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ON THE EXISTENCE OF EQUIVALENT LOCAL MARTINGALE MEASURES

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We study the existence of an equivalent local martingale measure for discrete time stochastic processes, an approach different from that in the current literature, which concerns itself with equivalent martingale measures. We give a new definition for "no arbitrage" and show that this condition is necessary for there to exist an equivalent local martingale measure. We conjecture that if a process satisfies our no arbitrage condition, then there exists an equivalent martingale measure for this process. We show that if a process is unbounded enough, then there always exists an equivalent local martingale measure for it, which is consistent with our conjecture. We also prove that our conjecture holds for a special class of processes, which includes a recent example from the literature.

We also consider families of stochastic processes and give a necessary condition for a family of processes to have an equivalent local martingale measure. We exhibit a family of processes for which there exists an equivalent local martingale measure but not an equivalent martingale measure.

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Chapter 1

Introduction

It is well known that one can't "win for sure " betting on a martingale. In ([4]), Dalang, Morton and Willinger give a result which intuitively can be interpreted as "if one can't win for sure betting on a process then it must be a martingale under some equivalent measure". This new measure is called an *equivalent martingale measure*.

The concept of an equivalent martingale measure was first introduced in the two fundamental papers, Harrison & Kreps [8] and Harrison & Pliska [9]. In these papers, the relationship between the economic notion of *arbitrage*, and the probabilistic idea of an *equivalent martingale measure* was first developed. The concept of an equivalent martingale measure now plays a vital role in mathematical finance. In general, necessary and sufficient conditions for the existence of an equivalent martingale measure are not known. Before stating the known results on equivalent martingale measures we will need some definitions.

From now on $(\Omega, \mathcal{F}, P)$ will be a probability space, and $F = \{\mathcal{F}_t : t \in T\}$ will be a filtration for this space. By a filtration, we mean an increasing sequence of sub-σ-fields of $\mathcal{F}$, i.e. for $s \leq t$

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

To begin with, we will consider a market with only finitely many assets. In chapter 2 we will generalize this to infinitely many securities. For the remainder of this chapter, $Z = \{Z_t : t \in T\}$ will be a $d+1$ dimensional stochastic process defined on $(\Omega, \mathcal{F}, P)$ and adapted to F. By adapted, we mean that for each t, the mapping $\omega \to Z_t(\omega)$ is $\mathcal{F}_t$-measurable.

Definition 1.0.1 Z is said to be a martingale (with respect to the filtration F) if

- $E\|Z_t\| < \infty \, \forall \, t$
- For every $s \leq t$ we have P a.s.

  $E[Z_t \mid \mathcal{F}_s] = Z_s$ a.s.

Let T be a stopping time with respect to the filtration F. (The random time T is a stopping time if for every t the event $\{T \leq t\} \in \mathcal{F}_t$.) We will denote the "stopped" process $\{Z_{t \wedge T} : t \in T\}$ by Z^T.

Definition 1.0.2 If there exists a nondecreasing sequence of stopping times, $\{T_n\}_{n=1}^{\infty}$, such that $P(\lim_{n\to\infty} T_n = \infty) = 1$ and for each n, Z^{T_n} is a martingale (with respect to F) then Z is called a *local martingale* . When such a sequence of stopping times exist, we will say that the sequence $\{T_n\}_{n=1}^{\infty}$ *reduces* Z.

Remark 1.0.3 Note that under our definition of local martingale, if Z is a local martingale then Z_0 must be integrable. This is consistent with the definition given in Karatzas and Shreve [13]. In Dellacherie and Meyer [6] a more general definition for a local martingale is given. Our results would not be affected by using the definition of Dellacherie and Meyer.

Definition 1.0.4 A measure Q defined on $(\Omega, \mathcal{F})$ is said to be *eqiuvalent* to P if for all $A \in \mathcal{F}$.

$$Q(A) = 0 \iff P(A) = 0.$$

We will denote this by $Q \sim P$.

Remark 1.0.5 If Q and P are countably additive, then $Q \sim P$ if and only if for every sequence of measurable sets $\{A_n\}$,

$$P(A_n) \to 0 \iff Q(A_n) \to 0$$

The next definition is a temporary definition that will be changed in section 2.

Definition 1.0.6 A probability measure, Q, is called an equivalent martingale measure for Z, (with respect to F), if

- $Q \sim P$ and
- Z is a martingale under Q.

1.1 Securities Market Models

What we give below is essentially the model of Harrison & Kreps [8]. The fundamental objects of the model are a set of trading dates, an information structure, and a price process. We will assume that the set of trading dates is the index set T. The information structure is represented by the filtration F. Thus the σ-field $\mathcal{F}_t$ represents the information available to an investor at time t. In this model, there are $d+1$ assets. The price process is given by the $d+1$ dimensional stochastic process Z. $Z_k(t)$ (the kth component of Z) represents the price of the kth asset at time t. We will need one component to serve as a numeraire. If one assumes that security zero is a riskless asset such as a bond or money market account, then it is reasonable to make the assumption that the price of the 0th security is strictly positive for all t. We will make the assumption that the 0th asset is strictly positive and that all prices have been normalized by the price of security zero, from which it follows that $Z_0(t, \omega) = 1$ for all ω and t. We will assume that the markets are frictionless, that is there are no trading costs, taxes, etc. Agents in this model buy and sell assets according to trading strategies. We will begin by considering simple trading strategies.

Definition 1.1.1 A simple trading strategy is a $d+1$ dimensional process $\phi = \{\phi_t : t \in T\}$ that satisfies

- There exists a finite integer N and a sequence of trading dates $0 = t_0 < t_1 < \cdots < t_N < \infty$ such that

$$\phi_t = \lambda_0 1_{[0]}(t) + \sum_{i=1}^{N} \lambda_i 1_{(t_{i-1},t_i]}(t)$$

  (Where $\lambda_i \in \mathcal{F}_{t_i}$.)

- For $n = 1, 2, \ldots, N-1$,

$$\phi_{t_n} Z_{t_n} = \phi_{t_{n+1}} Z_{t_n}.$$

A strategy ϕ represents a rule for holding assets, with $\phi^k(t, \omega)$ specifying the amount of asset k held at time t, when the state of the world is ω. The first condition requires that the portfolio held at time t, must be based on the information available at time t. With a simple trading strategy, agents are only allowed to trade a finite number of times, as is specified by the second condition. The third condition requires that a strategy be *self-financing*. That is, no new money may added to or removed from the portfolio between time 0 and the end of trading.

Definition 1.1.2 The *value process*, $V(\phi)$, associated with a trading strategy ϕ, is defined by:

$$V_t(\phi) = \phi_t \cdot Z_t$$

Let ϕ be a trading strategy, which trades at the dates, $t_0, t_1, \ldots, t_N$, we will denote the value of the process at the end of trading by, $V(\phi)$, i.e. $V(\phi) = V_{t_N}(\phi)$. We may express $V(\phi)$ as

$$\begin{aligned} V(\phi) &= \phi_{t_N} \cdot Z_{t_N} \quad (1.1) \\ &= \phi_{t_N} \cdot (Z_{t_N} - Z_{t_{N-1}}) + \phi_{t_N} \cdot Z_{t_{N-1}} \\ &= \phi_{t_N} \cdot (Z_{t_N} - Z_{t_{N-1}}) + \phi_{t_{N-1}} \cdot Z_{t_{N-1}} \\ &= \cdots \\ &= V_0(\phi) + \sum_{i=1}^{n} \phi_{t_N} \cdot (Z_i - Z_{i-1}) \\ &= V_0(\phi) + \sum_{i=1}^{n} \phi_{t_N} \cdot (Z_{t_N} - Z_{t_{N-1}}) \end{aligned}$$

Along with the value associated with a strategy, is the corresponding gain. The gain by time t associated with a strategy ϕ is

$$G_t(\phi) = V_t(\phi) - V_0(\phi)$$

Definition 1.1.3 A *simple arbitrage opportunity* is a trading strategy ϕ satisfying

- $P(V_0(\phi) = 0) = 1.$
- $P(V(\phi) \geq 0) = 1$ and
- $P(V(\phi) > 0) > 0.$

Remark 1.1.4 Since the definition of an arbitrage opportunity depends on the definition of trading strategies, different definitions of trading strategies give rise to different definitions of arbitrage. We have used the term simple arbitrage to separate this definition from one we will give later.

In Harrison and Kreps [8], it is established that if Ω is finite, then there exists an equivalent martingale measure for the price process Z if and only if there are no simple arbitrage opportunities.

Dalang, Morton and Willinger (see [4]) were able to remove the assumption that Ω is finite. Let $T = \{0, 1, 2, \ldots, n\}$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space (that is, $\mathcal{F}$ contains all the subsets of sets having probability zero). Z will be an arbitrary price process (i.e. the only assumption on Z is that the 0th component is identically 1). Let $\tilde{Z} = \{\tilde{Z}_t, t = 0, 1, \ldots, N\}$ be the d-dimensional process obtained by deleting the 0th component (ie. $Z = (1, \tilde{Z})$).

Theorem 1.1.5 *(Dalang, Morton, Willinger)*

The following are equivalent

1. *The market model contains no arbitrage opportunities.*
2. *For all $t \in \{1, 2, \ldots, N\}$ and all $\mathbf{R}^d$-valued $\mathcal{F}_{t-1}$-measurable random vectors α,*

$$\alpha \cdot (\tilde{Z}_t - \tilde{Z}_{t-1}) \geq 0 \; P-a.s. \implies \alpha \cdot (\tilde{Z}_t - \tilde{Z}_{t-1}) = 0 \; P-a.s.$$

3. *There exists an equivalent martingale measure for Z.*

Since the three statements of the theorem are equivalent, statements 1 and 2 are necessary and sufficient for there to exist an equivalent martingale measure for Z. Each of the three statements of the theorem is invariant under a change to an equivalent measure. Thus Theorem 1.1.5 completely characterizes all stochastic processes with a finite index set for which there exists an equivalent martingale measure.

1.2 Infinite Horizon Problems

We now turn to the question of whether there exists a similar result for processes with an infinite index set (either discrete or continuous time processes). When the index set is finite, any trading strategy is simple, and thus, the definition of arbitrage is clear. When the index set is infinite, there are many ways in which to define trading strategies. We will begin by showing that the absence of simple arbitrage is not sufficient for there to exist an equivalent martingale measure. The following example is well know in many different contexts.

Example 1.2.1 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with

$$P(X_1 = 1) = 1 - P(X_1 = -1) = p$$

for some $0 < p < 1$; $p \neq 1/2$. Let $Z_n = X_1 + \cdots + X_n$, and $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Suppose that there exists an equivalent martingale measure Q for Z. By the definition of a martingale

$E_Q[X_n \mid \mathcal{F}_{n-1}] = 0$ Q- a.s.

Since X_n only takes the values ± 1, we have that

$P(X_n = 1 \mid \mathcal{F}_{n-1}) = 1/2$ Q-a.s.

Hence, under Q, the X_n are i.i.d. with mean 0. By the strong law of large numbers, $X_n/n \rightarrow 2p - 1$ P-a.s., but under Q, $X_n/n \rightarrow 0$ Q-a.s., which contradicts the assumption that Q and P are equivalent. Thus for a process with an infinite index set the absence of simple arbitrage is not enough to guarantee the existence of an equivalent martingale measure.

