Expected Values of Mean Squares for a Diallel Crossing Design With Maturity Groups

by

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ABSTRACT

Expectations of sums of squares for a diallel crossing treatment design with the lines divided into groups, and under a random effects model, becomes increasingly complicated as the number of groups increases. These expectations have been derived for two and three groups for an arbitrary number of lines in a group. For three groups, the sum of squares for group general combining ability contains 18 variance components and that for group specific combining ability contains 17 variance components in the expected value. For $g > 2$ groups, there would be $2g + 3g(g-1)/2 + 3$ and $2g + 3g(g-1)/2 + 2$ variance components in the expected value of the two respective sums of squares under a random effects model. The associated solutions for effects have been obtained for g groups. These results are applied to an artificial numerical example. Some comments are presented relative to the suitability of random effects and fixed effects models. The problem arose in response to questions from the last two authors regarding an analysis for a diallel crossing experiment of 12 lines of maize divided into three maturity groups with four lines in each group.

1. INTRODUCTION

An experiment designed as a randomized complete block experiment design with 66 crosses was conducted. The treatment design involved a diallel crossing design for 12 maize lines crossed in all possible combinations. The 12 lines belonged to three maturity groups (early = E, medium = M, and late = L) with four lines in each maturity group (see Table 1). The objective of the experiment was to

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study specific and general combining ability for maturity groups and for lines within groups.

A linear model for the situation such as the one described here could be:

$$Y_{abcde} = \mu + \rho_a + G_b + G_c + S_{bc} + g_{bcd} + g_{bce} + s_{bcde} + \epsilon_{abcde} , \qquad (1)$$

where ρ_a and ϵ_{abcde} are random independent block and error effects distributed with mean zero and variances σ^2_ρ and σ^2_ϵ, respectively, μ is an effect common to every observation, G_b and G_c are the group general combining ability (gca) effects for maturity groups b and c, S_{bc} are the group specific combining ability (sca) effects of group b with group c, g_{bcd} and g_{bce} are the gca's of lines d and e, respectively, within the set of crosses from lines in groups b and c, s_{bcde} are the sca effects of lines d and e within the maturity groups set bc, $a = 1,2,\cdots,r$ replicates, $b \leq c = 1,2,\cdots,g$ groups, $d = 1,2,\cdots,n_b$, and $e = 1,2,\cdots,n_c$, where n_b equals the number of lines in group b and n_c is the number of lines in group c. Let n_{bcde} be 1 or 0, depending upon whether or not line d is crossed with line e in the set of n_b lines from group b crossed with the n_c lines of group c. Let n_{bc} be the number of crosses made in the set bc; then, $n_{bc} = n_b n_c$ for $b \neq c$ and $n_{bc} = n_{bb} = n_b(n_b - 1) / 2$ for $b = c$. Also, $n_{b.} = n_b(n_b - 1) / 2 + n_b \sum_{c \neq b} n_c = n_b[v - (n_b + 1)/2]$ is the total number of crosses with lines from group b, where $v = \sum_{b=1}^{g} n_b$. In the above experiment, $n_b = n_c = 4$ and $n_{b.} = 4(4-1) / 2 + 4(4+4) = 38$ crosses of early lines with themselves and with the lines in the medium and late maturity groups.

It should be noted that the above model and treatment design differs from the one given by Hinkelmann (1974) in several respects. First, the treatment design here contains lines of group b crossed with the other lines in group b whereas Hinkelmann's does not. He considers an equal number of lines n from each population whereas in our formulation the number (n_b and n_c) may vary with b and c. In addition, he considers general combining ability effects (gca) as g_{bd} and g_{ce} across all other groups, whereas we consider gca effects for the n_b lines of group b crossed with the n_c lines of group c, i.e., g_{bcd} and g_{bce}. The same is true for sca effects. This was done to accommodate the fact that gca and sca effects within the set of crosses bc could vary as bc varies. This could be particularly true for gca and sca effects within group (or population) and between group crosses.

To obtain solutions for the effects in this overparameterized model as given in Section 2, let us impose the following constraints on the solutions for the various effects in (1):

Table 1. Treatment design for 12 lines from three maturity groups (X denotes a cross and blank means no cross).

Line and maturity group

		Early = E				Medium = M				Late = L			
		1	2	3	4	5	6	7	8	9	10	11	12
E	1		X	X	X	X	X	X	X	X	X	X	X
	2			X	X	X	X	X	X	X	X	X	X
	3				X	X	X	X	X	X	X	X	X
	4					X	X	X	X	X	X	X	X
M	5						X	X	X	X	X	X	X
	6							X	X	X	X	X	X
	7								X	X	X	X	X
	8									X	X	X	X
L	9										X	X	X
	10											X	X
	11												X
	12												

$$\left.\begin{array}{l}\sum_{b=1}^{g}\hat{G}_b = \sum_{c=1}^{g}\hat{G}_c = \sum_{b=1}^{g}\hat{S}_{bc} = \sum_{c=1}^{g}\hat{G}_{bc} = 0 \\[2em] \sum_{d=1}^{n_b}\hat{g}_{bcd} = \sum_{e=1}^{n_c}\hat{g}_{bce} = \sum_{d=1}^{n_b}\hat{s}_{bcde} = \sum_{e=1}^{n_c}\hat{s}_{bcde} = 0\end{array}\right\} \qquad (2)$$

An effect $\hat{G}_b$ will be obtained from all lines in group b crossed with all other lines. That is, for the early maturity group of lines (1, 2, 3, and 4 in Table 1) , line 1, e.g., will be crossed with all other lines $2,3,\cdots,12$. The same will be true for lines 2, 3, and 4.

This set of $6+16+16=38$ crosses is used to illustrate how to estimate the general combining ability effect for group b = E, that is $\hat{G}_E$. The $\hat{G}_b$ and $\hat{S}_{bc}$ values for this example are obtained from the $g(g+1)/2=6$ groups E by E, E by M, E by L, M by M, M by L, and L by L. It could be argued that E by E, M by M, and L by L groups should be treated like selfs. If so, then the remaining three groups would be used to estimate G_b under the assumption that $S_{bb}=0$. At least four groups would be necessary to obtain estimates for S_{bc}. The partitioning of the degrees of freedom for $4(4-1)/2=6$ groups would then be:

Source of variation	Degrees of freedom	
Groups	5	
Selfs		2
Selfs vs. others		1
Among others (GCA for groups)		2

We are using the previous approach, equation (1), and the analysis as given in Table 2, i.e., G_b and S_{bc} are obtained from all groups and not as described above.

Expected values for mean squares in the analysis of variance (ANOVA) for a diallel crossing experiment have been given by Federer (1948, 1951, 1955) and Griffing (1956). An extensive discourse on statistical analyses for diallel crossing systems has been given by Randall (1976). Also, as mentioned above, Hinkelmann (1974) has considered two-level diallel crossing experiment analysis. The expected values of mean squares for the ANOVA in Table 2 are given in Sections 3 and 4.

A numerical example illustrating use of the solutions and expected values is given in Section 5. A discussion and summary of results appears in Section 6.

Table 2. ANOVA for the described experiment with random effects.

