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GRAPHIC GREEDOIDS AND THEIR DUALS

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ABSTRACT

Greedoids can be viewed as relaxations of matroids. But, unlike matroids, it appears difficult to construct duals of general greedoids. In this paper we consider a special class of greedoids, called graphic greedoids, and we show how to construct the duals of certain graphic greedoids. Our results generalize those for graphic matroids.

1. INTRODUCTION

Greedoids, which were first introduced by Korte and Lovasz in [1], can be viewed as relaxations of matroids. Given that there is a very nice duality theory for matroids, it is very natural to ask the following questions: (1) Is there a similar duality for general greedoids? (2) Is it possible to generalize the approach, by which dual matroids can be constructed, to greedoids? Unfortunately, the answers to these two questions are "no". The reason is that a matroid can be determined by its bases, but a greedoid cannot. In other words, greedoids which have the same bases might be quite different. Thus while a dual matroid can be constructed from only the bases of the primal matroid; greedoids require much more information.

Definitions and basic notations about matroid and greedoid are introduced in Section 2.

Section 3 is the main part of the paper. We show that a graphic matroid can be viewed as a family of generalized trees, or forests, rooted at $U = V(G)$, the node set of a graph G. As its natural generalization, a graphic greedoid is a family of generalized trees rooted at $U \subseteq V(G)$. Furthermore we can construct the dual of a graphic greedoid. Under some conditions, the dual of a graphic greedoid is still a graphic greedoid and we obtain a generalization of duality for graphic matroids. In particular, when a graphic greedoid is a graphic matroid, our results correspond to the well-known results for matroids.

2. MATROIDS AND GREEDOIDS

A matroid on a finite set E is an ordered pair (E,F) with $F \subseteq 2^E$ such that

(M1) $\phi \in$ **F**.

(M2) $X \subseteq Y \in F$ implies $X \in F$.

(M3) If $X,Y \in F$ and $|X| < |Y|$, then there exists an $e \in Y-X$ such that $X \cup \{e\} \in F$.

Let $G(V,E)$ be a connected graph. A graphic matroid (E,F) is defined by

(2.1) $$F = \{S \subseteq E(G): \ S \text{ has no cycles}\}.$$

A <u>language</u> L over a finite ground set (alphabet) E is a collection of finite sequences of elements, or letters, in E. These sequences are also called <u>strings</u> or <u>words</u>. Words are denoted by small Greek letters. Let E* be the collection of all possible strings over E. The underlying set of a word α is denoted by $\tilde{\alpha}$. $\tilde{L} \subseteq 2^E$ is the collection of all underlying set of L. Furthermore, a language is called <u>simple</u> if no letter is repeated in any word.

For a language L, we consider the following axioms.

(G1) $\phi \in L$.

(G2) If $\alpha \in L$ and $\alpha = \beta \cdot \gamma$ then $\beta \in L$.

(G3) If $\alpha, \beta \in L$ and $|\alpha| < |\beta|$ then there exists an $e \in \beta$ such that $\alpha \cdot e \in L$.

According to Korte and Lovasz's paper [1], a greedoid is viewed as either a simple language that satisfies (G1), (G2) and

(G3), or an unordered set system that satisfies (M1) and (M3). Thus greedoids are direct relaxations of matroids, obtained by omitting (M2) from matroid axioms (M1), (M2) and (M3).

For a greedoid (E,F), members of F are called <u>feasible</u> or <u>independent</u> sets. A maximal feasible subset of a set $X \subseteq E$ is called a <u>basis</u> of X. Bases of E are called bases of the greedoid (E,F). A greedoid (E,F) is called <u>full</u> if $E \in F$. We say that (E,F) is <u>normal</u> if for any $e \in E$, there exists a member S of F such that $e \in S$.

3. GRAPHIC GREEDOIDS AND THEIR DUALS

Let $G(V,E)$ be a connected graph, where V and E are the node set and the edge set in G respectively. For a subgraph X of G, let $V(X)$ and $E(X)$ denote the node set and edge set of X respectively. A subgraph is called a tree if it is connected and is without cycles.

<u>Definition.</u> A subgraph $X \subseteq E(G)$ of G is called a <u>generalized tree</u>, if all its components $X_1, X_2, \ldots, X_k$ are trees, where we assume $E(X_i) \neq \phi$, for $i = 1,2,\ldots,k$.

Obviously, if X is a tree, then it is a generalized tree. But the reverse is not true generally.