We will now look at two recent results for continuous time processes. The first result is due to Stricker (see [16]). Stricker considers a problem slightly different from ours.

Definition 1.2.2 The process (Z, F, P) is said to have the property $\mathcal{M}^p$, if there exists a probability measure $Q \sim P$ with $dQ/dP \in L^q(\Omega, \mathcal{F}, P)$ (where q is conjugate to p) and such that Z is a martingale with respect to Q.

Remark 1.2.3 In the early work of Harrison & Kreps and Harrison & Pliska, they assumed that all the random variables in the model were in L^2. There definition of an equivalent martingale measure was different from ours. Under there definition, a process had an equivalent martingale measure if it had the property $\mathcal{M}^2$.

Let the index set T be $[0,1]$. We assume that the probability space $(\Omega, \mathcal{F}, P)$ along with the filtration F satisfy the usual conditions. That is,

- F is right continuous, ie. if $t < 1$, then $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$ and
- $\mathcal{F}_0$ contains all null sets, ie. if $B \subset A$ and $P(A) = 0$ then $B \in \mathcal{F}_0$.

The vector valued process H is called a simple predictable process if it can be expressed in the form

$$H(t) = \sum_{i=0}^{n-1} \lambda_i 1_{(t_i, t_{i+1}]}(t)$$

where $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$ and $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^d)$ is $\mathcal{F}_{t_i}$ measurable.

In what follows, $1 \leq p < \infty$, and Z is a d-dimensional cadlag (right continuous with left hand limits) adapted process such that for each $t \in [0,1]$, $Z_t \in L^p$. Let $(H \cdot Z)_t$ denote the stochastic integrable of H with respect to Z, B_+ be the set of all bounded non-negative random variables, and L^p_+ be the set of all non-negative random variables in L^p.

Theorem 1.2.4 *(Stricker) :*

The following are equivalent

1. *(Z,P) has the property $\mathcal{M}^p$.*
2. *For every set $A \in \mathcal{F}$ such that $P(A) > 0$, $1_A \notin \overline{K - B_+}$.*
3. *$L^p_+ \cap \overline{K - B_+} = \{0\}$.*

$(K - B_+)$ is the set $\{k - b \mid k \in K,\ b \in B_+\}$.

The set K can be viewed as the set of all obtainable gains using simple predictable trading strategies. The set $K - B_+$ is the set of gains obtained using simple predictable trading strategies when investors are allowed to throw money away. (We shall come back to this rather surprising concept in chapter 5). Suppose that condition 3 is not satisfied. Then there exists a non-negative nonzero element of L^p, such that Y can be approximated by elements of $K - B_+$. Thus intuitively, if statement 3 does not hold, by using simple trading strategies and throwing money away when necessary, one can get arbitrarily close (in the L^p sense) to a sure win. Thus the third statement can be viewed as an "absence of arbitrage" condition. By the first statement, Z has the property $\mathcal{M}^p$ if and only if the no arbitrage condition holds.

For continuous processes, Stricker's Theorem simplifies to

Theorem 1.2.5 *(Stricker) :*

If Z is a continuous process then the following are equivalent

1. *(Z, P) has the property $\mathcal{M}^p$.*

2. *For every set $A \in \mathcal{F}$ such that $P(A) > 0$, $1_A \notin \overline{K - B_+}$.*

3. *$L^p_+ \cap \overline{K} = \{0\}$.*

We will now look at a recent result by Delbaen (see [5]). We will call the function g *elementary* if it takes only finitely many values.

Definition 1.2.6 A process $H : [0,1] \times \Omega \to \mathbf{R}$ is called "very simple" if there exists stopping times $0 = T_0 \leq T_1 \leq \ldots \leq T_n \leq T_{n+1} = 1$ and $\mathcal{F}_{T_k}$-measurable elementary functions $g_0, g_1, \ldots, g_n$ such that

$$H(t) = g_0 1_{[0]}(t) + \sum_{k=0}^{n} g_k 1_{(T_k, T_{k+1}]}(t)$$

Suppose that Z is a continuous bounded adapted process. Let
$K = \{(H \cdot Z)_1 \mid H \text{ is a very simple process } \}$.
From Theorem 5.1 in [5], one can conclude that the following are equivalent

1. If $\{ Y_n\}_{n\geq 1}$ is a sequence in K such that $\|Y_n\|_\infty \leq 1$, then

$$Y_n^- \to 0 \text{ in probability} \Longrightarrow Y_n^+ \to 0 \text{ in probability}$$

2. There exists an equivalent martingale measure for Z.

As with Stricker's result, one can view the set K as the set of all obtainable gains using very simple trading strategies (a very simple strategy is a trading strategy with the process ϕ being a very simple process). Suppose that condition 1 does not hold. Then there exists a non-negative, nonzero random variable Y, such that Y can be approximated (in probability) by elements of K. Thus condition 1 is a no arbitrage condition. By remark 1.0.5 condition 1 is invariant under a change to an equivalent measure. So Delbaen gives a necessary and sufficient condition for any continuous bounded process to have an equivalent martingale measure.

Remark 1.2.7 If one drops the assumption that Z is bounded, then Delbaen gets a necessary and sufficient condition for there to exist an equivalent measure under which Z is a local martingale.

1.3 Discrete Time Processes

We will now consider discrete time processes with an infinite index set which is what this thesis is concerned with. From now on, any stochastic process mentioned is assumed to have the index set $T = \{0, 1, \ldots\}$.

Up till now we have been concerned with the existence of an equivalent martingale measure (using definition 1.0.6) for a process Z. We will take a different approach and consider the existence of an equivalent measure under which Z is a local martingale. We will now weaken the definition of an equivalent martingale measure by only requiring that Z be a local martingale measure with respect to the measure.

Definition 1.3.1 A probability measure, Q, is called an equivalent martingale measure for Z (with respect to F) if

- $Q \sim P$ and
- Z is a local martingale under Q.

We will look at example 1.2.1 again.

Example 1.3.2 Recall that Z was a simple random walk with

$$P(X_1 = 1) = p = 1 - P(X_1 = -1).$$

We saw that if $p \neq 1/2$ then there was no equivalent martingale measure for this process. We now show that in this case one can win for sure (in some sense) by betting on this process. We will assume that $1/2 < p < 1$ (the case $1/2 > p > 0$ can be done with a symmetric argument). The idea for the following strategy is due to the work of Dubins and Savage (see [7] chapter 5 and section 10.5).

Let $\epsilon = 2p - 1$.

Set $W_0 = 1$.

For $n \geq 1$, set $\lambda_n = \epsilon W_{n-1} 1_{\{W_{n-1} \leq 2\}}$ and $W_n = W_{n-1} + \lambda_n X_n$

We then have that

$$\begin{aligned} W_n &= W_{n-1}(1 + \epsilon X_n 1_{\{W_{n-1} \leq 2\}}) \\ &\vdots \\ &= \prod_{k=1}^{n}(1 + \epsilon X_k 1_{\{W_{k-1} \leq 2\}}) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k (Z_k - Z_{k-1}) &= \sum_{k=1}^{n} \lambda_k X_k \\ &= W_n - 1 \end{aligned}$$

Claim $\lim_{n\to\infty} W_n$ *exists a.s and* $\lim_{n\to\infty} W_n \ \geq 2$.

Proof. It is enough to show that

$$\prod_{k=1}^{n}(1+\epsilon X_k) \ \rightarrow \ \infty \text{ a.s.} \tag{1.2}$$

Let $Y_n \ = \ \log(1+\epsilon X_n)$. Then $Y_1, Y_2, \ldots$ is an i.i.d. sequence of random variables. Let $\mu \ = \ EY_1$.

Lemma 1.3.3 *(Expected Value lemma) If* $p \ > \ 1/2$ *then* $\mu \ > \ 0$.

The proof is left to the appendix.

By the strong law of large numbers, $\left(\sum_{k=1}^{n} Y_k\right)/n \ \rightarrow \ \mu$ a.s. which implies that $\sum_{k=1}^{n} Y_k \ \rightarrow \ +\infty$ a.s. From this it follows that

$$\log\left(\prod_{k=1}^{n}(1+\epsilon X_n)\right) \ \rightarrow \ \infty \text{ a.s.}$$

Which implies equation 1.2.

Thus

$$\sum_{k=1}^{n} \lambda_k X_k \ \rightarrow \ 1 \text{ a.s.}$$

Note also that for every n,

$$\sum_{k=1}^{n} \lambda_k X_k \ > \ -1$$

□

Thus for the above process when there does not exist an equivalent martingale measure, one can win for sure by betting on the process. When an infinite number of trades (or bets) are allowed, it is well known that by using "doubling strategies" one can win for sure by betting on a martingale. What separates the strategy in example 1.3.2 from doubling strategies is that the value process in this example is uniformly bounded below. Thus this winning strategy seems more reasonable than a doubling strategy, and hence seems more reasonable to call an arbitrage opportunity. The results in Chapter 2 are motivated by this idea of a reasonable sure win.

In Chapter 2 a necessary condition for the existence of an equivalent martingale measure is given. This condition is a no arbitrage type condition. We will see that if there exists an equivalent martingale measure, then there are no opportunities such as those in example 1.3.2.

Chapters 3 and 4 are concerned with whether the condition given in Chapter 2 is sufficient. We will prove that this condition is sufficient for two classes of processes. Chapter 3 is concerned with processes for which all of the increments are unbounded. In chapter 4, we prove for a simple class of processes that the condition in chapter 2 is sufficient.

In chapter 5 we will look at two examples. The first example is a family of processes for which there exists an equivalent local martingale measure, but no equivalent martingale measure. The second example is concerned with the wildness of arbitrage opportunities for discontinuous processes.

Chapter 2

A Necessary Condition for the Existence of an Equivalent Martingale Measure

2.1 Introduction

In this chapter we will give a necessary condition for the existence of an equivalent martingale measure. In chapter 1 it was shown that the absence of simple arbitrage was not enough to guarantee that an equivalent martingale measure existed. We will begin by considering other definitions of allowable trading strategies (and hence a different definition of arbitrage). We will consider trading strategies that permit an infinite number of trades. If an infinite number of trades are allowed, some sort of boundedness requirement is necessary to eliminate "doubling strategies".

We will first consider strategies of the form $\phi = (\phi_1, \phi_2, \ldots)$ where ϕ is a bounded sequence. The following example is due to Back and Pliska ([3]).

Example 2.1.1 Consider a model with two assets. The $0th$ asset is a bond, whose price is equal to 1 for all t and ω. We will denote the price of the second asset, the risky asset, by Z. Let $\Omega = \{1, 2, \ldots\}$ and $0 < p < 1$.
Let P be given by,

$$P(\omega) = (1-p)p^{\omega-1}$$

Set $Z_0 = 1$ and

$$Z_n(\omega) = \begin{cases} (1/2)^n & \text{if } n < \omega \\ (\omega^2 + 2\omega + 2)(1/2)^\omega & \text{if } n \geq \omega \end{cases}$$

Thus on the event $\{\omega\}$; at each time n prior to ω, the price falls by 50 % ; at time ω, the process increases by $(\omega^2 + 2\omega)(1/2)^\omega$; after time ω the price is constant.

Back and Pliska show that there is no martingale measure for Z.

