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ADAPTIVE SEQUENTIAL PROCEDURES FOR SELECTING THE BEST OF SEVERAL NORMAL POPULATIONS.

by

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Abstract

There are k (≥ 2) competing normal populations with common known variance and unknown means $\theta_1, \theta_2, \ldots, \theta_k$. Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ denote the ordered values of the $\{\theta_i\}$. Nothing is known concerning the pairing of the $\{\theta_i\}$ and the $\{\theta_{[i]}\}$. In the location invariant identification problem, the differences $\{\theta_{[i]} - \theta_{[1]}\}$ are given and it is desired to select the population associated with $\theta_{[k]}$. Of particular interest is the slippage configuration $\theta_{[1]} = \theta_{[k-1]} = \theta_{[k]} - \delta^*$ where $\delta^* > 0$ is given. We restrict attention to procedures that guarantee that the population with the largest mean is correctly selected with probability at least P^* where $k^{-1} < P^* < 1$ is preassigned. Essentially, this requirement is satisfied by the stopping rule of Bechhofer, Kiefer and Sobel (Sequential Identification and Ranking Procedures, Univ. of Chicago Press 1968, Chap. 3) independently of the sampling rule and thus data dependent allocation rules can be considered. Unlike the case $k = 2$ (see Robbins and Siegmund (1974) J. Am. Statist. Assoc., 69, 132-139), when $k \geq 3$, substantial savings in expected total sample size can be obtained when adaptive sampling is used instead of the equal allocation rule ("vector at a time" sampling). Several procedures are proposed and investigated with respect to this criterion. Also their performance with regard to the alternative criterion of minimizing expected number of observations on the inferior populations is studied. Both theoretical and simulation results are presented. Comparisons are made with the performance of elimination-type procedures, such as that of Paulson (1964), Ann. Math. Statist., 35, 174-180.

Finally, for the ranking problem where the differences $\theta_{[i]} - \theta_{[1]}$

$(2 \leq i \leq k)$ are now unknown, it is shown that, when adaptive sampling rules are used, the slippage configuration is no longer necessarily least favorable for the usual indifference zone approach. This poses some interesting problems for future research.

1. Introduction

Adaptive sampling procedures have been the subject of considerable, theoretical interest in the methodology of sequential medical trials, where it is desired to compare k (≥ 2) treatments. A recent survey article by Hoel, Sobel and Weiss (1975) provides a comprehensive overview of the literature. Most authors have concentrated on the two-armed trial, $k = 2$, but studies with three or more arms are important. Gent (1976) describes a situation in which three independent nationwide trials involving aspirin substitutes were begun around the same time. He points out that, instead of conducting three separate 2-armed trials of drug vs. control, a single 4-armed study would have resulted in a considerable saving in the number of control subjects needed.

However, adaptive sampling might be advantageous in other areas of application. In a quality control situation there may be k batches of items which deteriorate in time (e.g. food, drugs, etc.) and due to records being lost it is not known which batch is the newest (oldest) unless sampling is undertaken. Alternatively there may be k apparently identical lots of items but of varying quality and it is desired to pick the superior one. Another application occurs when the observations are radar measurements on the nose cone, booster and other debris of a rocket after separation and it is desired to identify the nose cone as quickly as possible e.g. for guidance or for possible counter-measures if the nose cone is a warhead and the other objects are decoys. There have been several papers on the subject of how search radars could be improved by non-uniform scan controlled by a sequential detector - see e.g. Edrington and Petersen (1971).

Consider a k-ary communication channel with feedback. The receiver sequentially processes the signal corresponding to one of the k possible characters until it identifies it with some degree of certainty - the receiver then directs the transmitter to send the next character. In this application (and possibly others), there is a single input process with k hypotheses. If so, the k populations may be taken as the output processes of k filters. If the k possible signals are orthogonal and the noise Gaussian, then matched filters would accomplish the desired transformation. There are further applications to unsupervised learning and classification techniques -- see Nagy and Tolaba (1972).

An adaptive sequential procedure consists of three parts -- a <u>sampling rule</u> (also called <u>assignment</u> or <u>allocation algorithm</u>) for deciding from which population the next observation is to be taken, a <u>stopping rule</u> for deciding when sampling should cease, and a <u>terminal decision rule</u> for selecting a population as "best". Robbins and Siegmund (1974) treat the problem of deciding which of two normally distributed treatments with a common known variance has the larger mean response. They show how to construct a location invariant stopping rule and terminal decision rule of sequential probability ratio test type (SPRT) so that the error probabilities are essentially independent of the (symmetric) sampling rule used. In particular the SPRT can be constructed to guarantee that the probability (PCS) of correctly selecting the treatment with the larger mean is at least P^* wherever the difference of the means is no less than δ^*. Here $\delta^* > 0$ and $1/2 < P^* < 1$ are preassigned. They then consider various sampling rules with respect to two measures of performance, namely (A) expected total number of observations (ASN) and (B) expected total number of observations on the inferior population (ITN).

They showed that the ASN was minimized by pairwise (or vector-at-a-time VT) sampling. They also show how adaptive sampling rules which reduce the ITN necessarily cause a considerable increase in ASN. (As we shall demonstrate later, this result is in sharp contrast with the case $k \geq 3$ where the two goals (A), (B) are no longer in direct conflict.) Subsequently Louis (1975) obtained the optimal allocation rule for minimizing $ASN + \gamma \cdot ITN$ (for $\gamma > 0$) in the continuous time analogue of this $k = 2$ problem.

In this paper, we attempt to attack the k (≥ 3) treatment problem in the same spirit as Robbins and Siegmund. Bechhofer, Kiefer and Sobel (1968) [BKS] treat in great detail vector-at-a-time (VT) sampling for selecting the best of k populations for a stopping rule which is a generalization of the SPRT. However, when $k \geq 3$, great savings in ASN can be made if the sampling rule is allowed to be adaptive (i.e. data-dependent). As mentioned before, such rules can have the side benefit of producing a lower expected total number of observations on the $k - 1$ inferior treatments (ITN). This is intuitively clear; for consider the normal case with $k = 3$ and when one population mean is much smaller than the others. Very early on in experimentation, this population will become apparent and only two populations will be left in contention. Thus, compared to VT sampling, we might expect that the ASN can be reduced by almost half. Correspondingly larger savings could occur with larger values of k. In fact, although previous authors have treated small values of k (usually $k = 2$), there are applications where k may be as large as 1000. That ASN can be reduced by adaptive sampling is important since recently Byar et al. (1976) have suggested that it is ASN and not ITN which should be the more important criterion for clinical trials.