Source of variation*	Degrees of freedom			Expected value of mean square
Total	65r			—
Correction for mean	1			—
Block	r − 1			—
Treatment	65			—
Groups		5		—
E by E		5		—
GCA (E by E)			3	$\sigma_\epsilon^2 + r\,\sigma_{sEE}^2 + 2r\,\sigma_{gEE}^2$
SCA (E by E)			2	$\sigma_\epsilon^2 + r\,\sigma_{sEE}^2$
M by M		5		—
GCA (M by M)			3	$\sigma_\epsilon^2 + r\,\sigma_{sMM}^2 + 2r\,\sigma_{gMM}^2$
SCA (M by M)			2	$\sigma_\epsilon^2 + r\,\sigma_{sMM}^2$
L by L		5		—
GCA (L by L)			3	$\sigma_\epsilon^2 + r\,\sigma_{sLL}^2 + 2r\,\sigma_{gLL}^2$
SCA (L by L)			2	$\sigma_\epsilon^2 + r\,\sigma_{sLL}^2$
E by M		15		—
GCAE (M)			3	$\sigma_\epsilon^2 + r\,\sigma_{sEM}^2 + 4r\,\sigma_{gE(M)}^2$
GCA(E) M			3	$\sigma_\epsilon^2 + r\,\sigma_{sEM}^2 + 4r\,\sigma_{g(E)M}^2$
SCA (E by M)			9	$\sigma_\epsilon^2 + r\,\sigma_{sEM}^2$
E by L		15		—
GCAE (L)			3	$\sigma_\epsilon^2 + r\,\sigma_{sEL}^2 + 4r\,\sigma_{gE(L)}^2$
GCA(E) L			3	$\sigma_\epsilon^2 + r\,\sigma_{sEL}^2 + 4r\,\sigma_{g(E)L}^2$
SCA (E by L)			9	$\sigma_\epsilon^2 + r\,\sigma_{sEL}^2$
M by L		15		—
GCAM (L)			3	$\sigma_\epsilon^2 + r\,\sigma_{sML}^2 + 4r\,\sigma_{gM(L)}^2$
GCA(M) L			3	$\sigma_\epsilon^2 + r\,\sigma_{sML}^2 + 4r\,\sigma_{g(M)L}^2$
SCA (M by L)			9	$\sigma_\epsilon^2 + r\,\sigma_{sML}^2$
Block × treatment = error	65(r − 1)			σ_ϵ^2
Effects ($G_b + G_c + S_{bc}$, elim. μ)	5			
GCA for groups		2		(See Section 4)
Interaction = SCA for groups		3		

* GCA = general combining ability.

SCA = specific combining ability.

GCAE(M) = GCA effect for group E from E by M group, etc.

2. SOLUTIONS FOR EFFECTS IN EQUATION (1)

The various totals associated with the normal equations from a least squares analysis are given below. The constraints in (2) plus $\sum_{a=1} \hat{\rho}_a = 0$ are used to obtain the solutions. The usual dot notation for summations is used here.

$$Y_{\cdot bcde} = r(\mu + G_b + G_c + S_{bc} + g_{bcd} + g_{bcd} + s_{bcde}) \, , \tag{3}$$

$$Y_{\cdot bcd \cdot} = r\, n_c(\mu + G_b + G_c + S_{bc} + g_{bcd \cdot}) \text{ for } b \neq c \tag{4}$$

$$= r(n_c - 1)(\mu + 2G_b + S_{bb}) + r(n_b - 2) g_{bbd \cdot} \text{ for } b = c \, , \tag{5}$$

$$Y_{\cdot bc \cdot e} = r\, n_b(\mu + G_b + G_c + S_{bc} + g_{bc \cdot e}) \text{ for } b \neq c \tag{6}$$

$$= r(n_b - 1)(\mu + 2G_b + S_{bb}) + r(n_b - 2) g_{bb \cdot e} \text{ for } b = c \, , \tag{7}$$

and

$$Y_{\cdot bc \cdot \cdot} = r\, n_b n_c(\mu + G_b + G_c + S_{bc}) \text{ for } b \neq c \tag{8}$$

$$= r\, n_b(n_b - 1)(\mu + 2G_b + S_{bb}) \big/ 2 \text{ for } b = c \, . \tag{9}$$

Solutions for specific and general combining ability effects in set bc are:

$$\hat{s}_{bcde} = \bar{y}_{\cdot bcde} - \bar{y}_{\cdot bcd \cdot} - \bar{y}_{\cdot bc \cdot e} + \bar{y}_{\cdot bc \cdot \cdot} \text{ for } b \neq c \tag{10}$$

$$= \bar{y}_{\cdot bcde} - (\bar{y}_{\cdot bbd \cdot} + \bar{y}_{\cdot bb \cdot e}) \big/ r(n_b - 2) + n_b \bar{y}_{\cdot bb \cdot \cdot} \big/ (n_b - 2) \text{ for } b = c \, , \tag{11}$$

$$\hat{g}_{bcd \cdot} = \bar{y}_{\cdot bcd \cdot} - \bar{y}_{\cdot bc \cdot \cdot} \text{ for } b \neq c \tag{12}$$

$$= \left[Y_{\cdot bbd \cdot} - r(n_b - 1) \bar{y}_{\cdot bb \cdot \cdot} \right] \big/ r(n_b - 2) \text{ for } b = c \, , \tag{13}$$

and

$$\hat{g}_{bc \cdot e} = \bar{y}_{\cdot bc \cdot e} - \bar{y}_{\cdot bc \cdot \cdot} \text{ for } b \neq c \tag{14}$$

$$= \left[Y_{\cdot bb \cdot e} - r(n_b - 1) \bar{y}_{\cdot bb \cdot \cdot} \right] \big/ r(n_b - 2) \text{ for } b = c \, . \tag{15}$$

The remaining totals associated with the normal equations are:

$$Y_{\cdot b \cdot \cdot \cdot} = r\, n_b(\mu + G_b) \sum_{c \neq b} n_c + r\, n_b \sum_{c \neq b} n_c(G_c + S_{bc}) + r\, n_b(n_b - 1)(\mu + 2G_b + S_{bb}) \big/ 2 \, , \tag{16}$$

$$Y_{\cdot \cdot c \cdot \cdot} = r\, n_c(\mu + G_c) \sum_{b \neq c} n_b + r\, n_c \sum_{b \neq c} n_b(G_b + S_{bc}) + r\, n_c(n_c - 1)(\mu + 2G_c + S_{cc}) \big/ 2 \, , \tag{17}$$

and

$$Y_{\cdot \cdot \cdot \cdot \cdot} = r \sum_{b=1}^{g} n_b(n_b - 1)(\mu + 2G_b + S_{bb}) \big/ 2 + r \sum_{b < c=2} \sum^{g} n_b n_c(\mu + G_b + G_c + S_{bc}) \, . \tag{18}$$

Solutions for $\hat{G}_b$, $\hat{\mu}$, and $\hat{S}_{bc}$ are:

$$\hat{G}_b = \sum_{c=1}^{g} (\bar{y}_{\cdot bc \cdot \cdot} - \hat{\mu}) \Big/ g = \sum_{c=1}^{g} \bar{y}_{\cdot bc \cdot \cdot} \Big/ g - \hat{\mu} \,, \tag{19}$$

$$\hat{\mu} = \left[\sum_{c=1}^{g} \bar{y}_{\cdot bb \cdot \cdot} + 2 \sum_{b\,<} \sum_{c=2}^{g} \bar{y}_{\cdot bc \cdot \cdot} \right] \Big/ g^2 \,, \tag{20}$$

and

$$\hat{S}_{bc} = \bar{y}_{\cdot bc \cdot \cdot} - \hat{\mu} - \hat{G}_b - \hat{G}_c \,. \tag{21}$$

3. SUMS OF SQUARES AND EXPECTED VALUES FOR CROSSES AMONG LINES OF GROUP bc.

The general and specific combining ability sums of squares for crosses among lines of groups b and c for $b \neq c$ are the same as for a two-factor factorial. The expected value of these sums of squares for both random and fixed effects, may be obtained from Federer (1955, Section VIII–5.1), Federer and Plaisted (1962), or Hinkelmann (1974). The last author also considers the case where some effects are fixed and some are random. Herein we only consider a random or a fixed effects model, which is not always appropriate. The various sums of squares for n_b lines crossed with n_c lines in all combinations, is:

Group main effect or general combining ability

$$\sum_{d=1}^{n_b} \frac{Y^2_{\cdot bcd \cdot}}{r n_c} - \frac{Y^2_{\cdot bc \cdot \cdot}}{r n_b n_c} \tag{22}$$

with $n_b - 1$ degrees of freedom and

$$\sum_{e=1}^{n_c} \frac{Y^2_{\cdot bc \cdot e}}{r n_b} - \frac{Y^2_{\cdot bc \cdot \cdot}}{r n_b n_c} \tag{23}$$

with $n_c - 1$ degrees of freedom.

Interaction or specific combining ability:

$$\sum_{d=1}^{n_b} \sum_{e=1}^{n_c} \frac{Y^2_{\cdot bcde}}{r} - \sum_{d=1}^{n_b} \frac{Y^2_{\cdot bcd \cdot}}{r n_c} - \sum_{e=1}^{n_c} \frac{Y^2_{\cdot bc \cdot e}}{r n_b} + \frac{Y^2_{\cdot bc \cdot \cdot}}{r n_b n_c} \tag{24}$$

with $(n_b - 1)(n_c - 1)$ degrees of freedom.