They consider trading strategies of the form, (W, ϕ) where $\phi = (\phi_1, \phi_2, \ldots)$ is a bounded sequence. In their paper they show that there are no arbitrage opportunities for this model. The absence of arbitrage in this model follows from three factors: the time of the price increase is unpredictable; when the price increases, the size of the increase can be arbitrarily small; only bounded trading strategies are permitted. Hence under this definition of trading strategies, the absence of arbitrage is not sufficient for there to exist an equivalent martingale measure.

Another approach is to define a trading strategy such that unlimited negative wealth is never allowed.

Definition 2.1.2 We will call a process $\phi = (\phi_1, \phi_2, \ldots)$ an admissable trading strategy if

- For every N, the process up till time N is a simple trading strategy as defined in definition 1.1.1.
- There exists M and l_ϕ such that for each $n \geq M$, $V_n(\phi) \geq l_\phi$ a.s.
- $V_\infty(\phi) = \lim_n V_n(\phi)$ exists a.s.

We will see in section 3 that with this definition of trading strategies the no arbitrage condition is necessary for there to exist an equivalent martingale measure for Z, but the following example shows that it is not sufficient.

Example 2.1.3 This example is a modification of the Back and Pliska example. Let $\Omega = \{\alpha, 1, 2, 3, \ldots\}$, and let P be defined by

$$P(\omega) = \begin{cases} 1/2 & \omega = \alpha \\ 2^{-(n+1)} & \omega = n \end{cases}$$

Let $Z_0 = 0$ and for $n \geq 1$

$$Z_n(\omega) = \begin{cases} n & \text{if } \omega = \alpha \text{ or } \omega > n \\ \omega - 2 & \text{if } \omega \neq \alpha \text{ and } \omega \leq n \end{cases}$$

It is easily verified that the only possible martingale measure for Z is given by, $Q(\alpha) = 0$ and $Q(n) = 2^{-n}$, $n = 1, 2, \ldots$ Thus, since $P(\alpha) > 0$, Q is not equivalent to P.

Proposition 2.1.4 *If trading strategies are defined by definition 2.1.2, then the process in example 2.1.3 does not admit any arbitrage opportunities.*

Proof. Let ϕ be such a strategy with $V_0(\phi) = 0$ and suppose that $V_\infty(\phi) \geq 0$. Then

$$V_n(\phi) = \sum_{i=1}^{n} \lambda_i (Z_i - Z_{i-1})$$

Since on the event $\{1, 2, \ldots, n-1\}$, $Z_n = Z_{n-1}$, we may assume that the λ_n are constant. For any n, if $\lambda_n > 0$, then

$$P(\ V_\infty(\phi) < V_{n-1}(\phi)\ ) > P(\ \{n\} \mid \{1, 2, \ldots, n-1\}^c) > 0 \tag{2.1}$$

Thus applying this when $n = 1$, we have that $\lambda_1 \leq 0$. Hence on the event $\{1\}^c$, $V_1(\phi) \leq 0$. Applying this argument inductively, we have that for every n, $\lambda_n \leq 0$.

But on the event, $\{\alpha\}$,

$$V_\infty = \sum_{n=1}^{\infty} \lambda_n \leq 0.$$

Thus $\lambda_n = 0 \quad \forall\, n$. Hence there are no arbitrage opportunities for this process.

□

2.2 A Necessary Condition

In this section we will give a condition necessary for the existence of an equivalent martingale measure. We will begin by proving several simple lemmas about local martingales. In what follows $X = \{X_n\}_{n=0}^{\infty}$ will be a real-valued stochastic process defined on $(\Omega, \mathcal{F}, P)$ and adapted to the filtration F.

Lemma 2.2.1 *If X is a local martingale and for some n and l, $X_n \geq l$ a.s., then $X_k \geq l$ for $0 \leq k \leq n$.*

Proof. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of stopping times that reduce X. Let

$$A = \{X_{n-1} < l\}.$$

Then since $P(T_k > n-1) \to 1$ as $k \to \infty$, there exists m such that $P(\ A \cap \{T_m \geq n\}\ ) > P(A)/2$. Let

$$B = A \bigcap \{T_m \geq n\}.$$

Then $B \in \mathcal{F}_{n-1}$. If $P(B) > 0$ then by the definition, of conditional expectation,

$$\begin{aligned} lP(B) &\leq EX_n 1_B \\ &= E(1_B E[X_n \mid \mathcal{F}_{n-1}]) \\ &= E(E[X_{n\wedge T_m} 1_B \mid \mathcal{F}_{n-1}]) \\ &= E(X_{(n-1)\wedge T_m} 1_B) \\ &< lP(B) \end{aligned}$$

Which is impossible. Thus $P(B) = 0$, hence $P(A) = 0$. The result follows by induction. □

Lemma 2.2.2 *If X is a local martingale and $X_n \geq l$, then, X_n is integrable.*

Proof. From the previous lemma, for each m, $X_{n\wedge T_m} \geq l$.

Since X is a local martingale (see remark 1.0.3) $E \mid X_0 \mid < \infty$

Since for each m, X^{T_m} is a martingale,

$E(X_{n\wedge T_m} - X_0) = 0.$

Thus

$$\begin{aligned} E \mid X_{n\wedge T_m} - X_0 \mid &= 2E[\,(X_{n\wedge T_m} - X_0)^-\,] \\ &\leq 2E[(X_{n\wedge T_m})^- + X_0^+] \\ &\leq 2(-l + E \mid X_0 \mid) < \infty \end{aligned}$$

Since $X_{n\wedge T_m} \to X_n$ a.s. as $m \to \infty$.

we have by Fatou's lemma,

$$E \mid X_n - X_0 \mid \leq \liminf_m E \mid X_{n\wedge T_m} - X_0 \mid \leq 2(-l + E \mid X_0 \mid) < \infty$$

It follows that $E \mid X_n \mid < \infty$ □

Lemma 2.2.3 *If X is a local martingale, and $X_n \geq l$, then $EX_n = EX_0$.*

Proof. $\mid X_{n\wedge T_m} \mid \leq \mid X_1 \mid + \mid X_2 \mid + \cdots \mid X_n \mid$. By the previous lemma

$$E \mid X_1 \mid + E \mid X_2 \mid + \cdots + E \mid X_n \mid < \infty$$

Thus by dominated convergence,

$$EX_n = \lim_{m\to\infty} EX_{n\wedge T_m} = EX_0$$

□

Lemma 2.2.4 *If X is a martingale, then the process $G = \{G_n\}_{n=1}^{\infty}$ given by $G_n = \sum_{i=1}^{n} \lambda_i(X_i - X_{i-1})$, where for each i, $\lambda_i \in \mathcal{F}_{i-1}$, is a local martingale.*

Proof. Define the stopping times $\{S_k\}_{k=1}^{\infty}$ by $S_k = \inf\{i \mid \mid \lambda_{i+1} \mid > k\}$.

$$\begin{aligned} E \mid G_{n\wedge S_k} \mid &= E \mid \textstyle\sum_{i=1}^{n} \lambda_i(X_i - X_{i-1}) 1_{\{S_k \geq i\}} \mid \\ &\leq E \mid \textstyle\sum_{i=1}^{n} k(X_i - X_{i-1}) 1_{\{S_k \geq i\}} \mid \\ &\leq \textstyle\sum_{i=1}^{n} kE \mid (X_i - X_{i-1}) \mid 1_{\{S_k \geq i\}} \\ &\leq \textstyle\sum_{i=1}^{n} kE \mid (X_i - X_{i-1}) \mid \\ &< \infty \end{aligned}$$

So for each n, $G_{n\wedge T_k}$ is integrable.

$$\begin{aligned} E[G_{n\wedge S_k} \mid \mathcal{F}_{n-1}] &= E[G_{(n-1)\wedge S_k} + \lambda_n(X_n - X_{n-1})1_{\{S_k\geq n\}} \mid \mathcal{F}_{n-1}] \\ &= G_{(n-1)\wedge S_k} + \lambda_n 1_{\{S_k>n-1\}} E[\, X_n - X_{n-1} \mid \mathcal{F}_{n-1}] \\ &= G_{(n-1)\wedge S_k} + 0 \end{aligned}$$

Thus for every k, $G^{S_k} = \{G_{n\wedge S_k}\}_{n=1}^{\infty}$ is a martingale. It's clear that $P(\lim_k S_k = +\infty) = 1$. Hence G is a local martingale. □

Corollary 2.2.5 *If X is a local martingale, then G is a local martingale.*

Proof. Let $\{T_n\}$ be a sequence of stopping times that reduce X. For each n, X^{T_n} is a martingale, thus by the previous lemma $G^{S_n\wedge T_n}$ is a martingale. Since $T_n \to \infty$ a.s and $S_n \to \infty$ a.s we have that $S_n \wedge T_n \to \infty$ a.s □

$(\Omega, \mathcal{F}, P)$ and adapted to F.

Let K be the set of obtainable gains, that is

$$K = \{V(\phi) - V_0(\phi) : \phi \text{ is a simple trading strategy}\}$$

By equation 1.1 we may express $V_n(\phi)$ as

$$V_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \lambda_i(Z_i - Z_{i-1}).$$

We can then express K as

$$K = \left\{\sum_{i=1}^{n} \lambda_i(Z_i - Z_{i-1}) \mid n < \infty,\ \lambda_i \in \mathcal{F}_{i-1}\right\}.$$

We will often use the second formulation of K.

Let $K_b = \{Y \in K \mid Y \text{ is bounded below }\}$. We want to close the set K under a certain kind of limits which we will now define.

Definition 2.2.6 For a set of random variables, B, we will call Y a $*$point of B if there exists a sequence of random variables $Y_1, Y_2, \ldots \in B$ such that

- $Y_n \rightarrow Y$ a.s.
- The Y_n are bounded below, that is there exists a number l such that
  $Y_n \geq l$ a.s.

Definition 2.2.7 A set of random variables B, is called $*$closed if it contains all of it's $*$points.

Note that if Y is a $*$point of B under the measure P, and $Q \sim P$, then Y is $*$point of B under the measure Q.

Lemma 2.2.8 *Let $\mathcal{C}$ be a collection of $*$closed sets, and let*

$$A = \bigcap_{\mathcal{C}} C$$

Then A is $*$closed.

Proof. Let Y be a $*$point of A, and let $Y_n \in \mathcal{C}$ be as in the definition of a $*$point. Then for every C in $\mathcal{C}$, $\{Y_n\}_{n=1}^{\infty} \subset C$. Hence, Y is $*$point of each $C \in \mathcal{C}$. It follows that $Y \in C$ for every $C \in \mathcal{C}$

$\Longrightarrow Y \in A$

Which implies that A is $*$closed. □

Definition 2.2.9 Let $\mathcal{C}_B = \{C \mid B \subset C,\ C \text{ is } *\text{closed }\}$. We define the $*$closure, B^*, of B by

$$B^* = \bigcap_{\mathcal{C}_B} C.$$

Remark 2.2.10 For an arbitrary set of random variables B, B^* is not necessarily the union of B and its set of $*$points.

Let H_+ be the set of all non-negative random variables defined on $(\Omega, \mathcal{F}, P)$.