In the next section, we describe a k-population location invariant (LI) identification problem and propose a stopping/terminal decision rule with the property that its PCS is independent of the sampling rule used. In Section 3, we describe some adaptive allocation rules, and also, in Section 4, the elimination-type procedures of Paulson(1964) which can also be viewed as an adaptive rules. In Section 5, these rules are compared via Monte Carlo simulation. In Section 6, we investigate the ranking problem associated with the LI identification problem of Section 2. However, unlike the case $k = 2$, or the case of $k \geq 2$ with VT sampling, we find that the usual slippage configuration is no longer "least favorable" and thus the results of Section 2 cannot be extended. This is a somewhat surprising result (at least to the authors). All the above ideas will be made more precise in the sections that follow.

2. The location invariant (LI) identification problem

Observations X_{ij} $(1 \leq i \leq k,\ j \geq 1)$ are available sequentially from each of k (≥ 2) populations $\Pi_1, \Pi_2, \ldots, \Pi_k$. The observations $\{X_{ij}\}$ are independent with probability density function $f(x - \theta_i)$ where θ_i is a real location parameter. We let $\theta_{[1]} \leq \cdots\cdots \leq \theta_{[k-1]} \leq \theta_{[k]}$ denote the ranked values of the $\{\theta_i\}$. We assume that nothing is known a priori concerning the pairings of the $\{\theta_{[i]}\}$ and the $\{\Pi_i\}$. Also we define $\tau_i = \theta_i - \theta_{[1]}$, and let $\tau_{[i]} = \theta_{[i]} - \theta_{[1]}$ be the ranked values of the $\{\tau_i\}$. If the values of the $\theta_{[i]}$ are known (unknown) then we say we have an identification (ranking) problem. Here we shall consider the location invariant (LI) identification problem where only the differences $\{\tau_{[i]}\}$ are known. We shall discuss the construction of adaptive sequential procedures which guarantee that the probability of correctly selecting (PCS) that population associated with $\tau_{[k]}$ is at least P^*, where $k^{-1} < P^* < 1$ is prespecified (we assume $\tau_{[k]}$ is unique i.e. $\tau_{[k]} > \tau_{[k-1]}$). As in Robbins and Siegmund (1974), by invariance we can restrict consideration to procedures based on the maximal invariants $Y_{ij} = X_{ij} - X_{11}$, $1 \leq i \leq k$, $j \geq 1$. Suppose that we have taken n_i observations from Π_i and define $N = \sum_1^k n_i$.

The likelihood function based on the $\{Y_{ij}\}$ is given by

$$L^*(\tau_1, \tau_2, \ldots, \tau_k) = \int_{-\infty}^{\infty} \prod_{i=1}^{k} \prod_{j=1}^{n_i} f(y_{ij} - \tau_i + \tau_1 + z)dz$$

where of course $y_{11} = 0$.

Define $L(\alpha) = L^*(\tau_{[\alpha 1]}, \tau_{[\alpha 2]}, \ldots, \tau_{[\alpha k]})$ where $\alpha \in S_k$, the group of permutations of k elements. Let $S_{k-1}(i,j)$ denote the subgroup of permutations $\alpha \in S_k$ for which $\alpha i = j$. Define

$$Q_i = \frac{\sum_{\alpha \varepsilon S_{k-1}(i,k)} L(\alpha)}{\sum_{\alpha \varepsilon S_k} L(\alpha)}$$

which is the likelihood that $\tau_i = \tau_{[k]}$ and Π_i is the "best" population. (Q_i is also the a posteriori probability of this event, assuming the $\{\Pi_i\}$ are all equally likely to be the best, a priori.) Finally, define

$$Q = \max_{1 \leq i \leq k} Q_i.$$

The BKS stopping rule (and terminal decision rule) is as follows:

> "Terminate sampling the first time that $Q \geq P^*$ and select that population associated with the largest value of Q_i." (P_{BKS})

Theorem. If P_{BKS} is used for the LI identification problem above then $P[CS] \geq P^*$ for any symmetric sampling rule such that the procedure terminates almost surely.

Proof. This is essentially contained in BKS (Chap. 3).

Remark 1. Scale parameters. The case where $f(x|\theta_i) = f(x/\theta_i)$ is analogous and can be handled either by a log transformation or by considering quotients instead of differences.

For the remainder of this paper we will treat the example in which the observations $\{X_{ij}\}$ are normal with means $\{\theta_i\}$ with known variance σ^2. For $k = 2$, this setup has been recently treated by Robbins and Siegmund (1974) and Louis (1975). Here it is easy to show that

$$L^*(\tau_1, \tau_2, \ldots, \tau_k) \propto \exp - \frac{1}{2\sigma^2} [\sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{..} - \tau_i + \bar{\tau})^2]$$

where $\bar{x}_{..} = N^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}$ and $\bar{\tau} = N^{-1} \sum_{i=1}^{k} n_i \tau_i$. Note that $\bar{x}_{..}$ and $\bar{\tau}$ are not invariant under permutations of the subscripts.

We specialize to the δ^*-slippage configuration (Karlin and Truax, 1960) for which $\tau_{[i]} = 0$ $(1 \leq i \leq k-1)$ and $\tau_{[k]} = \delta^* > 0$. While of interest in its own right (e.g. see some of the examples in Section 1), this configuration also is the "least favorable" in the associated ranking problem with VT sampling - see Section 6. Substituting for $\tau_{[i]}$, $L(\alpha)$ simplifies and for $\alpha \in S_{k-1}(i,k)$, we have

$$L(\alpha) = \exp - \frac{1}{2\sigma^2} \left[\sum_{h=1}^{k} \sum_{j=1}^{n_i} (x_{hj} - \bar{x}_{..})^2 - 2\delta^* z_i \right]$$

where

$$z_i = n_i(\bar{x}_{i.} - \bar{x}_{..}) - \frac{n_i(N-n_i)}{2N} \delta^*$$

and $\bar{x}_{i.} = n_i^{-1} \sum_{j=1}^{n_i} x_{ij}$.