The expected value for the sum of squares in (22) is

$$\left(n_b-1\right)\left(\sigma_\epsilon^2+r\,\sigma_{sbc}^2+r\,n_c\,\sigma_{gb(c)}^2\right). \tag{25}$$

The expected value for the sum of squares in (23) is:

$$\left(n_c-1\right)\left(\sigma_\epsilon^2+r\,\sigma_{sbc}^2+r\,n_b\,\sigma_{g(b)c}^2\right). \tag{26}$$

The expected value for the sum of squares in (24) is:

$$\left(n_b-1\right)\left(n_c-1\right)\left(\sigma_\epsilon^2+r\,\sigma_{sbc}^2\right). \tag{27}$$

In the above, σ_ϵ^2 is an error variance component, σ_{sbc}^2 is a variance component associated with specific combining ability for group bc, $\sigma_{gb(c)}^2$ is a variance component associated with general combining ability for lines in group b in the presence of lines from group c, and $\sigma_{g(b)c}^2$ is a variance component associated with general combining ability for lines in group c when crossed with lines of group b. The expected values of mean squares are given in Table 2 for the specific example when $n_b = 4$.

When $b = c$, i.e., group bb, the formulas for sums of squares and expected values for mean squares may be found in various places when there are $n_b(n_b-1)/2$ crosses among the n_b lines (e.g., Sprague and Tatum, 1942; Federer, 1951, 1955; Griffing, 1956; Federer and Henderson, 1956). The sums of squares for general combining ability among lines in group bb is:

$$\sum_{d=1}^{n_b} 4\left(\frac{n_b}{2}Y_{\cdot bbd\cdot} - Y_{\cdot bb\cdot\cdot}\right)^2 \Big/ r\,n_b^2(n_b-2) \tag{28}$$

with expected value

$$\left(n_b-1\right)\left(\sigma_\epsilon^2+r\,\sigma_{sbc}^2+r(n_b-2)\,\sigma_{gb(e)}^2\right) \tag{29}$$

and with n_b-1 degrees of freedom. Other formulas for computing the sum of squares in (28) are

$$\sum_{d=1}^{n_b} \hat{g}_{bbd\cdot}Y_{\cdot bbd\cdot} = \sum_{d=1}^{n_b} \frac{Y^2_{\cdot bbd\cdot}}{r(n_b-2)} - \frac{4Y^2_{\cdot bb\cdot\cdot}}{r\,n_b(n_b-2)}. \tag{30}$$

The sum of squares for specific combining ability for lines in group bb may be computed as

$$\sum_{d<}\sum_{e=2}^{n_b} \hat{s}_{bbde}Y_{\cdot bbde} = \sum_{d<}\sum_{e=2}^{n_b} \frac{Y^2_{\cdot bcde}}{r} - \frac{2Y^2_{\cdot bb\cdot\cdot}}{r\,n_b(n_b-1)}$$

$$-\sum_{d=1}^{n_b} \hat{g}_{bbd\cdot}Y_{\cdot bbd\cdot} = \sum_{d<}\sum_{e=2}^{n_b} \frac{Y^2_{\cdot bcde}}{r} - \sum_{d=1}^{n_b} \frac{Y^2_{\cdot bbd\cdot}}{r(n_b-2)} + \frac{2Y^2_{\cdot bb\cdot\cdot}}{r(n_b-1)(n_b-2)}, \tag{31}$$

with $n_b(n_b-3)/2$ degrees of freedom and with expected value equal to

$$\frac{n_b(n_b-3)}{2}\left(\sigma_\epsilon^2+r\,\sigma_{sbb}^2\right). \tag{32}$$

Several expectations for the mean square for line d crossed with all other lines have appeared in the literature (Federer, 1948, 1951, 1955; Henderson, 1948; Rojas and Sprague, 1952; Griffing, 1956). Federer and Henderson (1956) discuss all these forms and demonstrate that all involve different model assumptions. They show that all are correct but the appropriate choice for an experiment centers on which model assumption is correct for the situation being considered. A short bibliography was also given by these authors. A more complete one may be found in Randall (1976).

4. SUMS OF SQUARES AND EXPECTED VALUES FOR GROUPS

As stated in the introduction, the treatment plan used for the maize experiment differs from that presented by Hinkelmann (1974), and as far as is known has not appeared elsewhere in the literature. Thus, it is necessary to develop formulae for computing sums of squares and their expected values. It is inappropriate in most cases to consider the group effects and their interactions as random effects. However, we present both cases, fixed and random, in the event that an experimental situation would have a random group effects situation. The sum of squares among the $g(g+1)/2$ groups may be computed as:

$$\sum_{b \leq} \sum_{c=1}^{g} Y^2_{\cdot bc\cdot\cdot} \Big/ r\, n_{bc} - Y^2_{\cdot\cdot\cdot\cdot\cdot} \Big/ r\, n_{\cdot\cdot}\,, \tag{33}$$

where $n_{bc} = n_b(n_b - 1)/2$ for $b = c$, $n_{bc} = n_b n_c$ for $b \neq c$, and $n_{\cdot\cdot}$ is the total number of crosses in the experiment, i.e., $n_{\cdot\cdot} = \left(\sum_{i=1}^{g} n_i\right)\left(\sum_{i=1}^{g} n_i - 1\right) \Big/ 2 = v(v-1)2$; for the maize experiment, this is $12(11)/2 = 66$ crosses. The sums of squares for the group general combining ability and specific combining ability effects are, respectively:

$$r \sum_{b=1}^{g} n_b \left[v - (n_b + 1) \Big/ 2 \right] \hat{G}_b^2 \tag{34}$$

and

$$r \sum_{b \leq} \sum_{c=1}^{g} n_{bc}\, \hat{S}_{bc}^2\,. \tag{35}$$

For the random effects situation, the expectations of these sums of squares is quite complex. It was necessary to use the software package *Mathematica* to handle the tedious algebra. For all sums of squares except equations (34) and (35), $\hat{\mu}$ and $\bar{y}_{\cdot\cdot\cdot\cdot\cdot}$ were the same but for these last two $\hat{\mu} \neq \bar{y}_{\cdot\cdot\cdot\cdot\cdot}$ and this means that the sums of squares in (34) and (35) will *not* add to the one in (33).

A further complication of this particular situation is that the coefficient of σ_ϵ^2 in (34) and (35) will *not* be degrees of freedom.

Owing to the complexities, it was decided to present results for g = 2 and g = 3 rather than for a general value of g. For g = 2,

$$\hat{G}_1 = -\hat{G}_2 = \frac{1}{2^2}(\bar{y}_{\cdot 11\cdot\cdot} - \bar{y}_{\cdot 22\cdot\cdot})$$

and

$$\hat{S}_{11} = \frac{1}{2^2}(\bar{y}_{\cdot 11\cdot\cdot} - 2\bar{y}_{\cdot 12\cdot\cdot} + \bar{y}_{\cdot 22\cdot\cdot}) = \hat{S}_{22} = -\hat{S}_{12}\,.$$

The expected value for the sum of squares in (35) is given in Table 3 and involves nine variance components. This sum of squares has one degree of freedom but the coefficient of σ_ϵ^2 for $n_1 = 3$ and $n_2 = 4$ is 35/32. If all effects except ϵ_{abcde} are fixed, then the expected value of the sum of squares in (35) is

$$\frac{1}{2^4}\left\{\left[\frac{2}{n_1(n_1-1)} + \frac{4}{n_1 n_2} + \frac{2}{n_2(n_2-1)}\right]\left[\frac{v(v-1)}{2}\right]\sigma_\epsilon^2 + \frac{r\,v(v-1)}{2}\left(S_{11} - 2S_{12} + S_{22}\right)^2\right\}\,.$$

The expected value of the sum of squares in (34) for g = 2 is given in Table 4. There are seven variance components in the expectation. Because $\bar{y}_{\cdot 12\cdot\cdot}$ is not included, the variance components σ_{s12}^2, $\sigma_{g1(2)}^2$, and $\sigma_{g(1)2}^2$ do not appear in the expectation. As $n_1 = n_2$ becomes large, the coefficient of σ_ϵ^2 approaches 3/4. Holding n_1 constant and letting n_2 become large, the coefficient of σ_ϵ^2 becomes large, implying that as soon as the proportion of observations contained in group 2 approaches one, the amount of information on the difference $\hat{G}_1 - \hat{G}_2$ approaches zero.

