- Li et al. 1997 – high frequency groundwater wave

$$\frac{\partial \phi}{\partial t} = -\frac{K}{n_e \cos\beta}\frac{\partial \phi}{\partial n} \boxed{- \frac{B}{n_e \cos\beta}\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial n}\right)}, z = \eta(x, t)$$



Kong et al. 2012

- Capillarity and capping effects (Z_0)



$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[K(\psi)\frac{\partial \Phi}{\partial x}\right] + \frac{\partial}{\partial z}\left[K(\psi)\frac{\partial \Phi}{\partial z}\right]$$

$$\Phi = \psi + z$$

$$\theta = (\theta_s - \theta_r)e^{\alpha\psi} + \theta_r$$

$$K(\psi) = K_s e^{\alpha\psi}$$

(1) hydrostatic

$$\Phi = h \qquad \psi = h - z$$

(2) non-hydrostatic

$$\psi = h - z + \boxed{P}$$

(1) Hydrostatic case

- Integration from 0 to Z_0 → governing equation

$$n_e\left[1-e^{\alpha(h-Z_0)}\right]\frac{\partial h}{\partial t}$$
$$=K_s\frac{\partial}{\partial x}\left\{h\frac{\partial h}{\partial x}+\frac{1}{\alpha}\left[1-e^{\alpha(h-Z_0)}\right]\frac{\partial h}{\partial x}\right\}-K_s e^{\alpha(h-Z_0)}\frac{\partial h}{\partial x}\frac{\partial Z_0}{\partial x}$$

- Recovery of the Boussinesq equation (with α → infinity)

$$n_e\frac{\partial h}{\partial t}=K_s\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)$$

(2) Non-hydrostatic case

- Integration from 0 to Z_0 → governing equation
  - Determination of *P*

$$\frac{\partial\theta}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \qquad w=-K_s e^{\alpha(h-z)}\frac{\partial P}{\partial z} \qquad w=\int_z^{Z_0}\frac{\partial\theta}{\partial t}dz+\int_z^{Z_0}\frac{\partial u}{\partial x}dz$$

$$w=n_e\frac{\partial h}{\partial t}\left[e^{\alpha(h-z)}-e^{\alpha(h-Z_0)}\right]+K_s\left[\left(\frac{\partial h}{\partial x}\right)^2+\frac{1}{\alpha}\frac{\partial^2 h}{\partial x^2}\right]\left[e^{\alpha(h-Z_0)}-e^{\alpha(h-z)}\right]$$

$$=\left[e^{\alpha(h-z)}-e^{\alpha(h-Z_0)}\right]\left\{n_e\frac{\partial h}{\partial t}-K_s\left[\left(\frac{\partial h}{\partial x}\right)^2+\frac{1}{\alpha}\frac{\partial^2 h}{\partial x^2}\right]\right\}$$

$$P(z)=-\int_z^{Z_0}\frac{\partial P}{\partial z}dz=-\left[\frac{n_e}{K_s}\frac{\partial h}{\partial t}-\left(\frac{\partial h}{\partial x}\right)^2-\frac{1}{\alpha}\frac{\partial^2 h}{\partial x^2}\right]\int_z^{Z_0}\left[e^{\alpha(z-Z_0)}-1\right]dz$$

$$=\left[\frac{1}{\alpha}\frac{\partial^2 h}{\partial x^2}-\frac{n_e}{K_s}\frac{\partial h}{\partial t}+\left(\frac{\partial h}{\partial x}\right)^2\right]\left[\frac{1}{\alpha}-\frac{1}{\alpha}e^{\alpha(z-Z_0)}-Z_0+z\right]$$

$$\frac{\partial P}{\partial x}=\left[\frac{1}{\alpha}\frac{\partial^3 h}{\partial x^3}+2\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial x^2}-\frac{n_e}{K_s}\frac{\partial^2 h}{\partial x\partial t}\right]\left[\frac{1}{\alpha}-\frac{1}{\alpha}e^{\alpha(z-Z_0)}-Z_0+z\right]$$