Let $z_{[1]} < \ldots < z_{[k]}$ denote the ranked values of the $\{z_i\}$. (Ties occur with probability zero.)

Then

$$\frac{1-Q}{Q} = \sum_{i=1}^{k-1} \exp[-\delta^*(z_{[k]} - z_{[i]})/\sigma^2]$$

and P_{BKS} is equivalent to:

(2.1) "Stop the first time that $(1-Q)/Q \leq (1-P^*)/P^*$ and select that population associated with $z_{[k]}$.

It should be noted that for non VT sampling the population associated with $z_{[k]}$ is not necessarily the same as the one associated with $\max_{1 \leq i \leq k} (\bar{x}_{i.})$.

We now address the question of sufficient conditions for the almost sure termination of rule (2.1).

Lemma 1. For the δ^*-slippage LI-identification problem, we suppose that the sampling rule used is such that:

(2.2) $\quad P[n_i(N) \to \infty \quad \text{as} \quad N \to \infty \text{ for each } 1 \leq i \leq k] = 1$

Then $Q \to 1$ as $N \to \infty$ almost surely. It follows that if the stopping rule (2.1) is used, termination will occur with probability one.

Proof. We first show that $\bar{X}_{i.} \to \theta_i$ a.s. for $1 \leq i \leq k$ as $N \to \infty$. The sequence $\{X_{ij};\ j = 1,2,\ldots\}$ is i.i.d. $N(\theta_i, \sigma^2)$. Hence, by the strong law of large numbers, $\sum_{j=1}^{\nu} X_{ij}/\nu \to \theta_i$ a.s. as $\nu \to \infty$. The sequence $\{\sum_{j=1}^{n_i(N)} X_{ij}/\nu;\ \nu = 1,2,\ldots\}$ by "stuttering", i.e. repeating each value several times before going on to the next. Hence $\bar{X}_{i.} \to \theta_i$ almost surely.

Now, without loss of generality assume $\tau_i = \tau_{[i]}$ $(1 \leq i \leq k)$. Then

$$Z_k = n_k(\bar{X}_{k.} - N^{-1} \sum_{i=1}^{k} n_i \bar{X}_{i.}) - n_k(N - n_k)\delta^*/2N$$

Hence

$$(\frac{1}{N-n_k} + \frac{1}{n_k})Z_k = \bar{X}_{k.} - \delta^*/2 - (\sum_{i=1}^{k-1} n_i \bar{X}_{i.} / \sum_{i=1}^{k-1} n_i),$$

which by the argument above converges almost surely to some strictly positive limit as $N \to \infty$. Thus we must have $Z_k \to +\infty$ a.s. $N \to \infty$. Similarly for $i \neq k$ we can write

$$(\frac{1}{N-n_i} + \frac{1}{n_i})Z_i = \bar{X}_{i.} - (n_k \bar{X}_{k.}/(N - n_i))$$
$$- (\sum_{\substack{\ell=1 \\ \ell \neq i}}^{k-1} n_\ell \bar{X}_{\ell.} / \sum_{\substack{\ell=1 \\ \ell \neq i}}^{k-1} n_\ell) - \delta^*/2$$

and the lim sup of the RHS is strictly negative almost surely as $N \to \infty$. Thus $\limsup Z_i = -\infty$ a.s., and $(1 - Q)/Q \to 0$ a.s. which proves the lemma.

Remark 2. If the sampling rule is such that condition (2.2) is not satisfied then Q may not converge to unity. For example, suppose that, with non-zero probability, $n_i(N) \to n_i < \infty$ for some $i(1 \leq i \leq k - 1)$ and $n_k(N) \to \infty$ as $N \to \infty$. Then $Z_k - Z_i = n_k \sum_{j \neq k} n_j(\bar{X}_{k.} - \bar{X}_{j.} - \delta^*/2)/N - n_i \sum_{j \neq i} n_j(\bar{X}_{i.} - \bar{X}_{j.} - \delta^*/2)/N$ remains finite and $Q \not\to 1$.

3. Sampling rules

In this section we describe several sampling (allocation) rules to be used in conjunction with the stopping rule and terminal decision rule given by (2.1). For convenience, we assume that all rules initially take one observation from each population.

(i) VT At each stage a block of k observations is taken, one from each population. In the medical context, within each block, patients are assigned at random to each of the k treatments. A closely related rule is the one in which observations are taken singly, one observation from each population in turn. The ASN of this rule must be smaller than that of the VT rule yet our experience suggests that the difference is small. Extensive tables for VT sampling are available (BKS, Chap. 18), and those results provide a yardstick against which our adaptive sampling results can be compared.

(ii) RAND Q This is a randomized allocation rule in which the next observation is taken from Π_i with probability Q_i $(1 \leq i \leq k)$. Recall that for the LI identification problem Q_i can be considered the posterior probability that Π_i is "best". (Note that the nonrandomized rule that always samples from the population associated with $\max_i Q_i$ may not terminate by Remark 2.) For $k = 2$ and Bernoulli observations, Simon, Weiss and Hoel (1975) have investigated this rule.

(iii) GRS Robbins and Siegmund (1974 Sect. 3) proposed a non-randomized rule for $k = 2$. Here we describe one particular generalization to $k \geq 2$. From the stopping rule (2.1), we note that sampling has certainly terminated if

$$(k-1)\exp[-\delta^*(z_{[k]} - z_{[k-1]})/\sigma^2] \leq (1 - P^*)/P^*$$

i.e. if $z_{[k]} - z_{[k-1]} \geq b$

where $b = \sigma^2 \log[(k-1)P^*/(1-P^*)]/\delta^*$. The GRS rule depends on a parameter $c \geq b$ and is as follows: If

$$\frac{z_{[k]} - z_{[k-1]}}{c} \geq \frac{n_{(k)} - n_{(k-1)}}{N}$$

take the next observation from the population associated with $z_{[k]}$; otherwise take a vector of observations from the remaining $k - 1$ populations. Here $n_{(i)}$ denotes the sample size associated with $z_{[i]}$. For $k = 2$ this reduces to the rule of Robbins and Siegmund. For our simulations, two values of c were chosen, namely $c/b = 1$ and $c/b = 1.2$; these were the values used by Robbins and Siegmund.