For g = 3 groups, it is convenient to use solutions for G_b in terms of the group means, i.e.,

$$\hat{G}_1 = \frac{1}{3^2}\left[2\bar{y}_{\cdot 11\cdot\cdot} + \bar{y}_{\cdot 12\cdot\cdot} + \bar{y}_{\cdot 13\cdot\cdot} - \bar{y}_{\cdot 22\cdot\cdot} - \bar{y}_{\cdot 33\cdot\cdot} - 2\bar{y}_{\cdot 23\cdot\cdot}\right],$$

$$\hat{G}_2 = \frac{1}{3^2}\left[2\bar{y}_{\cdot 22\cdot\cdot} + \bar{y}_{\cdot 12\cdot\cdot} + \bar{y}_{\cdot 23\cdot\cdot} - \bar{y}_{\cdot 11\cdot\cdot} - \bar{y}_{\cdot 33\cdot\cdot} - 2\bar{y}_{\cdot 13\cdot\cdot}\right],$$

and

$$\hat{G}_3 = \frac{1}{3^2}\left[2\bar{y}_{\cdot 33\cdot\cdot} + \bar{y}_{\cdot 13\cdot\cdot} + \bar{y}_{\cdot 23\cdot\cdot} - \bar{y}_{\cdot 11\cdot\cdot} - \bar{y}_{\cdot 22\cdot\cdot} - 2\bar{y}_{\cdot 12\cdot\cdot}\right].$$

For g groups,

$$\hat{G}_b = \frac{1}{g^2}\left[(g-1)\bar{y}_{\cdot bb\cdot\cdot} + (g-2)\sum_{\substack{c=1\\ \neq b}}^{g} \bar{y}_{\cdot bc\cdot\cdot} - \sum_{\substack{c=1\\ \neq b}}^{g} \bar{y}_{\cdot cc\cdot\cdot} - 2\sum_{\substack{b' < c\\ \neq b}}\sum \bar{y}_{\cdot b'c\cdot\cdot}\right].$$

Table 3. Coefficients for components in $r\left(n_{11}S_{11}^2 + n_1n_2S_{12}^2 + n_{22}S_{22}^2\right)$.

Component	Coefficient $\times 2^4$ for n_1 and n_2 lines*	n_1,n_2 3,4	4,4
σ_ϵ^2	$\left[\frac{2}{n_1(n_1-1)} + \frac{4}{n_1n_2} + \frac{2}{n_2(n_2-1)}\right] \cdot \left[\frac{v(v-1)}{2}\right]$	$\frac{35}{32}$	$\frac{49}{48}$
σ_{s11}^2	$\frac{2r}{n_1(n_1-1)}\left[\frac{v(v-1)}{2}\right]$	$\frac{7r}{16}$	$\frac{7r}{24}$
σ_{s12}^2	$\frac{4r}{n_1n_2}\left[\frac{v(v-1)}{2}\right]$	$\frac{7r}{32}$	$\frac{7r}{16}$
σ_{s22}^2	$\frac{2r}{n_2(n_2-1)}\left[\frac{v(v-1)}{2}\right]$	$\frac{7r}{32}$	$\frac{7r}{24}$
σ_{g11}^2	$\frac{4r}{n_1}\left[\frac{v(v-1)}{2}\right]$	$\frac{7r}{4}$	$\frac{7r}{4}$
$\sigma_{g1(2)}^2$	$\frac{4r}{n_1}\left[\frac{v(v-1)}{2}\right]$	$\frac{7r}{4}$	$\frac{7r}{4}$
$\sigma_{g(1)2}^2$	$\frac{4r}{n_2}\left[\frac{v(v-1)}{2}\right]$	$\frac{21r}{16}$	$\frac{7r}{4}$
σ_{g22}^2	$\frac{4r}{n_2}\left[\frac{v(v-1)}{2}\right]$	$\frac{21r}{16}$	$\frac{7r}{4}$
σ_S^2	$3r\,v(v-1)$	$\frac{63r}{8}$	$\frac{21r}{2}$
$(S_{11}-2S_{12}+S_{22})^2$	$\frac{r\,v(v-1)}{2}$	$\frac{21r}{16}$	$\frac{7r}{4}$

* $n_1 + n_2 = v$.

Table 4. Coefficients for components in $r\left(n_1 . G_1^2 + n_2 . G_2^2\right)$.

Component	Coefficient $\times 2^{4*}$	n_1,n_2 3,4	4,4
σ^2_ϵ	$\left[\frac{2}{n_1(n_1-1)}+\frac{2}{n_2(n_2-1)}\right]\cdot[n_1 . + n_2 .]$	$\frac{33}{32}$	$\frac{11}{12}$
σ^2_{s11}	$\frac{2r}{n_1(n_1-1)}\ [n_1 . + n_2 .]$	$\frac{11r}{16}$	$\frac{11r}{24}$
σ^2_{s12}	0	0	0
σ^2_{s22}	$\frac{2r}{n_2(n_2-1)}\ [n_1 . + n_2 .]$	$\frac{11r}{32}$	$\frac{11r}{24}$
σ^2_{g11}	$4r\,[n_1 . + n_2 .]\big/ n_1$	$\frac{11r}{4}$	$\frac{11r}{4}$
$\sigma^2_{g1(2)}$	0	0	0
$\sigma^2_{g(1)2}$	0	0	0
σ^2_{g22}	$4r\,[n_1 . + n_2 .]\big/ n_2$	$\frac{33r}{16}$	$\frac{11r}{4}$
σ^2_S	$2r\,[n_1 . + n_2 .]$	$\frac{33r}{8}$	$\frac{11r}{2}$
σ^2_G	$8r\,[n_1 . + n_2 .]$	$\frac{33r}{2}$	22r
$(G_1-G_2)^2$	$r\,v\,[n_1 . + n_2 .]$	$\frac{33r}{4}$	11r

* $n_1 . = n_1(n_1-1)/2 + n_1 n_2 = n_1\left(v-(n_1+1)\big/2\right)$

$n_2 . = n_2(n_2-1)/2 + n_1 n_2 = n_2\left(v-(n_2+1)\big/2\right)$

$v = n_1 + n_2$

The solutions for $\hat{S}_{bc}$ in terms of the group means for g = 3 are:

$$\hat{S}_{11} = \frac{1}{3^2}\left[4\bar{y}_{\cdot 11\cdot\cdot} - 4\bar{y}_{\cdot 12\cdot\cdot} - 4\bar{y}_{\cdot 13\cdot\cdot} + \bar{y}_{\cdot 22\cdot\cdot} + \bar{y}_{\cdot 33\cdot\cdot} + 2\bar{y}_{\cdot 23\cdot\cdot}\right],$$

$$\hat{S}_{12} = \frac{1}{3^2}\left[5\bar{y}_{\cdot 12\cdot\cdot} - 2\bar{y}_{\cdot 11\cdot\cdot} - 2\bar{y}_{\cdot 22\cdot\cdot} - \bar{y}_{\cdot 13\cdot\cdot} - \bar{y}_{\cdot 23\cdot\cdot} + \bar{y}_{\cdot 33\cdot\cdot}\right],$$

$$\hat{S}_{22} = \frac{1}{3^2}\left[4\bar{y}_{\cdot 22\cdot\cdot} - 4\bar{y}_{\cdot 12\cdot\cdot} - 4\bar{y}_{\cdot 23\cdot\cdot} + \bar{y}_{\cdot 11\cdot\cdot} + \bar{y}_{\cdot 33\cdot\cdot} + 2\bar{y}_{\cdot 13\cdot\cdot}\right],$$

$$\hat{S}_{13} = \frac{1}{3^2}\left[5\bar{y}_{\cdot 13\cdot\cdot} - 2\bar{y}_{\cdot 11\cdot\cdot} - \bar{y}_{\cdot 12\cdot\cdot} + \bar{y}_{\cdot 22\cdot\cdot} - \bar{y}_{\cdot 23\cdot\cdot} - 2\bar{y}_{\cdot 33\cdot\cdot}\right],$$

$$\hat{S}_{23} = \frac{1}{3^2}\left[5\bar{y}_{\cdot 23\cdot\cdot} + \bar{y}_{\cdot 11\cdot\cdot} - \bar{y}_{\cdot 12\cdot\cdot} - 2\bar{y}_{\cdot 22\cdot\cdot} - \bar{y}_{\cdot 13\cdot\cdot} - 2\bar{y}_{\cdot 33\cdot\cdot}\right],$$

and

$$\hat{S}_{33} = \frac{1}{3^2}\left[4\bar{y}_{\cdot 33\cdot\cdot} + \bar{y}_{\cdot 11\cdot\cdot} + 2\bar{y}_{\cdot 12\cdot\cdot} - 4\bar{y}_{\cdot 13\cdot\cdot} + \bar{y}_{\cdot 22\cdot\cdot} - 4\bar{y}_{\cdot 23\cdot\cdot}\right],$$

Note that the above solutions are independent of the number of lines in group b as only means appear in the solution.