(3.1) <u>Definition.</u> For a given node set $\phi \neq U \subseteq V(G)$, a generalized tree X is said to be <u>rooted at</u> U, if <u>all</u> components $X_1, X_2, \ldots, X_k$ of X such that

$$V(X_i) \cap U \neq \phi, \quad i = 1,2,\ldots,k.$$

According to Definition (3.1), we have the following proposition immediately.

Proposition 3.1. If X is a tree on a graph $G(V,E)$ and $U \subseteq V(X)$, then X is rooted at U.

Definition. For a connected graph $G(V,E)$, a subgraph system $F(U)$ is called a generalized tree system, if $F(U)$ consists of all generalized trees which are rooted at $U \subseteq V(G)$.

(3.2) An example. Let $G(V,E)$ be given in Figure 3.1, and $U = \{v_1\}$. Then

$F(U) = \{\phi, \{e_1\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}$.

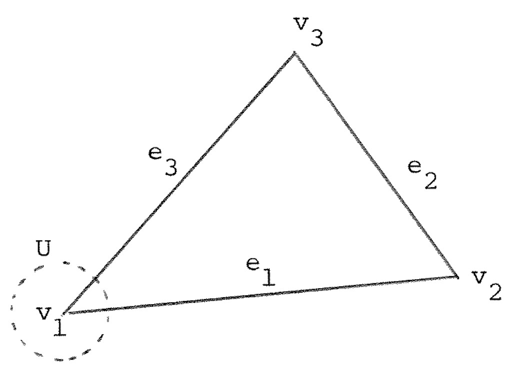

Figure 3.1

Proposition 3.2. For a connected graph $G(V,E)$, $F(V)$ is the graphic matroid.

Proof. By (2.1), it is clear that the proposition holds. □

Proposition 3.2 shows that a graphic matroid can be viewed as a generalized tree system.

Note that the reverse of Proposition 3.2 is not true generally. In other words, if $F(U)$ is a matroid, it is not necessarily the case that $U = V(G)$. For example, let $U_1 = \{v_1, v_2\}$ in Figure 3.1. then $F(U_1) = \{\phi, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}$. It is a graphic matroid. But $U_1 \neq V(G)$. If $F(U)$ is a matroid, the property about the node set $\mathbf{U}$ will be given in Proposition 3.6.

Lemma 3.3. For a connected graph $G(V,E)$ and $U \subseteq V(G)$, the generalized tree system $F(U)$ satisfies axiom (M3).

Proof. Assume $X, Y \in \mathbf{F(U)}$ *and* $|X| < |Y|$.

If Y is a tree, that is, Y consists of only one component, then we define

(3.3) $$Y_{\mathbf{U}} = \{e \in Y: \ e \text{ is rooted at } U\}.$$

(3.4) If there exists an $e \in Y_{\mathbf{U}}$ such that $e \notin X$ and $X \cup \{e\}$ has no cycles, then axiom (M3) holds.

Otherwise, for any $e \in Y_{\mathbf{U}}$ then

(3.5) either e is in X

(3.6) or $X \cup \{e\}$ has a cycle denoted by $C(e)$.

Thus we have

(3.7) $$Y_{\mathbf{U}} = Y^1_{\mathbf{U}} \cup Y^2_{\mathbf{U}}$$

where Y_U^1 consists of all elements in Y_U, say $e_1, e_2, \ldots, e_\ell$, which satisfy (3.6). Let $C_i = C(e_i)$, $i = 1, 2, \ldots, \ell$, and

$$C_i = X_i \cup \{e_i\}, \quad 1 \le i \le \ell, \tag{3.8}$$

where X_i is a certain subset of X. By (3.5) and (3.8), we see that

$$Y_U^2 \subseteq X \text{ and } \left|\bigcup_{i=1}^{\ell} C_i \cap X\right| \ge \left|\bigcup_{i=1}^{\ell} C_i \cap Y\right|. \tag{3.9}$$

Since for any $e_i \in Y_U^1 \subset Y_U$, $1 \le i \le \ell$, e_i is rooted at U, thus let $V(e_i) = \{v_i^1, v_i\}$, where $v_i \in U$. Define

$$Y_i = \{e \in (Y - Y_U) : e \text{ is rooted at } v_i^1\}, \quad 1 \le i \le \ell. \tag{3.10}$$

As $|X| < |Y|$, it is impossible that all Y_i, $1 \le i \le \ell$ are empty sets. Replacing Y_U by non-empty Y_i in their given order, the procedure from (3.4) to (3.9) is used repeatedly. (3.9) guarantees that (3.4) must occur eventually. Furthermore, (3.10) implies that such a subgraph $X \cup \{e\}$ is in $F(U)$. Therefore, axiom (M3) holds.