(iv) Bessler Bessler (1960 , Section 8) treated this problem with $k = 3$, but from a decision theoretic viewpoint in which there are costs assigned to incorrect terminal decisions and there is a cost c per observation -- there is no P^* requirement. He obtains asymptotic properties as $c \to 0$ for his procedure. Because Bessler's stopping rule does not guarantee error probability requirements we continue to use the stopping rule (2.1) and consider only Bessler's sampling rule. Let $\bar{x}_{[1].} \leq \bar{x}_{[2].} \leq \bar{x}_{[3].}$ denote the ranked values of the $\{\bar{x}_{i.}, 1 \leq i \leq 3\}$. Define $C(\underset{\sim}{x}) = [(\bar{x}_{[3].} - \bar{x}_{[1].})/(\bar{x}_{[3].} - \bar{x}_{[2].})]^2$. Bessler's allocation rule is randomized and prescribes that the next observation be taken from the population associated with $\bar{x}_{[i].}$ with probability λ_i (i=1,2,3), where $\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ maximizes $\lambda_2\lambda_3/(\lambda_2 + \lambda_3)$ subject to the constraints $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and

$$\frac{\lambda_2\lambda_3}{\lambda_2 + \lambda_3} = \frac{\lambda_1\lambda_3}{\lambda_1 + \lambda_3} \quad C(\underset{\sim}{x})$$

Bessler tabulates $\underset{\sim}{\lambda}$ as a function of $C(\underset{\sim}{x})$. He also conjectures some asymptotic optimality properties for the ASN functions. Although we are not using his stopping rule, we might still hope for low ASN values. In theory, Bessler's procedure can be generalized to $k > 3$.

(v) $\sqrt{k-1}:1:1$ Randomization. If we can assume that the τ_i are in the δ^*-slippage configuration, we might consider replacing $C(\underset{\sim}{x})$ by the quantity $C(\underset{\sim}{x})$ is estimating, namely $C(\underset{\sim}{\tau})$. But when $\tau = (0,0,\delta^*)$, $C(\underset{\sim}{\tau}) = 1$ and it is easy to show that $\lambda_1 = \lambda_2 = 1/(2 + \sqrt{2})$, $\lambda_3 = \sqrt{2}/(2 + \sqrt{2})$. Sampling rule (v) is therefore to sample from the population associated with $\max\limits_{1<i<3} Q_i$ (i.e. with $z_{[k]}$) with probability $\sqrt{2}/(2+\sqrt{2})$, and the other two each with probability $(2+\sqrt{2})^{-1}$. These are the same limiting proportions as those obtained by Bechhofer (1969) in the related fixed sample size multiple comparisons problem for two experimental treatments and a control. When $k > 3$, the analogous rule would be to sample from the population associated with $Z_{[k]}$ with probability $(1 + \sqrt{k-1})^{-1}$ and the remaining populations with equal probability.

If should be noted that rule (iii) (GRS) is the only one that does not involve any randomization. From the remarks of Byar et al. (1976) and Bailar (1976), randomization is important in clinical trials.

4. Paulson's procedure

Paulson (1964), considered a procedure based on VT sampling but featuring permanent elimination of non-contending populations. At the n'th stage of experimentation let I_n be the set of populations not yet eliminated (I_1 consists of all k populations). An observation is taken from each population in I_n and the cumulative sample sums $\{S_{in}: i \in I_n\}$ are computed where $S_{in} = \sum_{k=1}^{n} x_{ik}$. If

$$S_{in} < \max_{j \in I_n} S_{jn} - (a - n\lambda)^+, \text{ then}$$

Π_i is eliminated. Here $a > 0$ and $0 \leq \lambda \leq \delta^*$ are specified constants and $y^+ = \max(y,0)$. The procedure continues until only one population remains which is selected as best. A desirable property is that when $\lambda > 0$ the number of stages is bounded by $[a/\lambda] + 1$. Here $[y]$ denotes the integer part of y. Note that the stopping rule is different from (2.1).

Paulson proved that if $a = \frac{\sigma^2}{\delta^* - \lambda} \log \frac{k-1}{1-P^*}$, then $PCS \geq P^*$ whenever $\tau_{[k]} - \tau_{[k-1]} \geq \delta^*$. Paulson originally suggested the choice of $\lambda = \delta^*/4$, whereas Fabian (1974) recommended $\lambda = \delta^*/2$ since asymptotically ($P^* \to 1$) this choice minimizes the maximum ASN over all parameter configurations. The unbounded procedure when $\lambda = 0$ is also of interest. In fact, Paulson's procedure is conservative in that the PCS guaranteed is considerably greater than P^* even in the LF δ^*-slippage configuration. Fabian (1974) proposed a modification of Paulson's procedure whereby this "overprotection" is reduced and consequently so is the ASN. This modification consists of replacing "$1 - P^*$" in the formula for "a" by a quantity $\beta > 1 - P^*$. For

$\lambda = 0$, $\beta = (1 - P^*)/[1 - (1 - P^*)/(k - 1)]$; for $\lambda = \delta^*/4$, β is the solution to the equation

$$1 - P^* = \beta[1 - \frac{1}{2}(\beta/(k - 1))^{1/3}];$$

finally for $\lambda = \delta^*/2$, $\beta = 2(1 - P^*)$.

A more detailed comparison via simulation of the BKS (with VT sampling) and Paulson procedures with regard to PCS and ASN for a wide range of P^* values can be found in unpublished papers of Ramberg (1966), Bechhofer and Ramberg (1977). The asymptotic relative efficiency of the 2 procedures has been studied by Perng (1969). Paulson's procedure has been extended by Hoel and Mazumdar (1968) and Hoel (1971).