The expected value of the sum of squares in (35) is given in Table 5. Here we note that 17 variance components are included. For g = 4 groups, there would be 28 variance components involved, and for g groups there would be $2g + 3g(g-1)/2 + 2$ variance components included in the expected value of the sum of squares in (35).

The coefficients of variance components for the sum of squares in (34) are given in Table 6. There are 18 variance components involved here. For g groups, the number would be $2g + 3g(g-1)/2 + 3$. The coefficient of σ_ϵ^2 is close to two for the two examples in the table, i.e., $n_1 = n_2 = 3$ and $n_3 = 4$ and $n_1 = n_2 = n_3 = 4$. For the sum of squares in Table 5, the coefficient of σ_ϵ^2 is 257/81 and 55/18 for these two examples and is close to 3, the degrees of freedom. As was stated in Table 6, the coefficient of σ_ϵ^2 may be obtained by summing the six sets of coefficients for σ_{sbc}^2 and dividing by r, the number of replicates. This fact is useful since it is much easier to check the formulas of each σ_{sbc}^2 than for the sum of all six. In going to $g \geq 4$ groups this idea will become increasingly useful.

For the fixed effects case for all effects except ϵ_{abcde}, all terms drop out except for SSG in Table 6 and for the sum of squares for the three contrasts of the $\hat{S}_{bc}$. The coefficients for σ_ϵ^2 remain the same as given in Tables 5 and 6.

Table 5. Coefficients for components in $r \sum_{b \leq} \sum_{c=1}^{3} n_{bc} \hat{S}^2_{bc}$.

Component	Coefficient $\times 3^4$ for n_1, n_2, and n_3 lines*	n_1,n_2,n_3 3,3,4	n_1,n_2,n_3 4,4,4
σ^2_ϵ	(sum of coefficients for σ^2_{sbc} divided by r)	$\frac{257}{81}$	$\frac{55}{18}$
σ^2_{s11}	$\frac{r}{n_{11}}[16n_{11}+4n_1n_2+n_{22}+4n_1n_3+n_2n_3+n_{33}]$	$\frac{51r}{81}$	$\frac{14r}{27}$
σ^2_{s12}	$\frac{r}{n_1n_2}[16n_{11}+25n_1n_2+16n_{22}+n_1n_3+n_2n_3+4n_{33}]$	$\frac{41r}{81}$	$\frac{r}{2}$
σ^2_{s13}	$\frac{r}{n_1n_3}[16n_{11}+n_1n_2+4n_{22}+25n_1n_3+n_2n_3+16n_{33}]$	$\frac{53r}{108}$	$\frac{r}{2}$
σ^2_{s22}	$\frac{r}{n_{22}}[n_{11}+4n_1n_2+16n_{22}+n_1n_3+4n_2n_3+n_{33}]$	$\frac{51r}{81}$	$\frac{14r}{24}$
σ^2_{s23}	$\frac{r}{n_2n_3}[4n_{11}+n_1n_2+16n_{22}+n_1n_3+25n_2n_3+16n_{33}]$	$\frac{53r}{108}$	$\frac{r}{2}$
σ^2_{s33}	$\frac{r}{n_{33}}[n_{11}+n_1n_2+n_{22}+4n_1n_3+4n_2n_3+16n_{33}]$	$\frac{23r}{54}$	$\frac{14r}{27}$
σ^2_{g11}	$\frac{4r}{n_1}[16n_{11}+4n_1n_2+n_{22}+4n_1n_3+n_2n_3+n_{33}]$	$\frac{68r}{27}$	$\frac{28r}{9}$
$\sigma^2_{g1(2)}$	$\frac{r}{n_1}[16n_{11}+25n_1n_2+16n_{22}+n_1n_3+n_2n_3+4n_{33}]$	$\frac{41r}{27}$	2r
$\sigma^2_{g(1)2}$	$\frac{r}{n_2}[16n_{11}+25n_1n_2+16n_{22}+n_1n_3+n_2n_3+4n_{33}]$	$\frac{41r}{36}$	2r
$\sigma^2_{g1(3)}$	$\frac{r}{n_1}[16n_{11}+n_1n_2+4n_{22}+25n_1n_3+n_2n_3+16n_{33}]$	$\frac{53r}{27}$	2r
$\sigma^2_{g(1)3}$	$\frac{r}{n_3}[16n_{11}+n_1n_2+4n_{22}+25n_1n_3+n_2n_3+16n_{33}]$	$\frac{53r}{36}$	2r
σ^2_{g22}	$\frac{r}{n_2}[n_{11}+4n_1n_2+16n_{22}+n_1n_3+4n_2n_3+n_{33}]$	$\frac{68r}{27}$	$\frac{28r}{9}$
$\sigma^2_{g2(3)}$	$\frac{r}{n_2}[4n_{11}+n_1n_2+16n_{22}+n_1n_3+25n_2n_3+16n_{33}]$	$\frac{53r}{27}$	2r
$\sigma^2_{g(2)3}$	$\frac{r}{n_3}[4n_{11}+n_1n_2+16n_{22}+n_1n_3+25n_2n_3+16n_{33}]$	$\frac{53r}{36}$	2r
σ^2_{g33}	$\frac{4r}{n_3}[n_{11}+n_1n_2+n_{22}+4n_1n_3+4n_2n_3+16n_{33}]$	$\frac{23r}{9}$	$\frac{28r}{9}$
σ^2_S	$r[36(n_1n_2+n_1n_3+n_2n_3)+54(n_{11}+n_{22}+n_{33})]$	$\frac{68r}{3}$	$\frac{100r}{3}$

* $n_{bb} = n_b(n_b-1)/2$.

Table 6. Coefficients for components in $r \sum_{b=1}^{3} n_b . G_b^2$ for n_1, n_2, and n_3 lines where $n_{bb} = n_b(n_b - 1)/2$.