If y consists of several components, then we can use the above argument on each component of Y. □

From Lemma 3.3 and the fact that $\phi \in F(U)$, we immediately obtain the following theorem.

Theorem 3.4. For a connected graph $G(V,E)$ and $U \subseteq V(G)$, the generalized tree system $F(U)$ is a greedoid.

Definition. Because of this theorem, such an $F(U)$ is called *graphic greedoid*, and U is called the *root set* of $F(U)$.

Proposition 3.5. For a graphic greedoid $F(U)$ on $G(V,E)$, bases of $F(U)$ are spanning trees on $G(V,E)$.

Definition. For a given graphic greedoid $F(U)$ on $G(V,E)$, a node set S_F is called the *separate set* of $F(U)$, if

(3.11) $S_F = \{v \in V(G)$: there is $e \in E(G)$ such that e meets v and $\{e\} \notin F(U)\}$.

Proposition 3.6. If a graphic greedoid $F(U)$ is normal, then $F(U)$ is a matroid if and only if its separate sets $S_F = \phi$.

Proof. Assume $F(U)$ is a normal matroid, hence for any $e \in E(G)$, $\{e\} \in F(U)$. By (3.11), we see that $S_F = \phi$. Conversely, if $S_F = \phi$, (3.11) implies that for any $e \in E(G)$, $\{e\} \in F(U)$. If $\phi \neq X \in F(U)$, then by the definition of $F(U)$ and the fact that each edge e in X is in $F(U)$, we see that any subset of X is still a member of $F(U)$. Therefore, $F(U)$ is a matroid. $\square$

Definition. For a given generalized tree system $F(U)$ on $G(V,E)$, a node $v \in V(G)$ is called *F-maximal over U* if for all $v_1 \in (\bar{v} \cap U) \cup (\bar{v} \cap S_F)$, then

(3.12) $|\overline{v_1} \cap U| \le |\overline{v} \cap U|$,

where $\overline{v_1} = V(G) - v_1$, $\overline{v} = V(G) - v$ and S_F is given by (3.11).

Furthermore, a node set U^* is called _generalized complement of U over $F(U)$_ if

(3.13) $U^* = \{v \in V(G):\ v$ is F-maximal over $U\}$.

Proposition 3.7. If $F(U)$ is a normal matroid, then $U^* = V(G)$. Otherwise, $U^* = \overline{U} = V(G) - U$.

Proof. By Proposition 3.6, if $F(U)$ is a matroid, then $S_F = \phi$ and $(\overline{v} \cap U) \cup (\overline{v} \cap S_F) = (\overline{v} \cap U)$. If $v \in \overline{U}$, then $|v \cap U| = |U|-1$. Moreover, for all $v_1 \in (\overline{v} \cap U)$, we have $|\overline{v_1} \cap U| = |U|-1$. Thus (3.12) holds and $v \in U^*$. If $v \notin \overline{U}$, then $|\overline{v} \cap U| = |U|$, then (3.12) always holds. Hence $v \in U^*$. We see that $U^* = V(G)$. If $F(U)$ is not a matroid, then, by Proposition 3.6, $S_F \neq \phi$. If $v \in U$, then $|\overline{v} \cap U| = |U|-1$. By (3.12) and the definition of $F(U)$, it is easy to see that the fact that $v \in U$ implies $v \notin S_F$. Thus for any $v_1 \in S_F$, then $v_1 \notin U$. Furthermore, $|\overline{v_1} \cap U| = |U|$. Hence (3.12) does not hold. And so $v \notin U^*$. For any $v \in \overline{U}$, (3.12) always holds. Thus $v \in U^*$. We see that $U^* = \overline{U}$. □

Definition. A connected graph $G(V,E)$ is called a _discrete cycle graph_, or DC-graph for short, if for any two cycles C_1 and C_2 in G, then $E(C_1) \cap E(C_2) = \phi$. For example, the graph in Figure 3.2 is a DC-graph

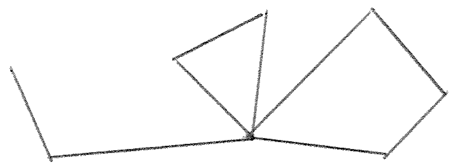

Figure 3.2

Obviously, if G is a DC-graph and B is its spanning tree, then $\overline{B} = E(G)-B$ has no cycles.

<u>Definition.</u> For a graphic greedoid $F(U)$ based on a DC-graph $G(V,E)$, $F^*(U)$ is called the <u>dual</u> of $F(U)$ if

(3.14) $F^*(U) = \{S \subseteq E(G):$ S satisfies (D1) and (D2)$\}$,

where

(D1) There is a basis B of $F(U)$ such that $S \cap B = \phi$.