(2) Non-hydrostatic case

- Integration from 0 to Z_0 → governing equation
  - Correction due to P

$$\int_h^{Z_0} K_s e^{\alpha(h-z)} \frac{\partial P}{\partial x} dz$$
$$= K_s \left[\frac{1}{\alpha} \frac{\partial^3 h}{\partial x^3} + 2 \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} - \frac{n_e}{K_s} \frac{\partial^2 h}{\partial x \partial t} \right] \int_h^{Z_0} \left\{ e^{\alpha(h-z)} \left[\frac{1}{\alpha} - \frac{1}{\alpha} e^{\alpha(z-Z_0)} - Z_0 + z \right] \right\} dz$$
$$= K_s \left[\frac{1}{\alpha} \frac{\partial^3 h}{\partial x^3} + 2 \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} - \frac{n_e}{K_s} \frac{\partial^2 h}{\partial x \partial t} \right] \left[-\frac{2}{\alpha^2} e^{\alpha(h-Z_0)} + \frac{h-Z_0}{\alpha} e^{\alpha(h-Z_0)} + \frac{2}{\alpha^2} + \frac{h-Z_0}{\alpha} \right]$$
$$\approx \frac{n_e}{\alpha^2} \left[2e^{\alpha(h-Z_0)} - 2 + \alpha(Z_0 - h) e^{\alpha(h-Z_0)} + \alpha(Z_0 - h) \right] \frac{\partial^2 h}{\partial x \partial t}$$

  - Governing equation

$$\mathrm{F} n_e \frac{\partial h}{\partial t} = K_s \frac{\partial}{\partial x}\left(\mathrm{M} h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial x}\left(\mathrm{N} \frac{\partial^2 h}{\partial x \partial t} \right)$$

$$\mathrm{F} = 1 - e^{\alpha(h-Z_0)}$$

$$\mathrm{M} = 1 + \frac{1}{\alpha h}\left[1 - e^{\alpha(h-Z_0)} \right]$$

$$\mathrm{N} = \frac{n_e}{\alpha^2} \left[2e^{\alpha(h-Z_0)} - 2 + \alpha(Z_0 - h) e^{\alpha(h-Z_0)} + \alpha(Z_0 - h) \right]$$

(2) Non-hydrostatic case

- Integration from 0 to Z_0 → governing equation
  - Correction due to P

$$\int_h^{Z_0} K_s e^{\alpha(h-z)} \frac{\partial P}{\partial x} dz$$

$$= K_s \left[\frac{1}{\alpha} \frac{\partial^3 h}{\partial x^3} + 2 \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} - \frac{n_e}{K_s} \frac{\partial^2 h}{\partial x \partial t} \right] \int_h^{Z_0} \left\{ e^{\alpha(h-z)} \left[\frac{1}{\alpha} - \frac{1}{\alpha} e^{\alpha(z-Z_0)} - Z_0 + z \right] \right\} dz$$

$$= K_s \left[\cancel{\frac{1}{\alpha} \frac{\partial^3 h}{\partial x^3}} + 2 \cancel{\frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2}} - \frac{n_e}{K_s} \frac{\partial^2 h}{\partial x \partial t} \right] \left[-\frac{2}{\alpha^2} e^{\alpha(h-Z_0)} + \frac{h-Z_0}{\alpha} e^{\alpha(h-Z_0)} + \frac{2}{\alpha^2} + \frac{h-Z_0}{\alpha} \right]$$

$$\approx \frac{n_e}{\alpha^2} \left[2e^{\alpha(h-Z_0)} - 2 + \alpha(Z_0 - h) e^{\alpha(h-Z_0)} + \alpha(Z_0 - h) \right] \frac{\partial^2 h}{\partial x \partial t}$$