Theorem 2.2.11 *If there exists a measure Q, equivalent to P, such that Z is a local martingale with respect to Q, then*

$$K^* \cap H_+ \ = \ \{0\}.$$

Proof. Let K_b be as above. Clearly, $K^* \cap H_+ \ = \ \{0\}$ if and only if $K_b^* \cap H_+ \ = \ \{0\}$.
Let Q be as in the hypothesis of the theorem. Now define the set A by

$$A \ = \ \{Y \mid E_Q Y \ \leq \ 0\}.$$

It easily follows from Fatou's lemma that A is $*$closed. By lemma 2.2, $K_b^* \ \subset \ A$. Also, $A \cap H_+ \ = \ \{0\}$, which implies that

$$K^* \cap H_+ \ = \ \{0\}.$$

□

Conjecture 2.2.12 There exists an equivalent martingale measure for Z if and only if

$$K^* \bigcap H_+ \ = \ \{0\}.$$

That is, if $K^* \cap H_+ \ = \ \{0\}$, then there exists an equivalent martingale measure for Z.

Let $\hat{K}$ be the set K along with all of its $*$points. As remarked earlier, in general, $\hat{K}$ is not the same set as K^*. When K is the set of achievable gains associated with Z, it is not clear what the relation between $\hat{K}$ and K^* is, but we state the following conjecture.

Conjecture 2.2.13 If K is the set of gains associated with Z, then

$$\hat{K} \bigcap H_+ = \{0\} \qquad \Longleftrightarrow \qquad K^* \bigcap H_+ = \{0\}.$$

Remark 2.2.14 As was stated in the first section, if trading strategies are defined as in definition 2.1.2, then Theorem 2.2.11 shows that the absence of arbitrage is necessary for there to exist an equivalent martingale measure.

2.3 Families of Stochastic Processes

In this section we will generalize the results of section 3 to a family of stochastic processes. Again let $(\Omega, \mathcal{F}, P)$ be some probability space, and F be a filtration for this space. Let $V = \{Z^\alpha;\ \alpha \in A\}$ be a family of discrete time processes on $(\Omega, \mathcal{F}, P)$ and F; that is for each $\alpha \in A$, $Z^\alpha = \{Z^\alpha(n)\}_{n=1}^\infty$ is an adapted process defined on $(\Omega, \mathcal{F}, P)$. In the financial setting, V represents the prices of a family of assets available to investors.

Definition 2.3.1 If V is any family of stochastic processes defined on some probability space and adapted to the filtration F, we say that P is a (local) martingale measure for V if P is a (local) martingale measure for every process in V.

Let

$$K = \{\sum_{k=1}^{n} \lambda_k^\alpha (Z_k^\alpha - Z_{k-1}^\alpha) \mid \alpha \in A, \lambda_k^\alpha \in \mathcal{F}_{k-1}\}$$

K represents the set of outcomes obtained by investing in a single asset using simple trading strategies. Let K_l be the set of all finite linear combinations of elements of K. As in section 2, let K_l^* be the $*$closure of K_l, and H_+ be the set of all non-negative random variables on $(\Omega, \mathcal{F}, P)$.

Theorem 2.3.2 *If there exists an equivalent local martingale measure for V, then*

$$K_l^* \cap H_+ = \{0\}$$

Proof. Let $Q \sim P$ be a local martingale measure for V. Let B be the set of elements of K_l that are bounded below.

Lemma 2.3.3 *If $X \in B$ then $E_Q X = 0$.*

Proof. If $X \in K_l$, then X can be written as $X = G^1(m_1) + G^2(m_2) + \cdots + G^n(m_n)$, where G^k is a process of gains for some Z^α. Thus by corollary 2.2.5, we have that for each k, G^k is a local martingale, as well as $\left(G^k\right)^{m_k}$ (the process stopped at the time m_k) . Hence $\left(G^1\right)^{m_1} + \cdots + (G^n)^{m_n}$ is a local martingale. It follows by lemma 2.2, that $E_Q X = 0$.

□

The result follows by the same proof as theorem 2.2.11.

□

Remark 2.3.4 Artzner and Heath [1] (remark 3, section 2.3) by citing Harrison & Pliska [10] and Jacod [11], state that is very natural to consider examples with infinitely many discontinuous price processes.

Chapter 3

Unbounded Stochastic Processes

3.1 Introduction

In this section we will examine whether Conjecture 2.2.12 is true. In order to motivate the results of this chapter, we will begin with an example.

Example 3.1.1 Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. normal random variables with mean and variance both 1 and let:

$$Z_n = X_1 + X_2 + \ldots + X_n.$$

The increments of this process are all unbounded above and below. Thus if K is the set of obtainable claims (as defined in section 2.2), then the random variable equal to 0 a.s. is the only element of K that is bounded below. This implies that $K^* = K$, which implies that

$$K^* \bigcap H_+ = \{0\}.$$

Therefore if conjecture 2.2.12 holds, then there must be an equivalent martingale measure for the process $Z = \{Z_n\}$.

One's first attempt to construct an equivalent martingale measure might be to use Girsanov's Theorem. However, if one tries to construct an equivalent martingale measure following Girsanov's Theorem, one gets a measure under which the process Z is the sum of i.i.d. normal mean zero random variables. It is easily seen by the strong law of large numbers that this new measure is mutually singular with respect to the original measure.

In section 2.3 we will show how to construct an equivalent martingale measure for this process. The construction holds for a class of processes that we will call *totally unbounded.*

Throughout the remainder of this chapter, $Z = \{Z_n\}_{n=0}^{\infty}$, will be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and adapted to the filtration $F = \{\mathcal{F}_n\}_{n=0}^{\infty}$.

Definition 3.1.2 A random variable is called totally unbounded if it is unbounded above and below. A stochastic process, Z, is called totally unbounded if all of its increments are conditionally totally unbounded, that is, for every $n \geq 1$ and for any constant M,

$$P(Z_n > M \;|\mathcal{F}_{n-1}) > 0 \; a.s \text{ and } P(Z_n < -M \;|\mathcal{F}_{n-1}) > 0 \; a.s.$$

It is clear that if conjecture 2.2.12 is to hold, then every totally unbounded process must have an equivalent martingale measure.

3.2 Integrability

In order to construct the equivalent martingale measure in section 3, we need a measure equivalent to P under which Z is an integrable process. In this section, we will construct such a measure.

Proposition 3.2.1 *There exists a probability measure P_1, equivalent to P, such that Z is integrable with respect to P_1.*

Proof. We will construct the probability measure P_1 by constructing its Radon-Nikodym derivative in the form:

$$\frac{dP_1}{dP} \equiv \rho_1 \rho_2 \cdots$$

Intuitively, the component ρ_k changes the measure so as to make the kth increment integrable. The ρ_k must be constructed such that when they are multiplied together, the infinite product converges, and this product is the Radon-Nikodym derivative of an equivalent probability measure. This will be established via a Borel-Cantelli argument.

For $n \geq 1$, let X_n be the nth increment of the process. That is

$$X_n = Z_n - Z_{n-1}.$$

Choose the integers $N_1, N_2, \ldots$ such that:

$$P(|X_k| > 2^{N_k}) \leq 2^{-(k+1)}$$

For $k \geq 1$, let

$$A_k = \{|X_k| > 2^{N_k}\}, \text{ and } C = \bigcap_{k \geq 1} A_k^c.$$

We then have that: $P(C) \geq 1/2$.

On the event C, the increments of the process are bounded. For each component, we will shift mass from where the component takes large values to the set C. For each $k \geq 1$ and $n \geq N_k$, let

$$A_{k,n} = \{2^n < |X_k| \leq 2^{n+1}\}.$$

Then

$$A_k = \cup_{n=N_k}^{\infty} A_{k,n}$$

Set $\mu_1 = P$. Assuming that $\mu_1, \mu_2, \ldots, \mu_n$ have been defined, let

$$\alpha_n = \sum_{k=N_n}^{\infty} \left(4^{-k} 1_{\{\mu_n(A_{n,k}) > 4^{-k}\}} + \mu_n(A_{n,k}) 1_{\{\mu_n(A_{n,k}) \leq 4^{-k}\}}\right)$$

Define ρ_n by:

$$\rho_n(\omega) = \begin{cases} \frac{4^{-k}}{\mu_n(A_{n,k})} & \omega \in A_{k,n},\ n \geq k, \text{ and } \mu_n(A_{n,k}) > 4^{-k} \\ 1 + \frac{\mu_n(A_n) - \alpha_n}{\mu_n(C)} & \omega \in C \\ 1 & \text{otherwise} \end{cases}$$

Define μ_{n+1} by:

$$d\mu_{n+1} = \rho_n d\mu_n = \rho_1 \rho_2 \cdots \rho_n dP$$

By the definition of ρ_n, we have that:

$$\begin{aligned} E_{\mu_n} \rho_n &= \mu_n(C^c \cup A_n^c) + \left(1 + \tfrac{\mu_n(A_n) - \alpha_n}{\mu_n(C)}\right)\mu_n(C) \\ &\quad + \textstyle\sum_{k=N_n}^{\infty} \left(\frac{4^{-k}}{\mu_n(A_{n,k})} 1_{\{\mu_n(A_{n,k}) > 4^{-k}\}} + 1_{\{\mu_n(A_{n,k}) \leq 4^{-k}\}}\right)\mu_n(A_{n,k}) \\ &= \mu_n(C^c \cap A_n^c) + \mu_n(C) + \mu_n(A_n) - \alpha_n + \alpha_n \\ &= 1 \end{aligned}$$

Note that for each n, ρ_n and μ_n have the following properties:

- For $\omega \in C$, $\rho_n(\omega) \geq 1$ and hence that $\mu_n(C) \geq \mu_{n-1}(C)$ (since $\alpha_n \leq \mu_{n-1}(A_n)$)

- If $\omega \in C^c$ then $\rho_n(\omega) \leq 1$ and thus for any measurable set B, $\mu_n(B \cap C^c) \leq \mu_{n-1}(B \cap C^c)$.

Lemma 3.2.2 *$\prod_{k=1}^{\infty} \rho_k(\omega)$ exists a.s., and is strictly positive with P-probability 1.*

Proof. Let $B = \bigcap_{k \geq 1} A_k$.
Since $\lim_{k\to\infty} P(A_k) = 0$, we have that $P(B) = 0$.
If $\omega \in C^c \cap B^c$ then for large k, $\rho_k(\omega) = 1$. Thus for all $\omega \in C^c \bigcap B^c$,

$$\prod_{k=1}^{\infty} \rho_k(\omega) \text{ exists and is strictly positive.}$$

For $\omega \in C$, we have that:

$$\begin{aligned} \rho_k(\omega) &= 1 + \tfrac{\mu_{k-1}(A_k) - \alpha_k}{\mu_{k-1}(C)} \\ &\leq 1 + \tfrac{2^{-(k+1)}}{P(C)} \\ &\leq 1 + \tfrac{2^{-(k+1)}}{1/2} \\ &= 1 + 2^{-(k)}. \end{aligned}$$

Thus for $\omega \in C$,

$$\prod_{k=1}^{\infty} \rho_k(\omega) \leq \prod_{k=1}^{\infty} (1 + 2^{-(k)}) < \infty.$$

□

Now set

$$\rho(\omega) = \prod_{k=1}^{\infty} \rho_k(\omega) .$$

Lemma 3.2.3 $E\rho = 1$.