(D2) S is a generalized tree rooted at U^*, i.e. $S \in F(U^*)$, where U^* is given by (3.13).

<u>Proposition 3.8.</u> If $F(U)$ is a normal graphic matroid on a DC-graph $G(V,E)$, then

(3.15) $F^*(U) = \{S \subseteq E(G):$ S satisfies (D1)$\}$.

<u>Proof.</u> If S satisfies (D1), then there is a basis B of $F(U)$ such that $B \cap S = \phi$. Since G is a DC-graph, by Proposition 3.5, we see that $\overline{B}$ has no cycles. But $S \subseteq \overline{B}$, S has no cycle too. Moreover, by Proposition 3.7, we see that $U^* = V(G)$. Thus S also satisfies (D2). □

Since (3.15) is in accordance with the dual matroid of $F(U)$, $F^*(U)$ given by (3.14) can be viewed as a generalization of duals of graphic matroids.

Theorem 3.9. For a DC-graph $G(V,E)$, let $F(U)$ be a generalized tree system on G. If non-empty node set U either is $V(G)$ or has the property that

(3.16) $\quad |V(C) \cap U| \leq 1$, for any cycle C in G,

then the following statements holds.

(3.17) (1) $F^*(U)$ is a greedoid, where $F^*(U)$ is given by (3.20).

(3.18) (2) $F^{**}(U) = F(U)$.

(3.19) (3) $|E(B^*)| + |E(B)| = |E(G)|$, where B^* and B are, respectively, bases of $F^*(U)$ and $F(U)$.

Proof. If $U = V(G)$, then $F(U)$ is a matroid. By Proposition 3.8, we see that $F^*(U)$ is the dual matroid of $F(U)$. Thus (3.17), (3.18) and (3.19) hold.

Let $X, Y \in F^*(U)$ and $|X| < |Y|$. Thus there are B_x and B_y which are bases of $F(U)$ such that $B_x \cap X = B_y \cap Y = \phi$. We see that if $e \in (Y-X)$, then e is in a cycle of G. Proposition 3.7 and property (3.16) guarantee that

(3.20) $\quad \{e\} \in F^*(U)$, where $U \subseteq V(G)$.

If $e \notin B_X$, then $(X \cup \{e\}) \cap B_X = \phi$ and $(X \cup \{e\}) \subseteq \overline{B}_X$, Since $G(V,E)$ is a DC-graph, thus $X \cup \{e\}$ has no cycles. This fact and (3.20) imply $X \cup \{e\} \in F^*(U)$. Therefore, axiom (M_3) holds.

If $e \in B_X$, then there is a unique cycle, say $C(e)$, in G such that $e \in C(e)$. Since $|C(e)-B_X| = 1$, let $u \in (C(e)-B_X)$. Clearly, $u \neq e$ and $(B_X-\{e\}) \cup \{u\}$ still is a basis of $F(U)$.

(3.21) If $u \notin X$, then $(X \cup \{e\}) \cap ((B-\{e\}) \cup \{u\}) = \phi$.

By the definition of $F^*(U)$, we see that $(X \cup \{e\}) \in F^*(U)$. Thus axiom (M3) holds. If (3.21) is not true, by the fact that $|X| < |Y|$, we can take a new $e_1 \in (Y-X)$ such that $e_1 \neq e$. Using e_1 instead of e and repeatedly applying the above procedure, the fact that $|X| < |Y|$ guarantees (3.21) must occur. Since $\phi \in F^*(U)$, (3.17) holds.

In order to prove (3.18), we have, by Proposition 3.7,

$$F^{**}(U) = \{S \subseteq E(G): (1) \text{ there is a basis } B^* \text{ of } F^*(U) \text{ such that } S \cap B^* = \phi \quad (2) \ S \in F(U^{**}) = F(U)\}.$$

It implies that $F^{**}(U) \subseteq F(U)$. On the other hand, if $S \in F(U)$, then there exists a basis B of $F(U)$ such that $S \subseteq B$. Because of (3.16) and that G is a DC-graph, $\overline{B} = E(G)-B$ is a basis of $F^*(U)$. Moreover, $\overline{B} \cap S = \phi$. Hence $s \in F^{**}(U)$ and so $F(U) \subseteq F^{**}(U)$, (3.18) holds.

The proof mentioned above also shows that statement (3.19) is true. □

<u>An example.</u>

Take example (3.2), since $U = \{v_1\}$, $U^* = \{v_2, v_3\}$,

$$F^*(U) = \{\phi, \{e_1\}, \{e_2\}, \{e_3\}\}.$$

$$F^{**}(U) = F(U).$$

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