  - Governing equation

$$\mathrm{F} n_e \frac{\partial h}{\partial t} = K_s \frac{\partial}{\partial x} \left(\mathrm{M} h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial x} \left(\mathrm{N} \frac{\partial^2 h}{\partial x \partial t} \right)$$

$$\mathrm{F} = 1 - e^{\alpha(h-Z_0)}$$

$$\mathrm{M} = 1 + \frac{1}{\alpha h} \left[1 - e^{\alpha(h-Z_0)} \right]$$

$$\mathrm{N} = \frac{n_e}{\alpha^2} \left[2e^{\alpha(h-Z_0)} - 2 + \alpha(Z_0 - h) e^{\alpha(h-Z_0)} + \alpha(Z_0 - h) \right]$$

(2) Non-hydrostatic case

- With $\alpha \rightarrow$ infinity (absence of the unsaturated zone), the governing equation also $\rightarrow$ BE

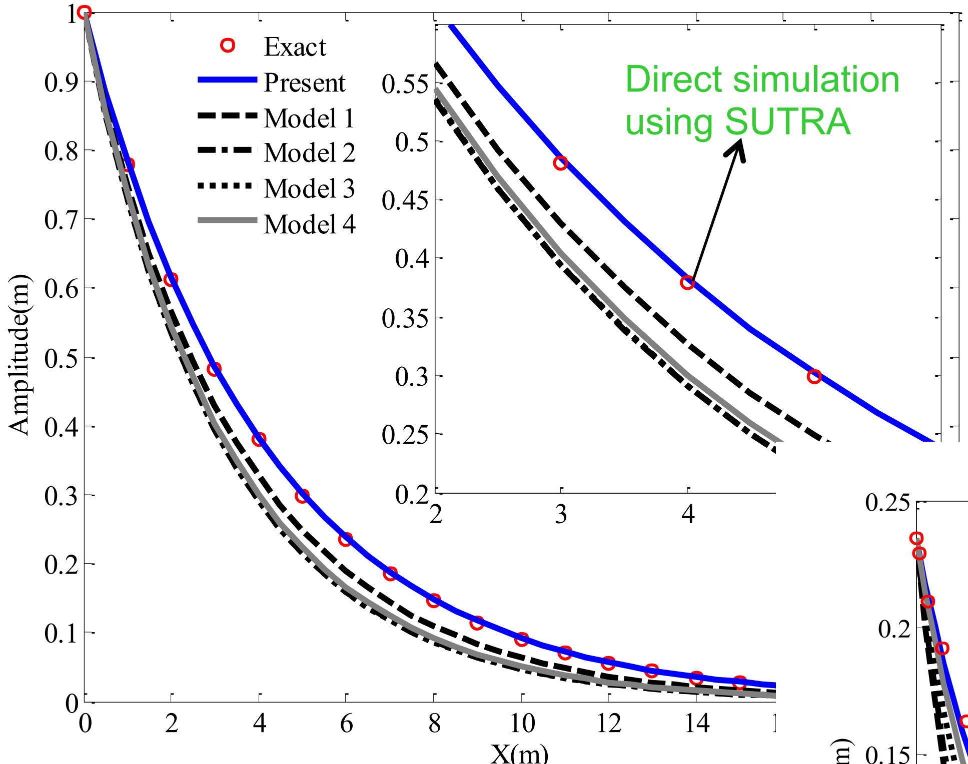
$$n_e \frac{\partial h}{\partial t} = K_s \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x} \right)$$