Component	Coefficient $\times 3^4$ – n_1, n_2, and n_3 lines[1]	n_1,n_2,n_3 3,3,4	4,4,4
σ_ϵ^2	$[n_1 . (4/n_{11} + 1/n_1n_2 + 1/n_1n_3 + 1/n_{22} + 4/n_2n_3 + 1/n_{33})$ $+ n_2 . (1/n_{11} + 1/n_1n_2 + 4/n_1n_3 + 4/n_{22} + 1/n_2n_3 + 1/n_{33})$ $+ n_3 . (1/n_{11} + 4/n_1n_2 + 1/n_1n_3 + 1/n_{22} + 1/n_2n_3 + 4/n_{33})]$	$\frac{515}{243}$	$\frac{209}{108}$
σ_{s11}^2	$r[4n_1 . + n_2 . + n_3 .] / n_{11}$	$\frac{50r}{81}$	$\frac{38r}{81}$
σ_{s12}^2	$r[n_1 . + n_2 . + 4n_3 .] / n_1n_2$	$\frac{56r}{243}$	$\frac{19r}{108}$
σ_{s13}^2	$r[n_1 . + 4n_2 . + n_3 .] / n_1n_3$	$\frac{25r}{162}$	$\frac{19r}{108}$
σ_{s22}^2	$r[n_1 . + 4n_2 . + n_3 .] / n_{22}$	$\frac{50r}{81}$	$\frac{38r}{81}$
σ_{s23}^2	$r[4n_1 . + n_2 . + n_3 .] / n_2n_3$	$\frac{25r}{162}$	$\frac{19r}{108}$
σ_{s33}^2	$r[n_1 . + n_2 . + 4n_3 .] / n_{33}$	$\frac{28r}{81}$	$\frac{38r}{81}$
σ_{g11}^2	$4r[4n_1 . + n_2 . + n_3 .] / n_1$	$\frac{200r}{81}$	$\frac{228r}{81}$
$\sigma_{g1(2)}^2$	$r[n_1 . + n_2 . + 4n_3 .] / n_1$	$\frac{56r}{81}$	$\frac{19r}{27}$
$\sigma_{g(1)2}^2$	$r[n_1 . + n_2 . + 4n_3 .] / n_2$	$\frac{56r}{81}$	$\frac{19r}{27}$
$\sigma_{g1(3)}^2$	$r[n_1 . + 4n_2 . + n_3 .] / n_1$	$\frac{50r}{81}$	$\frac{19r}{27}$
$\sigma_{g(1)3}^2$	$r[n_1 . + 4n_2 . + n_3 .] / n_3$	$\frac{25r}{54}$	$\frac{19r}{27}$
σ_{g22}^2	$4r[n_1 . + 4n_2 . + n_3 .] / n_2$	$\frac{200r}{81}$	$\frac{228r}{81}$
$\sigma_{g2(3)}^2$	$r[4n_1 . + n_2 . + n_3 .] / n_2$	$\frac{50r}{81}$	$\frac{19r}{27}$
$\sigma_{g(2)3}^2$	$r[4n_1 . + n_2 . + n_3 .] / n_3$	$\frac{25r}{54}$	$\frac{19r}{27}$
σ_{g33}^2	$4r[n_1 . + n_2 . + 4n_3 .] / n_3$	$\frac{168r}{81}$	$\frac{228r}{81}$

Table 6. Coefficients for components in $r \sum_{b=1}^{3} n_b . G_b^2$ for n_1, n_2, and n_3 lines where $n_{bb} = n_b(n_b - 1)/2$.

(continued)

Component	Coefficient $\times 3^4$ – n_1, n_2, and n_3 lines[1]	n_1,n_2,n_3 3,3,4	4,4,4
σ_S^2	$12r[n_1. + n_2. + n_3.]$	$\frac{104r}{9}$	$\frac{152r}{9}$
σ_G^2	$54r[n_1. + n_2. + n_3.]$	52r	76r
SSG^2	9r	$\frac{r}{9}$	$\frac{r}{9}$

[1] $n_1. = n_{11} + n_{12} + n_{13} = \frac{n_1(n_1-1)}{2} + n_1 n_2 + n_1 n_3 = n_1\left(v - (n_1+1)/2\right)$.

$n_2. = n_{12} + n_{22} + n_{23} = n_1 n_2 + \frac{n_2(n_2-1)}{2} + n_2 n_3 = n_2\left(v - (n_2+1)/2\right)$.

$n_3. = n_{13} + n_{23} + n_{33} = n_1 n_3 + n_2 n_3 + \frac{n_3(n_3-1)}{2} = n_3\left(v - (n_3+1)/2\right)$.

$v = n_1 + n_2 + n_3$.

[2] $SSG = \left[n_1.(2G_1 - G_2 - G_3)^2 + n_2.(-G_1 + 2G_2 - G_3)^2 + n_3.(-G_1 - G_2 + 2G_3)^2\right]$.

5. NUMERICAL EXAMPLE

To illustrate the use of the various formulae for estimation of the various effects, an example was constructed from known values of effects (Table 7). Then, the formulae, if correct, must give the same values for the effects that were used to construct the example in Table 7. The values of the parameters used to construct the example are given in Table 8. For example, the first yield given in Table 7 is constructed as follows:

$$\begin{aligned} Y_{aEE12} = 7 &= \hat{\mu} + \hat{G}_E + \hat{G}_E + \hat{S}_{EE} + \hat{g}_{EE1} + \hat{g}_{EE2} + \hat{s}_{EE12} + \epsilon_{aEE12} \\ &= 10 - 1 - 1 + 1 - 2 + 0 + 0 + 0 = 7 \ . \end{aligned}$$

The remaining yields in Table 4 were constructed similarly. The various effects estimated from the yields in Table 4 are:

	Formula
$\hat{s}_{EL11} = 11 - 48/4 - 27/3 + 120/12 = 0$	(10)
$\hat{s}_{EM11} = 7 - 21/3 - 27/3 + 72/9 = -1$	(10)
$\hat{g}_{EM1\cdot} = 21/3 - 72/9 = -1$	(12)
$\hat{g}_{EM\cdot 1} = 27/3 - 72/9 = 1$	(14)
$\hat{g}_{EL1\cdot} = 48/4 - 120/12 = 2$	(12)
$\hat{g}_{EL\cdot 1} = 27/3 - 120/12 = -1$	(14)
$\hat{g}_{EE1\cdot} = \left[16 - (3-1)(27/3)\right] \Big/ (3-2) = -2$	(13)
$\hat{g}_{EE2\cdot} = 18 - 2(9) = 0$	(13)
$\hat{g}_{EE3\cdot} = 20 - 2(9) = 2$	(13)
$\hat{g}_{LL1\cdot} = \left[40 - (4-1)(84/6)\right] \Big/ (4-2) = -1$	(13)
$\hat{g}_{LL2\cdot} = \left[40 - 3(14)\right] \Big/ 2 = -1$	(13)
$\hat{g}_{LL3\cdot} = \left[40 - 3(14)\right] \Big/ 2 = -1$	(13)
$\hat{g}_{LL4\cdot} = \left[48 - 3(14)\right] \Big/ 2 = 3$	(13)
$\hat{s}_{LL12} = 11 - 40/2 - 40/2 + 2(14) = -1$	(11)
$\hat{s}_{LL13} = 12 - 40/2 - 40/2 + 2(14) = 0$	(11)
$\hat{s}_{LL34} = 17 - 40/2 - 48/2 + 2(14) = 1$	(11)

Table 7. Yields for ten lines in three maturity groups.

	Early				Medium				Late				
	1	2	3	Total	1	2	3	Total	1	2	3	4	Total
Early													
1	–	7	9	16	7	7	7	21	11	13	12	12	48
2	–	–	11	18	11	7	9	27	9	11	10	10	40
3	–	–	–	20	9	7	8	24	7	9	8	8	32
Total				27	27	21	24	72	27	33	30	30	120
Medium													
1					–	7	4	11	11	9	9	11	40
2					–	–	10	17	13	15	13	15	56
3					–	–	–	14	12	12	11	13	48
Total								21	36	36	33	39	144
Late													
1									–	11	12	17	40
2									–	–	13	16	40
3									–	–	–	15	40
4									–	–	–	–	48
Total													84

Totals for groups

	Early	Medium	Late	Totals
Early	27	72	120	219
Medium	–	21	144	237
Late	–	–	84	348
Total				468

Table 8. Solutions $\hat{s}_{bcde}$, $\hat{g}_{bcd\,\cdot}$, $\hat{g}_{bc\,\cdot\,e}$, $\hat{G}_b$, $\hat{G}_c$, $\hat{S}_{bc}$ and $\hat{\mu}$ for yields in Table 7.