5. Simulation results for the LI identification problem

A Monte Carlo study was performed in order to compare performances with regard to ASN, ITN and PCS of the procedures described in Sections 3,4 for the δ^*-slippage problem. The values $P^* = 0.9$, $\delta^* = 0.2$, $\sigma^2 = 1$ were chosen along with two values of k, namely $k = 3, 10$. The results are displayed in Tables I,II. These values were chosen so that our results would be directly comparable with those tables in Chap. 18 of BKS. Ten procedures were considered namely (A) FIXED sample size (Bechhofer, 1954); (B) VT; (C) RAND Q; (D) GRS with $c/b = 1.0$; (E) GRS with $c/b = 1.2$; (F) the BESSLER sampling rule with stopping rule (2.1); (G) the $\sqrt{k-1}:1:1$ allocation rule; (H), (I), (J) Paulson's procedure with the Fabian modification with $\lambda/\delta^* = .5, .25, 0$ respectively. These are abbreviated PF(.5), PF(.25), PF(0).

In the tables that follow, the FIXED sample sizes of the non-sequential single stage procedure were taken from Table I of Bechhofer (1954). For $k = 10$, simulations were only performed for procedures (B), (C), (E), (I). This was for reasons of economy and because these seemed the most promising. In any event, Bessler's rule is not easy to obtain explicitly for $k = 10$.

The tables display $\{N_i, 1 \leq i \leq k\}$ where N_i is the mean number of observations taken from the population associated with $\tau_{[i]}$. Also displayed is the average total sample size ASN. The three entries in each cell are the estimated mean, its standard error (in parentheses), and the estimated standard deviation. Also tabulated are the observed proportion of correct selections and the mean value of Q upon termination. The latter yields a more precise unbiased estimate of the PCS for the δ^*-slippage configuration -- see BKS page 289. Q also gives

Table I. Simulation results

$k = 3$, $P^* = 0.9$, $\delta^* = 0.2$, $\sigma = 1$

δ^*-slippage configuration $\tau_{[1]} = \tau_{[2]} = \tau_{[3]} - 0.2$

(based on 1000 replications. VT results from BKS Table 18·5·5)

	FIXED	VT	RAND Q	GRS(1.0)	GRS(1.2)	BESSLER	$\sqrt{2}$:1:1	PF(.5)	PF(.25)	PF(0)
N_3	125	76.34 (1.0) 46.26	105.37 (2.3) 73.96	132.70 (2.6) 83.20	115.42 (2.3) 72.06	91.29 (2.4) 75.72	78.49 (1.56) 49.23	88.51 (1.1) 35.57	84.31 (1.3) 41.59	90.65 (1.7) 54.00
N_2	125	76.34 (1.0) 46.26	56.12 (1.7) 52.59	51.92 (1.8) 56.71	52.32 (1.6) 49.32	64.71 (2.3) 73.51	68.29 (1.44) 45.39	74.06 (1.1) 34.02	67.92 (1.2) 36.82	69.52 (1.5) 48.60
N_1	125	76.34 (1.0) 46.26	55.86 (1.7) 53.59	48.29 (1.6) 50.32	52.29 (1.5) 48.11	62.60 (2.1) 66.14	68.89 (1.43) 45.18	74.67 (1.1) 35.80	70.49 (1.3) 40.23	73.58 (1.7) 52.55
N_{TOT}	375	229.03 (3.1) 138.78	217.36 (4.7) 148.48	232.91 4.4) 140.35	220.33 (4.3) 135.84	218.61 (5.3) 166.36	215.30 (4.39) 137.07	237.23 (2.8) 89.06	222.73 (3.2) 101.65	233.76 (4.1) 128.52
Proportion of correct selections	-	0.914 (0.006)	0.915 (0.009)	0.918 (0.009)	0.886 (0.010)	0.918 (0.009)	0.887 (0.010)	0.884 (0.010)	0.928 (0.008)	0.923 (0.008)
Q	-	0.9125	0.908	0.906	0.906	0.908	0.909	0.916	0.923	0.924

Table II Simulation results

$k = 10$, $P^* = 0.9$, $\delta^* = 0.2$, $\sigma = 1$

δ^*-slippage configuration $\tau_{[1]} = \tau_{[9]} = \tau_{[10]} - \delta^*$

(based on 200 replications. VT results from BKS Table 18.4.5)

	FIXED	VT	RAND Q	GRS (1.2)	PF (.25)
N_{10}	223	145.3	179.9	190.8	175.9
N_9	223	145.3	62.9	66.0	107.3
N_8	223	145.3	59.8	68.1	112.4
N_7	223	145.3	55.8	69.5	104.1
N_6	223	145.3	61.5	65.6	107.3
N_5	223	145.3	63.1	67.3	114.0
N_4	223	145.3	63.2	65.6	106.3
N_3	223	145.3	59.7	67.6	110.4
N_2	223	145.3	65.9	70.1	105.9
N_1	223	145.3	60.1	67.8	111.0
N_{TOT}	2230	1453.3 (28.3) 713.4	731.9 (32.5) 459.9	798.4 (36.9) 521.3	1154.7 (24.0) 339.5
Proportion of correct selections	-	0.911 (0.010)	0.910 (0.020)	0.930 (.018)	0.925 (0.019)
Q	-	0.912	0.907	0.909	0.941

a measure of "excess" or "overshoot".

It can be seen from Tables I,II that the adaptive rules possess some distinct advantages over VT. For $k = 3$, the decrease in ASN is about 6% for $k = 3$ (Table I) and the savings increase to about 50% for $k = 10$ (Table II). Also there is the side benefit in that the ITN can be reduced by about 22% for $k = 3$ and the savings increases to 60% for $k = 10$. Most of the adaptive rules (C) - (G) perform about the same -- RAND Q seems to do slightly better with regard to ASN, GRS slightly better for ITN. Among the PF rules, the results here are consistent with Paulson's choice of $\lambda = \delta^*/4$. Of course, the simulations are only for $P^* = 0.9$, $\delta^*/\sigma = 0.2$, but we believe that the results are indicative of what happens for other $(P^*, \delta^*/\sigma)$ combinations.