Proof. We have that for $k \geq n$

$$\begin{aligned} \rho_k(\omega) &= 1 && \text{if } \omega \in C^c \bigcap A_n^c \\ \rho_k(\omega) &\geq 1 && \text{if } \omega \in C \end{aligned}$$

Let $\rho^a = \rho(\omega)$ and $\rho_n^a = \rho_n(\omega)$ for any $\omega \in C$ (since ρ and ρ_n are constant on C) . It follows that:

$$\begin{aligned} E\rho &= \rho^a P(C) + E\rho_1\rho_2\cdots\rho_n 1_{\{C^c \cap A_n^c\}} + E\rho 1_{\{A_n\}} \\ &\geq \rho_n^a P(C) + E\rho_1\rho_2\cdots\rho_n 1_{\{C^c \cap A_n^c\}} \\ &= \mu_{n+1}(A_n^c) \\ &\geq 1 - P(A_n) \\ &\geq 1 - 2^{-(n+1)} \end{aligned}$$

By Fatou's Lemma

$$\begin{aligned} E\rho &\leq \liminf_n E\rho_1\rho_2\cdots\rho_n \\ &= 1 \end{aligned}$$

□

Now define P_1 by,

$$dP_1 = \rho dP.$$

By definition $P_1 \ll P$. Since $P(\rho > 0) = 1$, P_1 is equivalent to P.

Lemma 3.2.4 *Z is integrable with respect to P_1.*

Proof.

$$\begin{aligned} E_{P_1}|X_n| &= E\rho|X_n| = E\rho|X_n|1_{\{|X_n| \leq 2^{N_n}\}} + \textstyle\sum_{k=N_n}^{\infty} E\rho|X_n|1_{\{A_{n,k}\}} \\ &\leq 2^{N_n} + \textstyle\sum_{k=N_n}^{\infty} 4^{-k}|X_n|1_{\{A_{n,k}\}} \\ &\leq 2^{N_n} + \textstyle\sum_{k=N_n}^{\infty} 4^{-k}2^{k+1} < \infty \end{aligned}$$

Thus

$$E_{P_1}|Z_n| \leq E_{P_1}|X_1| + E_{P_1}|X_2| + \cdots + E_{P_1}|X_n| < \infty$$

□

Thus, we have constructed a probability measure, P_1 , such that Z is integrable with respect to it, and P_1 is equivalent to P.

□

3.3 Constructing the Martingale Measure

We are now ready for the main result of this chapter. We will show how to construct an equivalent martingale measure for a stochastic process that is totally unbounded.

Theorem 3.3.1 *If Z is totally unbounded, then there exists a probability measure Q, equivalent to P, such that Z is a local martingale with respect to Q.*

The construction is similar to that in section 2, but there are some necessary differences. Again we will define the Radon-Nikodym derivative as

$$\frac{dQ}{dP} = \prod_{n=1}^{\infty} \rho_n$$

but now each ρ_n can be expressed as:

$$\rho_n = \frac{E[\frac{dQ}{dP} \, |\mathcal{F}_n]}{E[\frac{dQ}{dP} \, |\mathcal{F}_{n-1}]}$$

The ρ_n are constructed so as to make the conditional expectation of the increments 0. Again we will use a Borel Cantelli argument to make sure that we end up with an equivalent probability measure. Before proceeding with the proof, we will first prove several lemmas.

Notation: For the random variable X and any positive number a, let

$$X^a \equiv X 1_{\{|X| \leq a.\}}$$

Lemma 3.3.2 *Let X be a totally unbounded random variable with distribution μ defined on $(\Omega, \mathcal{G})$. Suppose that X is integrable. Then, for any n, there exists a $\mathcal{G}$-measurable function ρ_n such that:*

1. $\rho_n > 0$ *a.s.*
2. $P(\rho_n = 1) \geq 1 - 2^{-n}$
3. $E\rho_n|X| < 6E|X|$
4. $E\rho_n = 1$
5. $E\rho_n X = 0$

Remark 3.3.3 The proof of this lemma shows how to construct the components of the Radon-Nikodym derivative. The first condition is necessary for us to get an equivalent measure. We will apply the second condition to use the Borel Cantelli argument. It is the fourth and fifth conditions that cause the conditional expectations to work out correctly.

Proof. Let c_1 be such that $\mu([-c_1, c_1]) \geq 1 - 2^{-n}$.

Set $\epsilon_0 = 1$ - $\mu([-c_1, c_1])$.

Define c_2 by, $c_2 = 2E|X|/\epsilon_0$.

Let $a_1 = \mu([-c_2, -c_1) \cup (c_1, c_2])$

$a_2 \; = \mu((c_2, \infty))$, and

$a_3 \; = \mu((-\infty, c_2))$.

Define ϵ_2 by:

$$\epsilon_2 = \frac{3}{2}\max\{\frac{E|X^{c_2}|a_3}{EX1_{\{X<-c_2\}}}, \frac{E|X^{c_2}|a_2}{EX1_{\{X>c_2\}}}, \frac{2(a_2+a_3)}{3}\}$$

We have for the first term in the definition of ϵ_2 that:

$$\frac{3E|X^{c_2}|}{4E|X|1_{\{X\leq -c_2\}}} \leq \frac{3E|X|a_3}{4a_3c_2} \; = \; 3\epsilon_0/4$$

The same inequality holds for the second term. Thus since

$$a_2 + a_3 \; = \; \epsilon_0 - a_1 \; \leq \epsilon_0,$$

it follows that, $\epsilon_2 \; \leq \epsilon_0$. Set $\epsilon_1 \; = \epsilon_0 - \epsilon_2$.

We also have that: $\frac{\epsilon_1}{a_1} \leq 1$.

Define α by:

$$\alpha = \frac{EX^{c_1} + \frac{\epsilon_1}{a_1}E(X^{c_2} - X^{c_1}) + (\frac{\epsilon_2}{a_3}EX1_{\{X<-c_2\}})}{(\frac{\epsilon_2}{a_3}EX1_{\{X<c_2\}}) - (\frac{\epsilon_2}{a_2}EX1_{\{X>c_2\}})}$$

By the definition of ϵ_2, $0 < \alpha < 1$. Now define ρ_n by,

$$\rho_n = \begin{cases} 1 & \text{if } |X| \; \leq c_1; \\ \frac{\epsilon_1}{a_1} & \text{if } c_1 < |X| \leq c_2 \\ \frac{\alpha\epsilon_2}{a_2} & \text{if } X > c_2 \\ \frac{(1-\alpha)\epsilon_2}{a_3} & \text{if } X < -c_2 \end{cases}$$

By the construction, it is clear that ρ_n satisfies the first two statements of the lemma.

Lemma 3.3.4 $E\rho_n X = 0$

Proof.

$$\begin{aligned} E\rho_n X &= EX^{c_1} + \tfrac{\epsilon_1}{a_1}E(X^{c_2} - X^{c_1}) + \alpha\tfrac{\epsilon_2}{a_2}EX1_{\{X>c_2\}} + (1-\alpha)\tfrac{\epsilon_2}{a_3}EX1_{\{X<-c_2\}} \\ &= EX^{c_1} + \tfrac{\epsilon_1}{a_1}E(X^{c_2} - X^{c_1}) + \tfrac{\epsilon_2}{a_3}EX1_{\{X<-c_2\}} \\ &\quad + \alpha(\tfrac{\epsilon_2}{a_2}EX1_{\{X>c_2\}} - \tfrac{\epsilon_2}{a_3}EX1_{\{X<-c_2\}}) \\ &= 0 \end{aligned}$$

□

Lemma 3.3.5 $E\rho_n|X| \leq 6E|X|$

Proof. From above, we have that

$$(1-\alpha)\frac{\epsilon_2}{a_3}EX1_{\{X<-c_2\}} = -\left(EX^{c_1} + \frac{\epsilon_1}{a_1}E(X^{c_2} - X^{c_1}) + \alpha\frac{\epsilon_2}{a_2}EX1_{\{X>c_2\}}\right)$$

Thus

$$(1-\alpha)\frac{\epsilon_2}{a_3}E \mid X \mid 1_{\{X<-c_2\}} \leq E \mid X \mid^{c_2} + \frac{\alpha\epsilon_2}{a_2}EX1_{\{X>c_2\}} \tag{3.1}$$

$$\begin{aligned} E\rho_n \mid X \mid = \quad & E \mid X \mid^{c_1} + \frac{\epsilon_1}{a_1}E \mid X^{c_2} - X^{c_1} \mid + \alpha\frac{\epsilon_2}{a_2}EX1_{\{X>c_2\}} \\ & + (1-\alpha)\frac{\epsilon_2}{a_3}E \mid X \mid 1_{\{X<-c_2\}} \end{aligned} \tag{3.2}$$

We now check what happens in the three possible cases for ϵ_2.

<u>Case 1</u>: $\epsilon_2 = a_2 + a_3$

If $a_2 \geq a_3$ then substituting 3.1 into 3.2 we have,

$$\begin{aligned} E\rho_n|X| &\leq 2(E|X^{c_2}| + \tfrac{\alpha\epsilon_2}{a_2}EX1_{\{X>c_2\}}) \\ &\leq 2(E|X^{c_2}| + 2EX1_{\{X>c_2\}}) \\ &\leq 6E|X|. \end{aligned}$$

The same argument applies with $a_3 \geq a_2$.

$$\underline{\text{Case2}}: \epsilon_2 = \frac{3}{2}\frac{E|X^{c_2}|a_2}{EX1_{\{X>c_2\}}}$$

Again using 3.1 and 3.2 we have that:

$$\begin{aligned} E\rho_n|X| &\leq 2\left(E|X^{c_2}| + \tfrac{3E|X^{c_2}|}{2EX1_{\{X>c_2\}}}EX1_{\{X>c_2\}}\right) \\ &\leq 5E|X| \end{aligned}$$

$$\underline{Case3}: \epsilon_2 = \frac{3}{2}\frac{E \mid X \mid^{c_2} a_3}{EX1_{\{X<-c_2\}}}$$

The argument here is essentially the same as for the second case. $\square$

Thus ρ satisfies the conditions of the lemma.

$\square$

Lemma 3.3.6 *Suppose the functions ρ_n satisfy:*

- $0 < \rho_n < \infty$
- $E[\rho_{n+1}|\mathcal{F}_n] = 1$
- $P(\rho_{n+1} = 1|\mathcal{F}_n) \geq 1 - 2^{n+1}$

then $\prod_{n=1}^{\infty} \rho_n$ exists a.s., is strictly positive with probability 1, and $\forall\ k \geq 1$,

$$E[\prod_{n=k}^{\infty} \rho_n|\mathcal{F}_{k-1}] = 1$$

Proof. Since $P(\rho_n = 1) \geq 1 - 2^n$, it follows easily from the first Borel-Cantelli Lemma that,

$P(\rho_n \neq 1\ i.o.) = 0.$

Thus $\prod_{n=1}^{\infty} \rho_n$ exists and is strictly positive a.s.