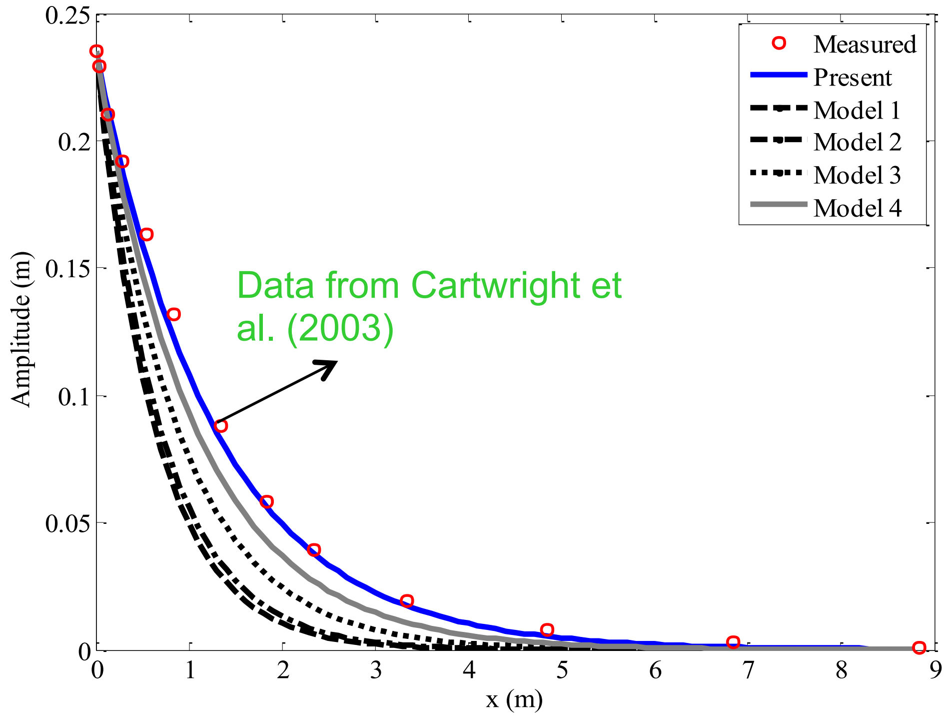
- With vertical flow also considered for the saturated zone

$$\mathrm{N} = \frac{n_e}{\alpha^2}\left[2e^{\alpha(h-Z_0)} - 2 + \alpha(Z_0 - h)e^{\alpha(h-Z_0)} + \alpha(Z_0 - h) \right] + \boxed{\frac{D^2}{3} n_e}$$

Liu and Wen 1997

Validation: damping





Analytical approximation

- Perturbation solution (1st order) based on $\varepsilon = A/D$

$$h = D + \exp\left(-x\,\mathrm{k}_{\mathrm{US}}\,\mathrm{F}_1\right)\cos\left(\omega t - x\,\mathrm{k}_{\mathrm{US}}\,\mathrm{F}_2\right)$$

$$\mathrm{k}_{\mathrm{US}} = \sqrt{\frac{\mathrm{R}_1\,\omega}{2\,\mathrm{R}_2}}$$

$$\mathrm{R}_1 = n_e\left[1 - e^{\alpha(D-Z_0)}\right]$$

$$\mathrm{R}_2 = K_s D + K_s\frac{1}{a}\left[1 - e^{\alpha(D-Z_0)}\right]$$

$$\mathrm{R}_3 = \frac{n_e}{\alpha^2}\left[2e^{\alpha(D-Z_0)} - 2 + \alpha\left(Z_0 - D\right)e^{\alpha(D-Z_0)} + \alpha\left(Z_0 - D\right) + \frac{D^2\alpha^2}{3}\right]$$