6. The ranking problem.

We now attempt the transition from the LI identification to the ranking problem where we assume that the ordered values $\tau_{[i]}$ are completely unknown. In the indifference zone formulation of the normal means ranking problem, we define the preference zone $PZ = \{\underset{\sim}{\tau} | \tau_{[k]} - \tau_{[k-1]} \geq \delta^*\} = \{\underset{\sim}{\theta} | \theta_{[k]} - \theta_{[k-1]} \geq \delta^*\}$ and seek procedures that guarantee the requirement that $PCS \geq P^*$ wherever $\underset{\sim}{\tau} \in PZ$. BKS (Theorem 6.1.1) prove that rule (2.1) with VT sampling satisfies this requirement. Paulson (1964) showed that his procedure also meets the requirement. (So does of course the fixed sample size rule of Bechhofer (1954).) All three procedures utilize the fact that, for a given set of procedure parameters $(P^*, \delta^*, \text{etc.})$, the infimum of the PCS over all $\underset{\sim}{\theta}$ in PZ is achieved in the δ^*-slippage configuration, which is therefore termed "the least-favorable" (LF) configuration. Thus for those procedures the ranking problem can be reduced to an LI identification problem. It might be reasonably conjectured that any procedure using stopping rule (2.1) and a symmetric sampling rule that assures termination w.p. 1 would also guarantee $PCS \geq P^*$ whenever $\underset{\sim}{\tau} \in PZ$. However our simulations (see Table VII) demonstrate this very intuitive conjecture to be false. In general there are configurations in PZ less favorable than the δ^*-slippage configuration. This was a rather surprising result - at least to the authors. A heuristic explanation for this phenomenon is given in the Appendix.

Monte Carlo simulations were carried out to investigate the performance of the ten procedures of Tables I,II when presented with parameter configurations other than δ^*-slippage. For $k = 3$, four alternate configurations were chosen, for $k = 10$ one alternate configuration. These were chosen so that easy comparison could be made with Tables 18.5 of BKS. Again $P^* = 0.9$, $\delta^* = 0.2$, and $\sigma = 1.0$. The results are presented in Tables III-VII, The format is the same as that used earlier in Tables I, II.

The FIXED, VT and PF rules do, of course, guarantee the probability requirement for the ranking problem. It is well known that, under certain parameter configurations, the sequential procedures lead to considerably lower ASN's than does the single-stage procedure. As between VT and PF, overall PF has of course a much lower ITN, and it appears to have a comparable or lower ASN. Also the overprotection (PCS actually attained) seems to be greater for the Paulson procedure even with the Fabian modification. Bechhofer and Ramberg (1977) compare the VT and PF procedures in much greater detail.

One desirable feature of any ranking procedure is that the ASN should not be greatly increased by the introduction of extra "non-contending" populations with means $\theta << 0$. The Paulson procedure does not have this feature since the continuation region is enlarged as k increases and it takes longer for the contending populations to be eliminated. On the other hand, the number of observations on the "contending" populations is fairly insensitive for procedures involving the BKS stopping rule.

Concerning the almost sure termination of the adaptive rules, Lemma 1 of Section 2 holds whenever $\tau_{[k]} - \tau_{[k-1]} > \delta^*/2$. The proof is unchanged. The simulation results from Table III suggest that this result should hold even if the restriction is removed. As with any of the procedures, if one is concerned about a large variance of the sample size, bounded procedures can be constructed which still enjoy some of the benefits of adaptive sequential sampling. To do this, define $P' + P'' = 1 + P^*$, use the adaptive sequential procedure but with P'

Table III Simulation results

$P^* = 0.9,\ \delta^* = 0.2,\quad \sigma = 1,\ k = 3$

Equal Means Configuration $\tau_{[1]} = \tau_{[2]} = \tau_{[3]}$

(based on 1000 replications. VT results from BKS Table 18·5·3)

	FIXED	VT	RAND Q	GRS (1.0)	GRS (1.2)	BESSLER	$\sqrt{2}$:1:1	PF (.5)	PF(.25)	PF(0)
N_3	125	130.10 (3.0) 95.59	111.50 (3.2) 100.95	130.33 (3.9) 123.76	121.74 (3.4) 106.75	118.63 (3.8) 119.52	119.08 (2.8) 87.79	93.23 (1.2) 38.97	103.82 (1.5) [illegible].89	121.59 (3.0) 94.93
N_2	125	130.10 (3.0) 95.59	113.55 (3.2) 101.01	123.58 (3.5) 110.16	116.74 (3.2) 100.63	125.48 (4.2) 133.85	119.00 (2.8) 88.16	98.16 (1.3) 40.02	104.92 (1.6) 51.[illegible]	114.51 (3.0) 93.97
N_1	125	130.10 (3.0) 95.59	113.67 (3.2) 100.36	125.26 (3.5) 111.95	116.66 (3.4) 107.30	119.79 (3.9) 122.11	118.78 (2.8) 87.82	98.49 (1.2) 38.60	106.02 (1.6) 49.45	117.98 (3.1) 98.53
N_{TOT}	375	390.29 (9.1) 286.77	338.72 (7.8) 247.98	379.08 (8.4) 265.58	355.15 (7.6) 239.19	363.90 (9.6) 304.89	356.86 (8.2) 259.85	294.61 (2.9) 91.23	314.76 (3.8) 120.09	354.09 (7.4) 232.93
Q	-	0.9121	0.907	0.906	0.907	0.908	0.908	0.873	0.878	0.892

Table IV Simulation results

$k = 3$, $P^* = 0.9$, $\delta^* = 0.2$, $\sigma = 1$

δ-slippage (GLF) configuration with $\delta = 0.36$ $\tau_{[1]} = \tau_{[2]} = \tau_{[3]} - 0.36$

(based on 1000 replications. VT results from BKS Table 18·5·7)