Let $\alpha_n = \prod_{k=n}^{\infty} \rho_k$

For any positive integer r,

$$\begin{aligned} E[\alpha_{n+1}|\mathcal{F}_n] &\geq E\rho_{n+1}\cdots\rho_{n+r}\alpha_{n+r+1}1_{\{\alpha_{n+r+1}=1\}} \\ &= E\rho_{n+1}\cdots\rho_{n+r}E[\alpha_{n+r+1}1_{\{\alpha_{n+r+1}=1\}}|\mathcal{F}_{n+r}] \\ &\geq E\rho_{n+1}\cdots\rho_{n+r}(1-2^{n+r+1}) \\ &= 1-2^{n+r+1} \end{aligned}$$

Thus $E[\alpha_{n+1}|\mathcal{F}_n] \geq 1$. Fatou's Lemma easily yields $E[\alpha_{n+1}|\mathcal{F}_n] \leq 1$, so we are done. □

Proof. (Of Theorem 1.2.1)

Let $X_n = Z_n - Z_{n-1}$.

Let $\mu_n(\cdot,\omega)$ be a regular conditional distribution of X_n given $\mathcal{F}_{n-1}$.

That is, for any Borel set A, $\mu(A,\omega)$ is a version of

$$E[1_{\{X_n \in A\}}|\mathcal{F}_{n-1}]$$

such that for almost all ω, $\mu_n(\cdot,\omega)$ is a measure that corresponds to a proper distribution function (see Ash [2]). Let

$$A_n = \{\omega \mid \mu_n(\cdot,\omega) \text{ is totally unbounded and integrable }\}.$$

Then $\forall\ n$, $P(A_n) = 1$. For $\omega \in A_n$ apply Lemma 1.2.2 to μ_n and n to define $\rho_n(\omega)$. For $\omega \notin A_n$, set $\rho_n(\omega) = 1$.

By Lemma 1.2.3, we may define ρ by:

$$\rho \equiv \prod_{n=1}^{\infty} \rho_n.$$

We have that $\rho > 0$ a.s., and $E\rho = 1$.

Now define the measure Q, by:

$$dQ = \rho dP.$$

It is clear that $Q \ll P$. Since $\rho > 0$ P-a.s., Q is equivalent to P. We now need to verify that Z is a local martingale with respect to Q.

Define the stopping times T_n by:

$$T_n \equiv \inf\{k \geq 1 \mid \rho_1\rho_2\cdots\rho_k > n\}$$

It's clear that the T_n are increasing, and

$$P(\lim_n T_n = \infty) = 1$$

Lemma 3.3.7 *For each n, $E_Q|Z_{k\wedge T_n}| < \infty$*

Proof. We have that:

$$\begin{aligned} E_Q|X_k|1_{\{T_n \geq k\}} &= E\rho|X_k|1_{\{T_n\geq k\}} \\ &\leq En\textstyle\prod_{l=k}^{\infty}\rho_l|X_k| \\ &= E\left[nE[\rho_k|X_k|\,|\mathcal{F}_{k-1}]\right] \leq 6nE\left[\,E[X_k\,|\mathcal{F}_{k-1}]\right] \\ &= 6nE|X_k| < \infty \end{aligned}$$

Thus

$$\begin{aligned} E_Q|Z_{k\wedge T_n}| &= E_Q|Z_0 + \textstyle\sum_{i=1}^{k} X_i 1_{\{T_n\geq i\}}\,| \\ &\leq E|Z_0| + \textstyle\sum_{i=1}^{k} E_Q|X_i|1_{\{T_n\geq i\}} < \infty \end{aligned}$$

□

Lemma 3.3.8 *If $E|X| < \infty$, and $E_Q|X| < \infty$,*

then $Y \equiv E[\alpha_{n+1}X|\mathcal{F}_n]$ is a version of $E_Q[X|\mathcal{F}_n]$.

(Where $\alpha_n = \prod_{k=n}^{\infty}\rho_k$).

Proof. Clearly Y is $\mathcal{F}_n$−measurable. If $A \in \mathcal{F}_n$ then:

$$\begin{aligned}
E_Q Y 1_A &= E\rho Y 1_A \\
&= E[E[\rho Y 1_A | \mathcal{F}_n]] \\
&= E\rho_1 \cdots \rho_n Y 1_A E[\alpha_n \, | \mathcal{F}_n] \\
&= E\rho_1 \cdots \rho_n Y 1_A \\
&= E(\rho_1 \cdots \rho_n 1_A E[\alpha_n X | \mathcal{F}_n]) \\
&= E(E[\rho X 1_A | \mathcal{F}_n]) \\
&= E(\rho X 1_A) \\
&= E_Q X 1_A
\end{aligned}$$

□

Lemma 3.3.9 *For each n, $(Z_{k \wedge T_n})_{k=0}^{\infty}$ is a Q-martingale.*

Proof.

$$\begin{aligned}
E_Q[Z_{k \wedge T_n} 1_{\{T_n \geq k\}} | \mathcal{F}_{k-1}] &= E_Q[Z_k 1_{\{T_n \geq k\}} | \mathcal{F}_{k-1}] \\
&= 1_{\{T_n \geq k\}} \left(Z_{k-1} + E_Q[X_k \, | \mathcal{F}_{k-1}] \right) \\
&= 1_{\{T_n \geq k\}} \left(Z_{k-1} + E[\alpha_k X_k | \mathcal{F}_{k-1}] \right) \\
&= 1_{\{T_n \geq k\}} [\, Z_{k-1} + 0] \\
&= 1_{\{T_n \geq k\}} Z_{k-1}
\end{aligned}$$

Using this, we have that:

$$\begin{aligned}
E_Q[Z_{k \wedge T_n} | \mathcal{F}_{k-1}] &= E_Q[Z_{k \wedge T_n} 1_{\{T_n \geq k\}} | \mathcal{F}_{k-1}] + E_Q[Z_{k \wedge T_n} 1_{\{T_n < k\}} | \mathcal{F}_{k-1}] \\
&= 1_{\{T_n \geq k\}} Z_{k-1} + E_Q[Z_{(k-1) \wedge T_n} 1_{\{T_n < k\}} | \mathcal{F}_{k-1}] \\
&= 1_{\{T_n \geq k\}} Z_{k-1} + 1_{\{T_n < k\}} Z_{(k-1) \wedge T_n} \\
&= Z_{(k-1) \wedge T_n}
\end{aligned}$$

□

Thus Z is a local martingale measure with respect to Q. Since Q is equivalent to P, we are done. □

Remark 3.3.10 It would be interesting to know when one can construct a martingale measure and not not just a local martingale measure. One such case occurs when under the original measure the process has independent increments. In this case, Q can be constructed so as to make the process a martingale.

Chapter 4

Binary Processes

4.1 Introduction

In this section we will give a special class of processes for which conjecture 2.2.12 holds. We will call this class of processes *binary*. This class includes the example studied by Back and Pliska (example 2.1.1). As before, Z will be a stochastic process defined on the probability space, $(\Omega, \mathcal{F}, P)$. We will assume that the filtration, F, is the filtration generated by Z. That is, $\mathcal{F}_n = \sigma(Z_1, Z_2, \ldots, Z_n)$.

Definition 4.1.1 Z is called binary, if

- Ω is countable.
- Given the values of the process up to time $n-1$, Z_n can take at most two distinct values.
- Z separates points of ω. That is if ω_1, $\omega_2 \in \Omega$ and

$$Z_n(\omega_1) = Z_n(\omega_2) \qquad \forall\, n$$

then $\omega_1 = \omega_2$. (The last assumption can be replaced by the assumption that $\mathcal{F}$ is the smallest σ-field on which Z can be defined.)

4.2 Constructing Martingale Measures for Binary Processes

In what follows, the sets K, K^* and H_+ will be as defined in section 2.2.

Theorem 4.2.1 *If Z is a binary process, then there exists an equivalent martingale measure for Z if and only if*

$$K^* \bigcap H_+ = \{0\} \tag{4.1}$$

From now on will refer to 4.1 as the no arbitrage hypothesis. Let X_n be the increments of the process, ie.

$$X_n = Z_n - Z_{n-1}$$

Lemma 4.2.2 *If $P(\{\omega\}) > 0$ then*

$$P(X_n > 0 \mid \mathcal{F}_{n-1})(\omega) > 0 \iff P(X_n < 0 \mid \mathcal{F}_{n-1})(\omega) > 0$$

That is if we have observed the process up to time n, then at time $n+1$, the process is either constant with probability 1, or can increase or decrease, each with positive probability.

Proof. Let

$$\lambda = P(\{\omega\} \mid \mathcal{F}_{n-1})$$

Suppose that $P(X_n > 0 \mid \mathcal{F}_{n-1})(\omega) > 0$, but $P(X_n < 0 \mid \mathcal{F}_{n-1})(\omega) = 0$. Then $\lambda X_n \in K$ and $\lambda X_n \in H_+$, which is a contradiction to the no arbitrage hypothesis.

□

Definition 4.2.3 Define the function $\pi_1 : \Omega \to \{0,1\}$ by:

$$\pi_1(\omega) = \begin{cases} 1 & X_1(\omega) \geq 0 \\ 0 & X_1(\omega) < 0 \end{cases}$$

For $n \geq 2$ define $\pi_n : \Omega \to \{0,1\}^n$ by

$$\pi_n(\omega) = \begin{cases} (\pi_{n-1}(\omega), 1) & X_n(\omega) \geq 0 \\ (\pi_{n-1}(\omega), 0) & X_n(\omega) < 0 \end{cases}$$

We will need π_n defined on $\{0,1\}^k$, $n \leq k \leq \infty$ as well as on Ω. For $x \in \{0,1\}^k$, $n \leq k \leq \infty$, set

$\pi_n(x) = (x_1, x_2, \ldots, x_n)$. i.e., π_n is the *natural projection* of x on the first n coordinates.

Definition 4.2.4 For $x \in \{0,1\}^n$, let

$$\pi_n^{-1}(\{x\}) = \{\omega \in \Omega \mid \pi_n(\omega) = x\}$$

Note that by the definition of a binary process, Z_n (and hence X_n) is constant on $\pi_n^{-1}(\{x)\})$.

Proof. (of Theorem 4.2.1)

Define the collection of sets $\mathcal{S}$ by,

$$\mathcal{S} = \{\emptyset\} \cup \{\Omega\} \cup \{\pi_n^{-1}(\{x\}) \mid x \in \{0,1\}^n, \, 1 \leq n < \infty\}$$

We will define the set function, Q, on $\mathcal{S}$, such that Q gives the sets in $\mathcal{S}$ the value that a martingale measure would. After showing that Q has an extension to all of $\mathcal{F}$, we will then verify that this extension is equivalent to P.

For the remainder of this chapter, e will always denote either 0 or 1. If its value is not specified, then e is to be interpreted as "either 0 or 1".

Set $\overline{e} = 1 - e$.

Set,

$Q(\emptyset) = 0$ and

$Q(\Omega) = 1$.

When $\pi_1^{-1}(\{e\}) \neq \emptyset$, then ω_e will denote an arbitrary element of $\pi_1^{-1}(\{e\})$.

If either $P(\pi_1^{-1}(\{e\})) = 1$ or $X_1(\omega_e) = 0$ (which implies $X_1(\omega_{\overline{e}}) = 0$) then set

$Q(\pi_1^{-1}(\{e\})) = P(\pi_1^{-1}(\{e\}))$

$Q(\pi_1^{-1}(\{\overline{e}\})) = P(\pi_1^{-1}(\{\overline{e}\}))$.

Otherwise, let

$$\alpha_1 = \frac{X_1(\omega_1)}{X_1(\omega_1) - X_1(\omega_0)}$$

Then by lemma 4.2.2, $\alpha_1 \in (0,1)$.