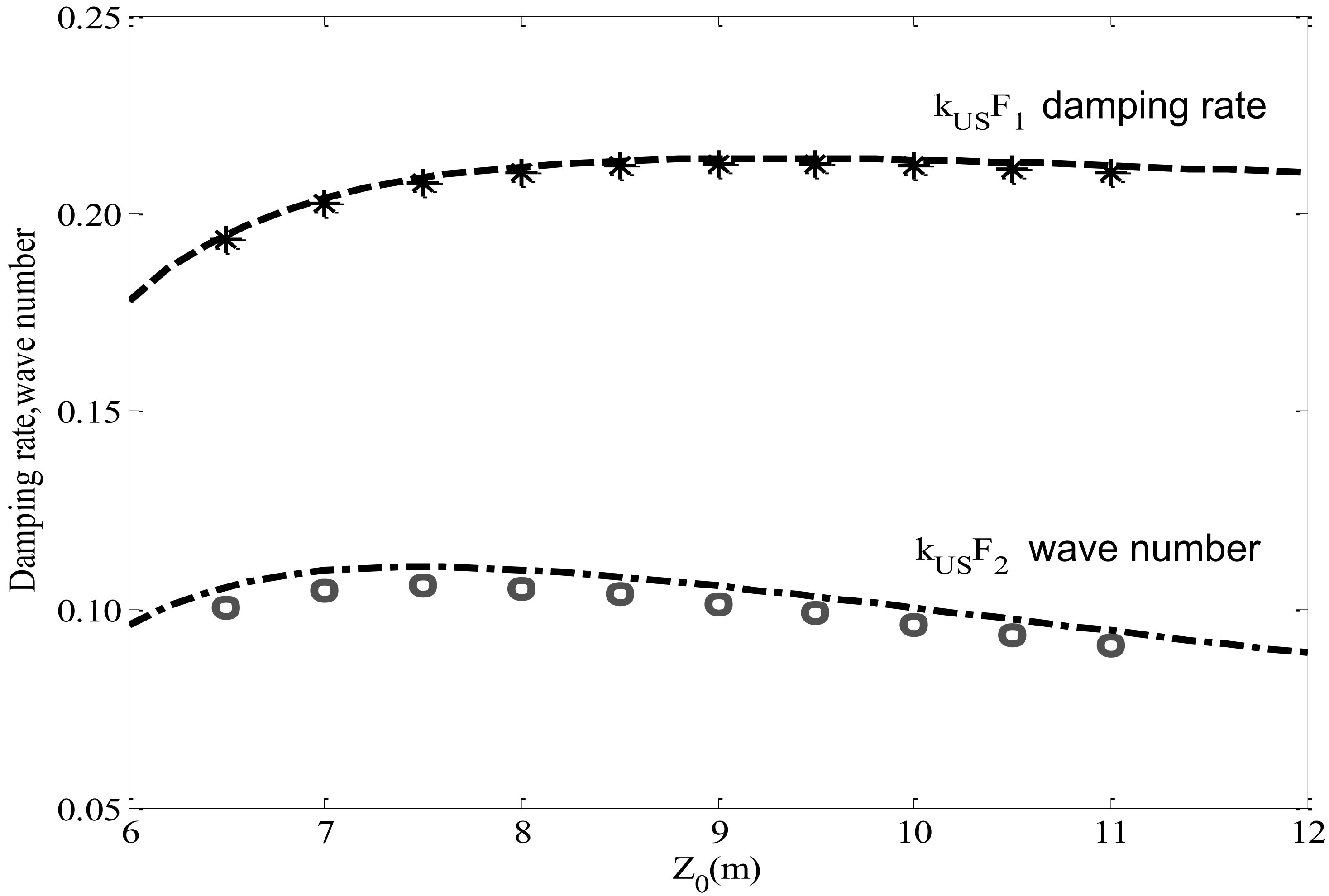
$$\mathrm{F}_1 = \sqrt{\frac{\mathrm{N}_{\mathrm{US}}}{\sqrt{1+\mathrm{N}_{\mathrm{US}}{}^2}} + \frac{\mathrm{N}_{\mathrm{US}}}{1+\mathrm{N}_{\mathrm{US}}{}^2}}$$

$$\mathrm{F}_2 = \sqrt{\frac{\mathrm{N}_{\mathrm{US}}}{\sqrt{1+\mathrm{N}_{\mathrm{US}}{}^2}} - \frac{\mathrm{N}_{\mathrm{US}}}{1+\mathrm{N}_{\mathrm{US}}{}^2}}$$