	FIXED	VT	RAND Q	GRS (1.0)	GRS (1.2)	BESSLER	$\sqrt{2}$:1:1	PF (.5)	PF (.25)	PF(0)
N_3	125	46.89 (0.9) 25.11	72.72 (1.2) 38.20	103.48 (1.7) 52.59	84.72 (1.3) 39.85	53.60 (1.0) 32.28	47.58 (0.8) 25.13	60.60 (0.7) 22.49	55.17 (0.8) 25.55	51.25 (0.9) 27.23
N_2	125	46.89 (0.9) 25.11	27.40 (0.7) 22.24	24.96 (0.7) 22.52	25.51 (0.7) 20.85	34.57 (1.2) 37.07	38.29 (0.7) 22.26	51.99 (0.7) 20.75	46.43 (0.7) 23.33	41.64 (0.8) 24.25
N_1	125	46.89 (0.9) 25.11	28.52 (0.8) 24.59	24.46 (0.7) 22.96	26.14 (0.7) 22.41	37.34 (1.2) 38.54	38.40 (0.7) 22.58	52.38 (0.7) 21.71	46.33 (0.8) 24.00	42.36 (0.8) 25.09
N_{TOT}	375	140.66 (2.7) 75.33	128.65 (2.2) 68.53	152.90 (2.2) 70.19	136.37 (2.1) 66.37	125.51 (2.6) 80.75	124.27 (2.1) 67.39	164.97 (1.8) 56.62	147.94 (2.0) 63.40	135.25 (2.1) 65.12
Proportion of correct solutions	-	0.984 (0.004)	0.986 (0.004)	0.983 (0.004)	0.984 (0.004)	0.986 (0.004)	0.992 (0.003)	0.995 (0.002)	0.988 (0.003)	0.991 (0.003)
Q	-	0.914	0.908	0.905	0.906	0.908	0.909	0.956	0.945	0.931

Table V Simulation results

$k = 3$, $P^* = 0.9$, $\delta^* = 0.2$, $\sigma = 1$

Configuration $\tau_{[1]} = \tau_{[2]} - 2\delta^* = \tau_{[3]} - 3\delta^*$

(based on 200 replications, VT results based on 400 replications)

	FIXED	VT	RAND Q	GRS (1.0)	GRS (1.2)	BESSLER	$\sqrt{2}:1:1$	PF (.5)	PF (.25)	PF (0)
N_3	125	56.28 (2.0) 39.74	74.89 (3.3) 46.19	83.96 (3.2) 45.78	70.73 (2.4) 33.28	64.15 (3.2) 45.76	56.77 (2.5) 34.80	78.27 (2.5) 35.57	72.73 (3.2) 45.38	76.29 (4.2) 59.34
N_2	125	56.28 (2.0) 39.74	41.64 (2.7) 38.81	36.41 (3.2) 45.84	36.63 (2.4) 34.36	58.51 (3.3) 47.23	51.95 (2.7) 37.64	78.04 (2.5) 35.81	72.27 (3.2) 45.84	75.65 (4.2) 59.82
N_1	125	56.28 (2.0) 39.74	17.44 (0.6) 8.81	18.81 (0.7) 10.43	21.42 (0.8) 11.82	16.74 (0.7) 9.58	45.36 (2.1) 29.57	32.95 (0.8) 11.04	28.10 (0.7) 9.90	25.23 (0.8) 11.25
N_{TOT}	375	168.85 (6.0) 119.23	133.97 (5.4) 76.14	139.18 (5.5) 63.18	128.78 (4.1) 58.43	139.41 (6.6) 92.79	154.08 (7.0) 99.50	189.26 (5.3) 74.49	173.11 (6.58) 93.12	177.18 (8.7) 122.43
Proportion of correct selections	-	0.927 (0.013)	0.920 (0.019)	0.900 (0.021)	0.895 (0.022)	0.960 (0.014)	0.925 (0.019)	0.965 (0.013)	0.970 (0.012)	0.985 (0.009)
Q	-	0.914	0.908	0.906	0.906	0.908	0.909	0.944	0.940	0.935

Table VI Simulation results

$k = 3$, $P^* = 0.9$, $\delta^* = 0.2$, $\sigma = 1$

Equally spaced configuration $\tau_{[1]} = \tau_{[2]} - \delta^* = \tau_{[3]} - 2\delta^*$

(based on 200 replications, VT results based on 400 replications)

	FIXED	VT	RAND Q	GRS (1.0)	GRS (1.2)	BESSLER	$\sqrt{2}$:1:1	PF (.5)	PF (.25)	PF (0)
N_3	125	56.82 (1.6) 32.88	79.91 (3.3) 47.09	104.13 (4.2) 59.77	90.99 (3.3) 46.40	78.16 (4.6) 65.45	66.03 (3.1) 43.40	72.75 (2.2) 30.46	72.32 (2.9) 41.53	76.12 (3.5) 49.81
N_2	125	56.82 (1.6) 32.88	52.58 (3.6) 50.58	42.46 (3.0) 42.85	39.83 (2.5) 35.81	74.92 (5.2) 72.87	60.01 (3.4) 47.83	71.19 (2.2) 31.13	70.04 (3.0) 42.09	73.58 (3.6) 50.91
N_1	125	56.82 (1.6) 32.88	27.75 (1.4) 19.75	26.03 (1.2) 16.51	28.61 (1.4) 19.18	27.93 (2.8) 40.16	54.97 (2.7) 37.61	44.62 (1.2) 17.28	39.63 (1.3) 18.86	37.93 (1.5) 20.69
N_{TOT}	375	170.46 (4.8) 110.6	160.24 (6.7) 95.01	172.61 (6.0) 85.08	159.43 (5.9) 83.19	181.02 (9.9) 139.97	181.01 (9.0) 126.68	188.56 (4.9) 69.30	181.99 (6.3) 89.47	187.63 (7.7) 108.36
Proportion of correct selections	-	0.9125 (0.014)	0.935 (0.017)	0.935 (0.017)	0.930 (0.018)	0.925 (0.019)	0.885 (0.023)	0.970 (0.012)	0.950 (0.015)	0.975 (0.011)
Q	-	.912	.907	.906	.906	.908	.909	.947	.936	.933

Table VII Simulation results

$k = 10$, $P^* = 0.9$, $\delta^* = 0.2$, $\sigma = 1$

Equally spaced configuration $\tau_{[i+1]} - \tau_{[i]} = \delta^*$ $(1 \leq i \leq 9)$