Now set

$$P(\pi_1^{-1}(0)) = \alpha_1 \text{ and } P(\pi_1^{-1}(1)) = 1 - \alpha_1$$

Now suppose for each $k \in \{1, 2, \ldots, n-1\}$ and for every $x \in \{0,1\}^k$

$Q(\pi_k^{-1}(\{x\}))$ has been defined.

Fix $x \in \{0,1\}^{n-1}$.

We will now define $Q(\pi_n^{-1}(\{(x,e)\}))$ in terms of $Q(\pi_{n-1}^{-1}(\{x\}))$. In what follows ω_e is any element of $\pi_n^{-1}(\{(x,e)\})$. ie. $\omega_1 \in \pi_n^{-1}(\{(x,1)\})$.

Case 1: If $P(\pi_{n-1}^{-1}(\{x\})) = 0$ (which implies that $Q(\pi_{n-1}^{-1}(\{x\})) = 0$), then set

$$Q(\pi_n^{-1}(\{(x,e)\})) = Q(\pi_n^{-1}(\{(x,\overline{e})\})) = 0$$

Case 2: If $X_n(\omega_e) = 0$ and $P(\pi_{n-1}^{-1}(\{x\})) > 0$ then set

$$Q(\pi_n^{-1}(\{(x,e)\})) = Q(\pi_{n-1}^{-1}(\{x\}))\frac{P(\pi_n^{-1}(\{(x,e)\}))}{P(\pi_{n-1}^{-1}(\{x\}))}$$

Case 3 : $P(\pi_n^{-1}(\{(x,e)\})) = 0$, and $P(\pi_n^{-1}(\{(x,\overline{e})\})) > 0$.

(Note that by lemma 4.2.2 this implies that $X_n(\omega_{\overline{e}} = 0$.)

Set

$$\begin{aligned} Q(\pi_n^{-1}(\{(x,e)\})) &= 0 \\ Q(\pi_n^{-1}(\{(x,\overline{e})\})) &= Q(\pi_{n-1}^{-1}(\{x\})) \end{aligned}$$

Case 4: $Q(\pi_{n-1}^{-1}(\{x\})) > 0$, and $X_n(\omega_1) \neq 0$, then set

$$\alpha_n = \frac{X_n(\omega_0)}{X_n(\omega_0) - X_n(\omega_1)}.$$

By lemma 4.2.2, $\alpha_n \in (0,1)$.

Now set

$$\begin{aligned} Q(\pi_n^{-1}(\{(x,1)\})) &= \alpha_n\, Q(\pi_{n-1}^{-1}(\{x\})) \\ Q(\pi_n^{-1}(\{(x,0)\})) &= (1-\alpha_n)Q(\pi_{n-1}^{-1}(\{x\})) \end{aligned}$$

From this construction, we can define Q, for every set in $\mathcal{S}$. Note that everywhere that α appears in the construction, we have that $\alpha \in (0,1)$. Thus for $x \in \{0,1\}^n$

$$Q(\pi_n^{-1}(\{x\})) = 0 \iff P(\pi_n^{-1}(\{x\})) = 0$$

Lemma 4.2.5 *$\mathcal{S}$ is a semi-algebra; that is,*

1. *$\emptyset$, $\Omega \in \mathcal{S}$*

2. *$\mathcal{S}$ is closed under finite intersection.*

3. *If $S \in \mathcal{S}$, then S^c is the finite union of disjoint sets in $\mathcal{S}$.*

Proof. To check 2, let S_1, S_2 be any two elments of $\mathcal{S}$.

It is trivial if for either $i = 1$ or 2 $S_i = \emptyset$ or Ω. So we may assume that for each i, $S_i \neq \emptyset$, or Ω. Then by the defintion of $\mathcal{S}$ there exists x_1, x_2 and n_1, n_2 with $x_1 \in \{0,1\}^{n_1}$ and $x_2 \in \{0,1\}^{n_2}$

such that

$S_1 = \pi_{n_1}^{-1}(\{(x_1)\}),$

$S_2 = \pi_{n_2}^{-1}(\{(x_2)\}).$

Suppose without loss of generality that $n_1 \leq n_2$.

Then

$$S_1 \bigcap S_2 = \{\omega \mid \pi_{n_1}(\omega) = x_1 \text{ and } \pi_{n_2}(\omega) = x_2\}$$

The set on the right hand side is either S_2, or is the empty set. Thus 2 holds.

To verify 3, let $S \in \mathcal{S}$, then for some n and $x \in \{0,1\}^n$ we have that

$$S = \pi_n^{-1}(\{x\}) = \{\omega \mid \pi_n(\omega) = x\}$$

Set

$$G = \{y \in \{0,1\}^n \mid y \neq x\}$$

Then G has finite cardinality and

$$S^c = \bigcup_{y \in G} \pi_n^{-1}(\{y\}) = \{\omega \mid \pi_n(\omega) \neq x\}$$

Hence, S^c is a finite union of elements of $\mathcal{S}$.

Thus $\mathcal{S}$ is a semi-algebra as claimed.

□

Lemma 4.2.6 *Q is an additive probability measure on $\mathcal{S}$.*

Proof. To show that Q is additive, let $S_1, S_2 \in \mathcal{S}$, be such that

$$S_1 \bigcap S_2 = \emptyset \text{ and}$$

$$S_1 \bigcup S_2 \in \mathcal{S}$$

Set

$$S = S_1 \bigcup S_2$$

Then for some $x \in \{0,1\}^n$, S has the form $S = \pi_n^{-1}(\{x\})$. Since the intersection of S_1 and S_2 is empty, and their union is $\pi_n^{-1}(\{x\})$, it follows that there exists $e_1, e_2, \ldots, e_k \in \{0,1\}$ such that

$$\begin{aligned} S_1 &= \pi_{n+k+1}^{-1}(\{(x, e_1, e_2, \ldots, e_k, e)\}) \\ S_2 &= \pi_{n+k+1}^{-1}(\{(x, e_1, e_2, \ldots, e_k, \overline{e})\}) \end{aligned}$$

and that

$$\pi_{n+i}^{-1}(\{(x, e_1, \ldots, e_i)\}) = \emptyset \qquad 1 \leq i \leq k$$

By the construction, we have that

$$Q(S) = Q(\pi_{n+k}^{-1}(\{(x, e_1, \ldots, e_k)\})$$

and there exists $\beta \in [0,1]$, such that

$$\begin{aligned} Q(S_1) &= \beta Q(S) \\ Q(S_2) &= (1-\beta) Q(S) \end{aligned}$$

This implies that

$$Q(S) = Q(S_1) + Q(S_2)$$

□

In order for there to exist a countably additive extension of Q to all of $\mathcal{F}$, we need that Q be countably additive on $\mathcal{S}$.

Definition 4.2.7 By countably additive on $\mathcal{S}$, we mean that if

- $S_1, S_2, \ldots \in \mathcal{S}$
- $S_1 \supset S_2 \supset S_3 \supset \cdots$
- $\cap_{n=1}^{\infty} S_n = \emptyset$

then $Q(S_n) \to 0$.

Lemma 4.2.8 *Q is countably additive on $\mathcal{S}$.*

Proof. Let $\{S_n\}$ be as in the definition of countable additivity. By the definition of $\mathcal{S}$, there exists for each k, n_k and $x_k \in \{0,1\}^{n_k}$ such that
$S_k = \pi_{n_k}^{-1}(\{x_k\})$.
For each n, let ω_n be an arbitrary element of S_n.
$S_k \supset S_{k+1}$ implies that $\pi_{n_k}(\omega_{k+1}) = x_k$. Hence, we may assume that $n_1 \leq n_2 \leq \cdots$
We then have that, for each k,
$\pi_{n_k}(x_{k+1}) = x_k$.
Thus there exists $x \in \{0,1\}^\infty$ such that for each k, $S_k = \pi_{n_k}^{-1}(\{\pi_{n_k}(x)\})$. (This says that the sequence of sets S_n correspond to essentially one sample path) . By adding some sets and removing duplicates if necessary, we may assume that $n_k = k$, ie.

$S_k = \pi_k^{-1}(\{\pi_k(x)\}).$

If $P(S_n) = 0$ for some n, then $Q(S_n) = 0$, and the lemma is clearly true. So, assume otherwise. By the construction of Q, there exists $\beta_1, \beta_2, \ldots \in (0, 1]$ such that

$$Q(S_n) = \beta_1 \beta_2 \cdots \beta_n$$

Suppose that

$$\lim_{n \to \infty} Q(S_n) > 0.$$

We will show that this violates the no arbitrage hypothesis.

Let $S_0 = \Omega$.

For each n let ω_n be an arbitrary element of S_n. For n such that $S_n \neq S_{n-1}$, let $\overline{x}_n = (\pi_{n-1}(x), 1) - (\pi_n(x))$ and $\overline{\omega}_n$ be an element of $\pi_n^{-1}(\overline{x}_n)$. Then we have that $\overline{\omega}_n \in S_{n-1}$, but $\overline{\omega}_n \notin S_n$. Note that if $X_n(\omega_n) \neq 0$, then $S_n \neq S_{n-1}$ and $\overline{\omega}_n$ exists.

We will now construct an investment strategy, that in the limit gives us a *sure win*. The strategy is such that if we win at time 1, then we stop. If we do not win at time 1 and at time n our gain is negative (represented by $-G_n$), we invest just enough so that if we win at time $n+1$, our total gain will be 0. Assume without loss of generality that $\beta_1 < 1$. This implies that $P(S_1^c) > 0$

Let

$$\lambda_1 = \frac{\operatorname{sgn}(X_1(\overline{\omega}_1))}{\mid X_1(\overline{\omega}_1) \mid}$$

and set

$G_1 \;=\; -\lambda_1 X_1$

For $n \;\geq\; 2$ let

$$\lambda_n(\omega) \;=\; \begin{cases} 0 & \text{if } X_n(\omega_n) \;=\; 0 \\ \frac{G_{n-1}}{X_n(\overline{\omega}_n)} 1_{\{\omega \in S_{n-1}\}} & \text{if } X_n(\omega_n) \;\neq\; 0 \end{cases}$$

and set

$$\begin{aligned} G_n &= G_{n-1} - \lambda_n X_n \\ &= G_{n-1}\left(\tfrac{X_n(\overline{\omega}_n) - X_n(\omega)}{X_n(\overline{\omega}_n)}\right) \end{aligned}$$

We then have that

$$G_n(\overline{\omega}_n) \;=\; G_{n-1}(\overline{\omega}_n)\left(\frac{X_n(\overline{\omega}_n) - X_n(\overline{\omega}_n)}{X_n(\overline{\omega}_n)}\right) \;=\; 0$$

On the event S_n, we have that

$$\begin{aligned} G_n(\omega_n) &= G_{n-1}\tfrac{X_n(\overline{\omega}_n) - X_n(\omega_n)}{X_n(\overline{\omega}_n)} \\ &= G_{n-1}/\beta_n \end{aligned}$$

Which implies that on the event S_n we have

$$\begin{aligned} G_n(\omega_n) &= \tfrac{X_1(\overline{\omega}_1)}{X_1(\omega_n)\beta_2 \cdots \beta_n} \\ & \tfrac{1-1/\beta_1}{\beta_2\beta_3\cdots\beta_n} \end{aligned}$$

Let

$$Y_n \;=\; \sum_{i=1}^{n} \lambda_i X_i \;=\; -G_n$$

Then we have that

$$Y_n(\omega) \;=\; \begin{cases} 1 & \omega \;\in\; S_1^c \\ 0 & \omega \;\in\; S_n^c \bigcup S_1 \\ \frac{1-1/\beta_1}{\beta_2\beta_3\cdots\beta_n} & \omega \;\in\; S_n \end{cases}$$

Since

$$\bigcap_{i=1}^{\infty} S_i = \emptyset,$$

we have that

$$Y_n \to Y \text{ a.s.}$$

where

$$Y(\omega) = \begin{cases} 1 & \omega \in S_1^c \\ 0 & \omega \in S_1 \end{cases}$$

For each n we have that

$$\begin{aligned} Y_n &= -G_n \\ &\geq -c/(\beta_2\beta_3\cdots\beta_n) \\ &= -cQ(S_1)/Q(S_n) \\ &\geq \lim_k Q(S_1)/Q(S_k) \end{aligned}$$

Where c is a positive constant. Thus if $\lim_n Q(S_n) > 0$, then the Y_n are bounded below (uniformly). We then have $Y \in K^*$, and $Y \in H_+$. Since Y is not 0, we have a contradiction to

$$K^* \bigcap H_+ = \{0\}$$

So it must be that $\lim_n Q(S_n) = 0$. Thus Q is countably additive. □

By proposition I.6.1 in Neveu [15] , there is a unique probability measure, $\tilde{Q}$, that is the extension of Q on $\sigma(\mathcal{S})$. From now on we will simply refer to $\tilde{Q}$ as Q.