	Early $\hat{s}_{EEde}$				Medium $\hat{s}_{EMde}$				Late $\hat{s}_{ELde}$				
	1	2	3	$\hat{g}_{EEd\,\cdot}$	1	2	3	$\hat{g}_{EMd\,\cdot}$	1	2	3	4	$\hat{g}_{ELd\,\cdot}$
d = 1	–	0	0	−2	−1	1	0	−1	0	0	0	0	2
2	–	–	0	0	1	−1	0	1	0	0	0	0	0
3	–	–	–	2	0	0	0	0	0	0	0	0	−2
				$\hat{g}_{EM\,\cdot\,e}$	1	−1	0	$\hat{g}_{EL\,\cdot\,e}$	−1	1	0	0	–
					$\hat{s}_{MMde}$			$\hat{g}_{MMd\,\cdot}$	$\hat{s}_{MLde}$				$\hat{g}_{MLd\,\cdot}$
d = 1					–	0	0	−3	1	−1	0	0	−2
2					–	–	0	3	−1	1	0	0	2
3					–	–	–	0	0	0	0	0	0
								$\hat{g}_{ML\,\cdot\,e}$	0	0	−1	1	–
									$\hat{s}_{LLde}$				$\hat{g}_{LLd\,\cdot}$
d = 1									–	−1	0	1	−1
2									–	–	1	0	−1
3									–	–	–	−1	−1
4									–	–	–	–	3

$\hat{\mu} = 10$

$\hat{G}_E = -1$

$\hat{G}_M = -1$

$\hat{G}_L = 2$

$\hat{S}_{EE} = 1$

$\hat{S}_{EM} = 0$

$\hat{S}_{EL} = -1$

$\hat{S}_{MM} = -1$

$\hat{S}_{ML} = 1$

$\hat{S}_{LL} = 0$

$$\hat{\mu} = \left[27/3 + 72/9 + 120/12 + 21/3 + 144/12 + 84/6\right] \Big/ 6 = 10 \qquad (20)$$

$$\hat{G}_E = (9 + 8 + 10)/3 - 10 = -1 \qquad (19)$$

$$\hat{G}_L = (10 + 12 + 14)/3 - 10 = 2 \qquad (19)$$

$$\hat{S}_{EE} = 9 + 10 - (9 + 8 + 10)/3 - (8 + 7 + 12)/3 = 1 \qquad (21)$$

$$\hat{S}_{EL} = 10 + 10 - (9 + 8 + 10)/3 - (10 + 12 + 14)/3 = -1 \qquad (21)$$

The various sums of squares given in Table 9 may be computed as follows. Again note that $\hat{\mu} \neq \bar{y}.....$, which may indicate whether $\hat{\mu}$ or $\bar{y}.....$ should be used to compute the sums of squares. Since this is a randomized complete block (RCB) designed experiment, and since treatments and blocks are orthogonal in an RCB design, arithmetic means are appropriate. The way the model is formulated in (1), μ is not orthogonal to G_b, G_c, and S_{bc}. This fact needs to be taken into account when computing sums of squares for the effects, but not for the blocks, treatments, and block $\times$ treatment sums of squares. The sum of squares for treatment is

$$\sum_{b \le c} \sum \sum_{d=1}^{n_b} \sum_{e=1}^{n_c} Y^2_{.bcde} \Big/ r - Y^2_{.....} \Big/ rn_{..}$$

$$= (7^2 + 9^2 + 7^2 + 7^2 + \cdots + 13^2 + 16^2 + 15^2) \Big/ 1 - 468^2/45$$

$$= 5220 - 4{,}867.2 = 352.8 \ .$$

The sum of squares for the $g(g+1)/2 = 6$ groups is

$$\sum_{b \le} \sum_{c=1}^{g} Y^2_{.bc..} \Big/ r\,n_{bc} - Y^2_{.....} \Big/ rn_{..}$$

$$= \frac{27^2}{3} + \frac{72^2}{9} + \frac{120^2}{12} + \frac{21^2}{3} + \frac{144^2}{12} + \frac{84^2}{6} - \frac{468^2}{45}$$

$$= 202.8 \ .$$

The sums of squares within groups are obtained in the usual manner as are those for $n_b \times n_c$ two-way tables for $b \neq c$. Those sums of squares for the groups where $b = c$ are obtained in the usual manner for diallel cross experiments with $n_b(n_b - 1)/2 = n_{bb}$ crosses. Since specific combining ability estimates for $n_b = 3$ with $n_{bb} = 3$ crosses are not possible, there are only general combining ability effects estimable for the E and M groups of this example. For the L group, the sum of squares for general combining ability effects is from (28),

Table 9. ANOVA for observations in Table 4.

Source of variation	d.f.		Sum of squares		Mean square	Expected value
Total	45		5,220		–	–
Correction for mean	1		4,867.2		–	–
Block	0				–	–
Treatment	44		352.8		–	–
Among groups	5		202.8		40.56	–
E × E	2		8		–	–
GCA		2		8	4.00	$\sigma_\epsilon^2 + r\,\sigma_{sEE}^2 + (3{-}2)r\,\sigma_{gE(E)}^2$
SCA		0		0	0	$\sigma_\epsilon^2 + r\,\sigma_{sEE}^2$
M × M	2		18		–	–
GCA		2		18	9.00	$\sigma_\epsilon^2 + r\,\sigma_{sMM}^2 + (3{-}2)r\,\sigma_{gM(M)}^2$
SCA		0		0	0	$\sigma_\epsilon^2 + r\,\sigma_{sMM}^2$
L × L	5		28		–	–
GCA		3		24	8.00	$\sigma_\epsilon^2 + r\,\sigma_{sLL}^2 + (4{-}2)r\,\sigma_{gL(L)}^2$
SCA		2		4	2.00	$\sigma_\epsilon^2 + r\,\sigma_{sLL}^2$
E × M	8		16		–	–
GCAE(M)		2		6	3.00	$\sigma_\epsilon^2 + r\,\sigma_{sEM}^2 + 3r\,\sigma_{gE(M)}^2$
GCA(E)M		2		6	3.00	$\sigma_\epsilon^2 + r\,\sigma_{sEM}^2 + 3r\,\sigma_{g(E)M}^2$
SCAEM		4		4	1.00	$\sigma_\epsilon^2 + r\,\sigma_{sEM}^2$
E × L	11		38		–	–
GCAE(L)		2		32	16.00	$\sigma_\epsilon^2 + r\,\sigma_{sEL}^2 + 4r\,\sigma_{gE(L)}^2$
GCA(E)L		3		6	2.00	$\sigma_\epsilon^2 + r\,\sigma_{sEL}^2 + 3r\,\sigma_{g(E)L}^2$
SCAEL		6		0	0.00	$\sigma_\epsilon^2 + r\,\sigma_{sEL}^2$
M × L	11		42		–	–
GCAM(L)		2		32	16.00	$\sigma_\epsilon^2 + r\,\sigma_{sML}^2 + 4r\,\sigma_{gM(L)}^2$
GCA(M)L		3		6	2.00	$\sigma_\epsilon^2 + r\,\sigma_{sML}^2 + 3r\,\sigma_{g(M)L}^2$
SCAML		6		4	0.67	$\sigma_\epsilon^2 + r\,\sigma_{sML}^2$
Block × treatment	0		0		0	σ_ϵ^2

$$\sum_{b=1}^{4} 4\left(\tfrac{4}{2}Y_{\cdot LLd\cdot} - Y_{\cdot LL\cdot\cdot}\right)^2 \Big/ 1(4^2)(4-2)$$

$$= \left\{[2(40)-84]^2 + [2(40)-84]^2 + [2(40)-84]^2 + [2(48)-84]^2\right\} \Big/ 8$$

$$= \{16 + 16 + 16 + 144\} \Big/ 8 = 24 \text{ with 3 degrees of freedom .}$$

The sum of squares for specific combining ability effects from (31) is

$$\sum_{d<} \sum_{e=2}^{4} \hat{s}_{LLde} Y_{\cdot LLde}$$

$$= [(-1)(11) + 0(12) + 1(17) + 1(13) + 0(16) + (-1)(15)]$$

$$= 4 \text{ with 2 degrees of freedom.}$$

These two sums of squares add to the total among the six crosses, i.e.,

$$11^2 + 12^2 + 17^2 + 13^2 + 16^2 + 15^2 - 84^2/6 = 28 = 24 + 4 \text{ .}$$

The remaining sums of squares are given in Table 9.