(based on 200 replications)

	FIXED	VT	RAND Q	GRS (1.2)	PF (.25)
N_{10}	223	64.8	33.4	28.4	121.6
N_9	223	64.8	26.3	19.7	120.1
N_8	223	64.8	19.1	17.0	65.6
N_7	223	64.8	9.6	9.2	44.8
N_6	223	64.8	7.7	6.6	34.9
N_5	223	64.8	7.2	6.4	28.3
N_4	223	64.8	5.9	6.4	23.1
N_3	223	64.8	5.8	6.4	19.8
N_2	223	64.8	5.5	6.4	17.6
N_1	223	64.8	4.7	6.4	15.8
N_{TOT}	2230	648.2 (27.5) 388.9	125.2 (3.7) 51.72	113.0 (3.0) 42.8	491.5 (9.3) 131.9
Proportion of correct elections	-	0.95 (0.015)	0.675 (0.033)	0.575 (0.035)	0.975 (0.011)
Q	-	0.912	0.908	0.908	0.939

Frequency population i selected out of 200 replications

	POP 10	POP 9	POP 8	POP 7
VT	190	10	0	0
RAND Q	135	52	13	0
GRS (1.2)	115	52	29	4
PF (.25)	195	5	0	0

replacing P^*, but take no more observations than prescribed by the fixed sample size rule (Bechhofer 1954) for $PCS = P''$. This idea was suggested by BKS (p. 227).

An advantage of the VT and PF rules not enjoyed by the adaptive rules is that there is a "time-blocking" effect. This might be important if, for instance, overall quality of treatment improves with time.

7. Directions of future research.

It is disappointing that, in the ranking problem, the BKS stopping rule (2.1) does not guarantee the requirement that PCS $\geq P^*$ whenever $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$ independently of the sampling rule when $k \geq 3$. Thus the adaptive procedures of Section 3 cannot be regarded as a satisfactory solution to the general ranking problem. It is possible that the δ^*-slippage configuration is least favorable within a restricted class of configurations such as perhaps the δ-slippage (GLF) configurations where $\delta \geq \delta^*/2$ is unknown. Alternatively it may be possible to construct sampling rules (other than VT) for which the δ^*-slippage configuration is least favorable. Further investigation is needed into these questions.

In the past, there seems to have been little work done on adaptive sampling for identification or ranking problems with $k \geq 3$. Many of the arguments used by authors for $k = 2$ do not seem to carry over easily. For Bernoulli sbservations on $k \geq 3$ populations, Sobel and Weiss (1972) have discussed "Play-the-Winner" sampling and inverse stopping rules -- see also Berry and Young (1977).

For the LI identification problem the potential savings in ITN and ASN might seem to justify the more complex allocation rules of the adaptive procedures. Other procedures that might also be of interest, but were not compared here, are those of Box and Hill (1967) and Blot and Meeter (1973). No attempt here was made to find optimal sampling rules for the LI identification problem -- it is possible that the approach of Kiefer and Sacks (1963) might lead to optimal fully sequential procedures.

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APPENDIX

Here we give heuristic arguments concerning some sufficient conditions in the ranking problem for the non-consistency of the procedure (2.1) when adaptive sampling rules are employed. This gives insight into why the δ^*-slippage configuration is not least favorable and it is possible for PCS $<$ P* for some configurations in the preference zone.

Recall $z_i = n_i(\bar{X}_{i.} - \bar{X}_{..} - (N - n_i)\delta^*/2)$. W.l.o.g. assume $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$. Suppose also $N >> 0$ with $n_j >> 0$ $1 \leq j \leq n$. Also let $n_i = \max_{1<j<k} (n_j)$ and $n_i >> n_k$. For the adaptive sampling rules discussed, this situation can occur with positive probability due to an "unlucky start." Suppose $n_i = C \cdot N$ and on some set Ω' of positive probability C tends to a limit α where α is bounded away from 0 and 1. (For the argument that follows all we really need is that $\liminf_{N\to\infty} C > 0$ and $\limsup_{N\to\infty} C < 1$ on some set Ω' of non-zero probability. There is also a condition analogous to (A1) if $\alpha = 1$ on some Ω' of non-zero probability.)

Then by use of the law of the iterated logarithm it is possible to show that on Ω'

(A1) $z_i - z_k \sim \alpha(1 - \alpha)N(\mu_i - \delta^*/2) - \alpha \sum_{j\neq i} n_j\mu_j + o(N)$. If the configuration $\{\mu_i\}$ is such that $z_i - z_k \to +\infty$ as $N \to \infty$ then $Q \to 1$, the procedure will stop but Π_i will be incorrectly selected as best. From (A1), this happens if

(A2) $N\mu_i - (1 - \alpha)N\delta^*/2 - \sum_{j=1}^{k} n_j\mu_j \to +\infty$.

For the δ-slippage (GLF) configuration this cannot happen since the expression (A2) tends to $-\infty$ and eventually $z_k > z_i$. (Of course, for $\tilde{\delta} = \delta^*$ (the LI identification problem) we already have the consistency of the procedure (2.1) for any symmetric sampling rule that terminates almost surely.)

However, for a general configuration of the $\{\mu_j\}$, it is quite possible for condition (A2) to hold (e.g. for the equally-spaced configuration with $\mu_{j+1} - \mu_j = \delta$, $1 \leq j \leq k - 1$).

The question arises concerning the kind of sampling rules for which this phenomenon can occur, i.e., roughly speaking, when we can have $n_i >> n_k$ for $i \neq k$ and $n_k >> 0$. Unfortunately it occurs for rules like RAND Q and GRS, which have the feature of being more likely to sample next from the population with the highest current value of z_j (with the aim of reducing ITN). For these rules, due to an unlucky start it is quite possible for $z_i > z_k$ after some stage of sampling. For such rules this is likely to imply that $n_i > n_k$. Then Π_i is more likely to be sampled on next and, if (A2) is satisfied, the difference $z_i - z_k$ is likely to be increased, thus perpetuating and worsening the condition.