$$\mathrm{N}_{\mathrm{US}} = \frac{\mathrm{R}_2}{\mathrm{R}_3\,\omega}$$

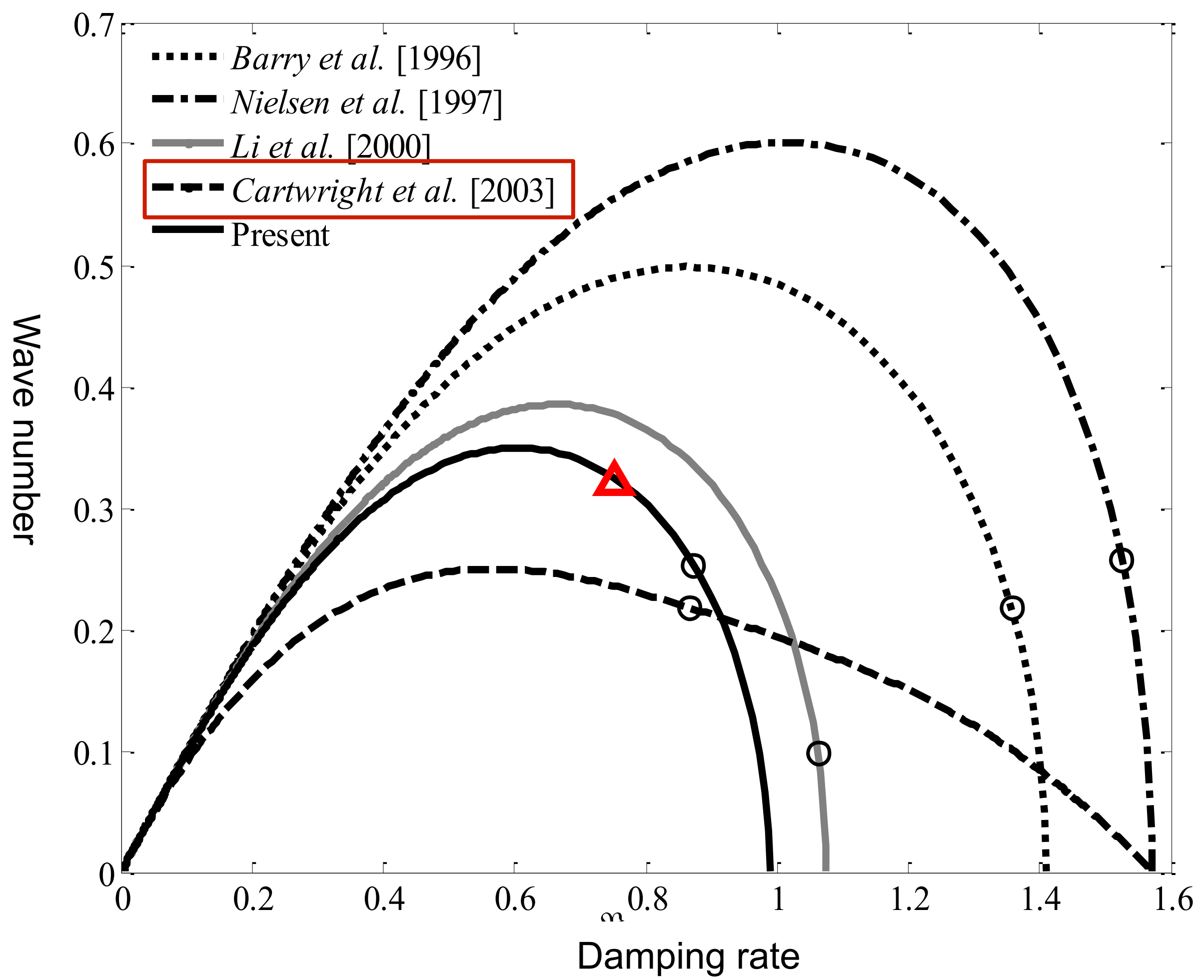
Validation: comparison with numerical solution

- Damping and phase shift (capping effect)