Using equations (34) and (35), the sums of squares for group general and group specific combining abilities, we obtain

$$r\sum_{b=1}^{3} n_{b\cdot}\hat{G}_b^2 = 24(-1)^2 + 24(-1)^2 + 30(2)^2 = 168$$

and

$$r\sum_{b=1}^{3} n_{bc}\hat{S}_{bc}^2 = 3(1)^2 + 9(0)^2 + 12(-1)^2 + 3(-1)^2 + 12(1)^2 + 6(0)^2 = 30 \text{ .}$$

The among-groups sum of squares is 202.8 and the sum of the above two sums of squares is $168 + 30 = 198$. The difference between these two sums of squares is $202.8 - 198 = 4.8$. The arithmetic mean $\bar{y}_{\cdot\cdot\cdot\cdot\cdot} = 10.4$ and $\hat{\mu} = 10$. $n_{\cdot\cdot}\left(\bar{y}_{\cdot\cdot\cdot\cdot} - \hat{\mu}\right)^2 = 45(10.4-10)^2 = 7.2$. Hence, the difference in the two sums of squares is not a simple comparison of $\bar{y}_{\cdot\cdot\cdot\cdot\cdot}$ with $\hat{\mu}$ but is some weighted average involving the G_b and S_{bc} effects as well.

For this example, the various estimated variance components are:

$$\hat{\sigma}_\epsilon^2 = 0 \text{ ,}$$

$$\hat{\sigma}_{sEE}^2 = (0-0)/1 = 0 \text{ ,}$$

$$\hat{\sigma}_{sMM}^2 = (0-0)/1 = 0 \text{ ,}$$

$$\hat{\sigma}_{sLL}^2 = (2-0)/(r=1) = 2 \text{ ,}$$

$$\hat{\sigma}_{sEM}^2 = (1-0)/1 = 1 \text{ ,}$$

$$\hat{\sigma}^2_{sEL} = (0-0)/1 = 0 \ ,$$

$$\hat{\sigma}^2_{sML} = (0.67-0)/1 = 2/3 \ ,$$

$$\hat{\sigma}^2_{gEE} = (4-0)/(3-2) = 4 \ ,$$

$$\hat{\sigma}^2_{gMM} = (9-0)/(3-2) = 9 \ ,$$

$$\hat{\sigma}^2_{gLL} = (8-2)/(4-2) = 3 \ ,$$

$$\hat{\sigma}^2_{gE(M)} = (3-1)/3 = 2/3 \ ,$$

$$\hat{\sigma}^2_{g(E)M} = (3-1)/3 = 2/3 \ ,$$

$$\hat{\sigma}^2_{gE(L)} = (16-0)/4 = 4 \ ,$$

$$\hat{\sigma}^2_{g(E)L} = (2-0)/3 = 2/3 \ ,$$

$$\hat{\sigma}^2_{gM(L)} = (16-2/3)/4 = 23/6 \ ,$$

$$\hat{\sigma}^2_{g(M)L} = (2-2/3)/3 = 4/9 \ ,$$

$$\hat{\sigma}^2_S = \Big(30-\tfrac{257}{81}(0)-\tfrac{51}{81}(0)-\tfrac{41}{81}(1)-\tfrac{53}{108}(0)-\tfrac{51}{81}(0)-\tfrac{53}{108}\left(\tfrac{2}{3}\right)-\tfrac{23}{54}(2)-\tfrac{68}{27}(4)-\tfrac{41}{27}\left(\tfrac{2}{3}\right)-\tfrac{41}{36}\left(\tfrac{2}{3}\right)$$
$$-\tfrac{53}{27}(4)-\tfrac{53}{36}\left(\tfrac{2}{3}\right)-\tfrac{68}{27}(9)-\tfrac{53}{27}\left(\tfrac{23}{6}\right)-\tfrac{53}{36}\left(\tfrac{4}{9}\right)-\tfrac{23}{9}(3)\Big)\Big/(68/3) = -1.36 \text{ (from Table 5)} \ ,$$

and

$$\hat{\sigma}^2_G = \Big(168-\tfrac{515}{243}(0)-\tfrac{50}{81}(0)-\tfrac{56}{243}(1)-\tfrac{25}{162}(0)-\tfrac{50}{81}(0)-\tfrac{25}{162}\left(\tfrac{2}{3}\right)-\tfrac{28}{81}(4)-\tfrac{200}{81}(4)-\tfrac{56}{81}\left(\tfrac{2}{3}\right)-\tfrac{56}{81}\left(\tfrac{2}{3}\right)$$
$$-\tfrac{50}{81}(4)-\tfrac{25}{54}\left(\tfrac{2}{3}\right)-\tfrac{200}{81}(9)-\tfrac{50}{81}\left(\tfrac{23}{6}\right)-\tfrac{25}{54}\left(\tfrac{4}{9}\right)-\tfrac{168}{81}(3)\Big)\Big/52 = 2.34 \text{ (from Table 6)} \ ,$$

For the fixed effects case for all effects except ϵ_{abcde} and ρ_a effects, the various variances of difference of G_b effects for $n_1 = n_2 = 3$ and $n_3 = 4$ are:

$$\mathrm{Var}\left[\hat{G}_1-\hat{G}_2\right] = \mathrm{Var}\left[\tfrac{1}{3}\left(\bar{y}_{.11..}+\bar{y}_{.13..}-\bar{y}_{.22..}-\bar{y}_{.23..}\right)\right]$$

$$= \frac{\sigma^2_\epsilon}{9r}\left[\frac{51}{81}+\frac{53}{108}+\frac{51}{81}+\frac{53}{108}\right] = 121\,\sigma^2_\epsilon \Big/ 486r \ ,$$

$$\mathrm{Var}\left[\hat{G}_1-\hat{G}_3\right] = \mathrm{Var}\left[\tfrac{1}{3}\left(\bar{y}_{.11..}+\bar{y}_{.12..}-\bar{y}_{.33..}-\bar{y}_{.23..}\right)\right]$$

$$= \frac{\sigma^2_\epsilon}{9r}\left[\frac{51}{81}+\frac{41}{81}+\frac{23}{54}+\frac{53}{108}\right] = 665\,\sigma^2_\epsilon \Big/ 2916r \ ,$$

$$\mathrm{Var}\left[\hat{G}_2-\hat{G}_3\right] = \mathrm{Var}\left[\tfrac{1}{3}\left(\bar{y}_{.22..}+\bar{y}_{.12..}-\bar{y}_{.33..}-\bar{y}_{.13..}\right)\right]$$

$$= \frac{\sigma^2_\epsilon}{9r}\left[\frac{51}{81}+\frac{41}{81}+\frac{23}{54}+\frac{53}{108}\right] = 665\,\sigma^2_\epsilon \Big/ 2916r \ .$$

When $n_1 = n_2 = n_3 = 4$, the variance of a difference between any two $\hat{G}_b$ values is

$$\mathrm{Var}\left[\hat{G}_1 - \hat{G}_2\right] = \mathrm{Var}\left[\hat{G}_1 - \hat{G}_3\right] = \mathrm{Var}\left[\hat{G}_2 - \hat{G}_3\right] = \frac{2\sigma_\epsilon^2}{9r}\left[\frac{14}{27} + \frac{1}{2}\right] = 55\,\sigma_\epsilon^2 \Big/ 243r\ .$$

Using the estimated value for σ_ϵ^2 and using the appropriate t-value, range value, or other selected value, a confidence interval on the difference of two $\hat{G}_b$ values may be obtained. Linear combinations of the $\hat{S}_{bc}$ values may be handled in a similar manner.

6. DISCUSSION

Using groups of lines in a diallel crossing system added complexity to the statistical analysis. A statistical peculiarity encountered was that the sums of squares for general and for specific combining ability did not add to that among-groups sum of squares. Also, the algebra became tedious and was facilitated using the software package *Mathematica.*

The experimenter wished to have variance components for general combining and for specific combining abilities for the groups of lines. This would imply that maturity group effect was a random effect. Although a rationalization could be postulated for maturity, it appears more appropriate to consider the maturity groups as fixed effects. For other situations, a random group effects may be quite plausible. The lines within a group could quite conceivably be considered to be random effects in that a random sample of lines was obtained for each maturity group.

Hinkelmann (1974) obtained general and specific combining ability effects over all the other groups. Although this appears to be a logical assumption in some cases, it would not be reasonable in general. For maturity groups, there appears to be little validity for believing that the gca and sca effects are the same in the early by medium and early by late maturity groups. Usually gca and sca effects within a group will not be the same as across groups. Based on this, it was decided to estimate the $\sigma^2_{gb(c)}$, $\sigma^2_{g(b)c}$, and σ^2_{sbc} variance components for each bc group. If the effects are as Hinkelmann (1974) postulated, it will be simple to combine them across groups omitting crosses within a group. Including the crosses within a group would add some complexity.

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