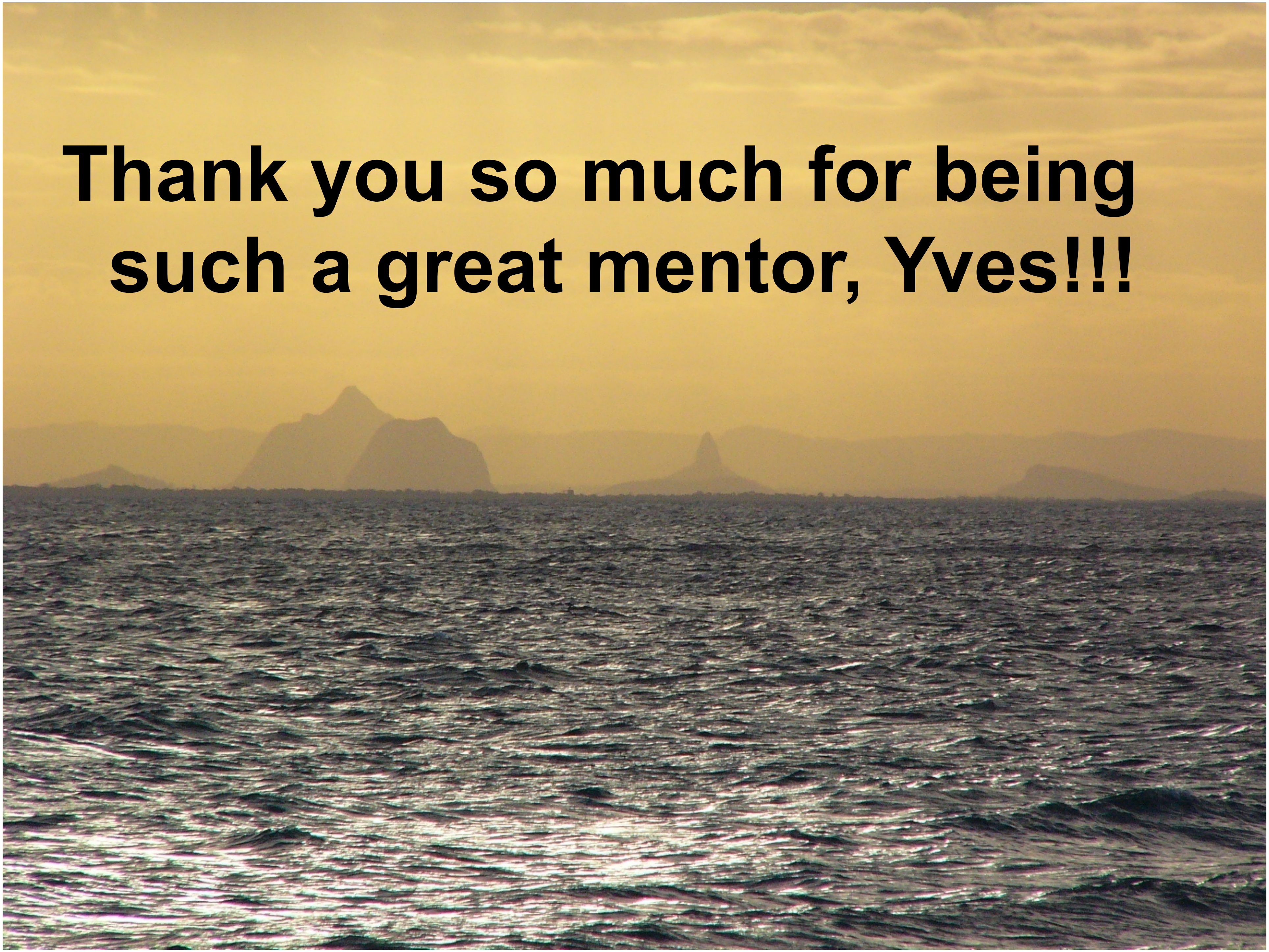
Wave characteristics

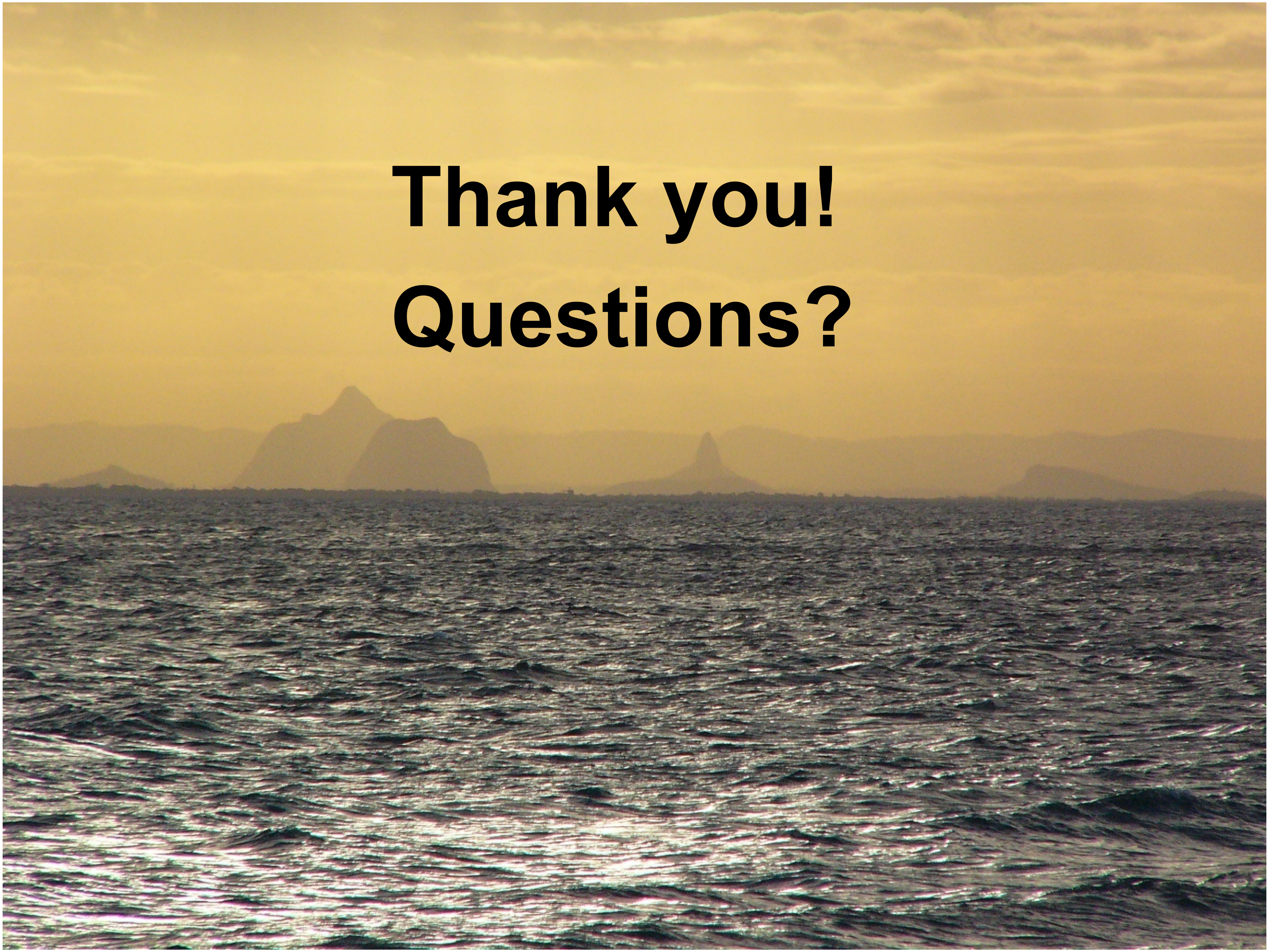
- Damping and phase shift: comparison with measurement



Circle calculated based on the wave dispersion relationships of various models using measured soil properties

Triangle determined based on measured water table fluctuations


Thank you so much for being
such a great mentor, Yves!!!


Thank you!
Questions?