For each $\omega \in \Omega$,

$$\{\omega\} = \bigcap_{n=1}^{\infty} \pi_n^{-1}(\{\pi_n(\omega)\}) \in \sigma(\mathcal{S})$$

Thus $\sigma(\mathcal{S}) = \mathcal{F}$.

It easily follows from the construction that under Q, Z is a martingale (with respect to F).

It remains to show that $Q \sim P$.

It is enough to verify that for each $\omega \in \Omega$

$$Q(\{\omega\}) = 0 \iff P(\{\omega\}) = 0$$

<u>Case 1</u>: $P(\pi_k^{-1}(\{\pi_k(\omega)\})) = 0$ for some k.

If $k = 1$, it follows from the construction of $Q(\pi_1^{-1}(e))$ that $Q(\pi_1^{-1}(\{\pi_1(\omega)\})) = 0$, which implies that $Q(\{\omega\}) = 0$. For $k > 1$, case 1 in the construction implies that

$$Q(\pi_k^{-1}(\{\pi_k(\omega)\}) = 0$$
$$\Longrightarrow Q(\{\omega\}) = 0$$

<u>Case 2</u>: $Q(\pi_k^{-1}(\{\pi_k(\omega)\}) = 0$ for some k.

This can only happen in the first three cases of the construction. Each of the first three cases implies that

$$P(\{\omega\}) = 0$$

<u>Case 3</u>: $Q(\{\omega\}) > 0$

Let $S_k = \pi_k^{-1}(\{\pi_k(\omega)\})$, we then have that

$$\bigcap_{k=1}^{\infty} S_k = \{\omega\}$$

Suppose that $P(\{\omega\}) = 0$. Then we have that the sets S_k decrease to a set of probability 0. Using the proof of lemma 4.2.3, we get a contradiction to the no arbitrage hypothesis. Hence it must be that $P(\{\omega\}) > 0$.

<u>Case 4</u>: $Q(\{\omega\}) = 0$, and for each k, $Q(\pi_k^{-1}(\{\pi_k(\omega)\}) > 0$

We will show that if $P(\{\omega\}) > 0$, then we can construct a sequence, $Y_n \in K$, such that the sequence is bounded below and converges to a nonzero positive random variable. The Y_n will be positive on the event $\{\omega\}$, and bounded below if $Q(\{\omega\}) = 0$.

For an arbitrary n, suppose that $\pi_n(\omega) = (\pi_{n-1}(\omega), e)$. Then when the set $\pi_n^{-1}(\{(\pi_{n-1}(\omega), \overline{e})\})$ is non-empty, $\overline{\omega}_n$ will denote an arbitrary element of this set. Note that when $\pi_n^{-1}(\{\pi_{n-1}(\omega), \overline{e})\})$ is empty we have that $X_n(\omega) = 0$.

To simplify the notation, when $\pi_n^{-1}(\{(\pi_{n-1}(\omega, \overline{e})\})$ is not empty let

$$\begin{aligned} b_n &= X_n(\omega) \\ a_n &= X_n(\overline{\omega}_n) \end{aligned}$$

If $\pi_n^{-1}(\{(\pi_{n-1}(\omega), \overline{e})\})$ is empty, let $a_n = b_n = 0$. Thus b_n is the change in the process at time n on the sample path corresponding to ω, and a_n is the change in the process at time n when the process follows the sample path corresponding to ω up to time $n-1$, but not up to time n. By lemma 4.2.2,

$$\operatorname{sgn}(a_n) = -\operatorname{sgn}(b_n)$$

By the construction

$$Q(\{\omega\}) = \prod_{n=1}^{\infty} \beta_n$$

Where

$$\beta_n = \begin{cases} 1 & b_n = 0 \\ \frac{a_n}{a_n - b_n} & b_n \neq 0 \end{cases}$$

By assumption, $\prod_{n=1}^{\infty} \beta_n = 0$.

Let $\gamma_1 = \frac{1-\beta_1}{b_1} 1_{\{b_1 \neq 0\}}$

and for $n \geq 2$

$$\gamma_n = 1_{\{b_n \neq 0\}} \frac{1-\beta_n}{b_n} \prod_{k=1}^{n-1} \beta_k$$

$$\begin{aligned}
\sum_{k=1}^{N} b_n \gamma_n &= \sum_{k=1}^{N} (1-\beta_n) \prod_{i=1}^{k-1} \beta_i \\
&= \sum_{k=1}^{N} \left[\prod_{i=1}^{k-1} \beta_i - \prod_{i=1}^{k} \beta_i \right] \\
&= 1 - \prod_{i=1}^{N} \beta_i
\end{aligned} \tag{4.2}$$

Thus,

$$\lim_N \sum_{k=1}^{N} b_k \gamma_k = 1$$

Hence for each n, we may choose M_n, such that

$$\sum_{k=n}^{M_n} \gamma_k b_k \geq (1 - 1/n) \sum_{k=n}^{\infty} \gamma_k b_k$$

Let $S_n = \pi_n^{-1}(\{\pi_n(\omega)\})$.

Set

$$\lambda_k^n(\omega) = \frac{\gamma_k}{\sum_{i=n}^{\infty} \gamma_i b_i} 1_{\{\omega \in S_{k-1}\}}$$

We are now ready to define the sequence of gains, Y_n.

Set

$$Y_n = \sum_{k=n}^{M_n} \lambda_k^n X_k$$

Then we have that for each n, $Y_n \in K$.

On the event $\{\omega\}$,

$$Y_n = \frac{\sum_{i=n}^{M_n} \gamma_i b_i}{\sum_{i=n}^{\infty} \gamma_i b_i} \geq 1 - 1/n,$$

and on S_{n-1}^c,

$$Y_n = 0.$$

Since

$$\{\omega\}^c = \bigcup_{n=1}^{\infty} S_n^c$$

it follows that

$$Y_n \to Y \text{ a.s.}$$

Where Y is 0 on $\{\omega\}^c$ and 1 on $\{\omega\}$. The last step is show that the Y_n are bounded below.

On the event $S_{k-1} \cap S_k^c$, for $k \geq n$ we have that

$$\begin{aligned} Y_n &= \frac{\sum_{i=n}^{k-1} \gamma_i b_i + \gamma_k a_k}{\sum_{i=n}^{\infty} \gamma_i b_i} \\ &\geq \frac{\gamma_k a_k}{\sum_{i=n}^{\infty} \gamma_i b_i} \\ &= \frac{\gamma_k a_k}{\prod_{i=1}^{n-1} \beta_i} \quad \text{(by equation 4.2)} \\ &= \frac{-\prod_{i=1}^{k} \beta_i}{\prod_{i=1}^{n-1} \beta_i} \\ &\geq -1 \end{aligned}$$

Thus for all n, $Y_n \geq -1$ a.s. Thus Y is a $*$point of the set K. Since Y is a positive nonzero random variable, we get a contradiction to the no-arbitrage hypothesis. Thus if $Q(\{\omega\}) = 0$, then we must have that $P(\{\omega\}) = 0$. We have now shown that $Q \sim P$, which completes the proof of Theorem 4.2.1.

□

Chapter 5

Concluding Examples

5.1 A Family of Processes without an Equivalent Martingale Measure

In this chapter we will look at two examples. The first example is a family of stochastic processes for which there exists an equivalent local martingale measure, but for which no equivalent martingale measure exists. This example along with the results in Chapter 2 suggest that in general the idea of no arbitrage should actually be related to the existence of an equivalent local martingale measure and not just an equivalent martingale measure.

Let $\Omega = \{1, 2, \ldots\} \times \{0, 1\}$ and let $\mathcal{F}$ be the collection of all subsets of Ω. We will write each element of Ω as $\omega = (\omega_1, \omega_2)$. We will assume that the original measure, P, is such that every point of Ω has positive probability.

Define the process Z by,

$$Z(0,\omega) = 2$$

$$Z(1,\omega) = \omega_1 \text{ and for } k \geq 2$$

$$Z(k,\omega) = \begin{cases} Z(1) + 2^{\omega_1} & \text{if} \quad \omega_2 = 1 \\ Z(1) - 2^{\omega_1} & \text{if} \quad \omega_2 = 0 \end{cases}$$

Now define the processes $\{X^n\}_{n=1}^{\infty}$ by

$$X^n(0) = 1$$

and for $k \geq 1$

$$X^n(k,\omega) = \begin{cases} 2^n & \text{if} \quad \omega_1 = n \\ 0 & \text{if} \quad \omega_1 \neq n \end{cases}$$

Let the filtration F be the σ-fields generated by the processes. V will be the family consisting of Z and $\{X^n\}_{n=1}^{\infty}$. Suppose that Q is a martingale measure for V. Since X^n is a martingale under Q for every n, we have that

$$1 = E_Q X^n(1) = 2^n Q(\omega_1 = n)$$

Hence, $Q(\omega_1 = n) = 2^{-n}$.

Since $E_Q[Z(2) \mid \mathcal{F}_1] = Z(1)$, we have that for each n, $Q(\,(n,1) \mid \omega_1 = n) = 1/2$.

Thus

$$Q(\,(n,1)\,) = Q(\,(n,0)\,) = 2^{-(n+1)}$$

It follows that Q is the only possible martingale measure for Q.

Now, $E_Q \mid Z_1 \mid < \infty$, but

$$E_Q \mid Z(2) - Z(1) \mid = \sum_{k=1}^{\infty} 2^n 2^{-n} = \infty$$

From which it follows that $E_Q \mid Z(2) \mid = \infty$. Thus since Z is not integrable, Q (and hence any equivalent measure) is not a martingale measure for V, but we will show that it is a local martingale measure for V.

Claim *Q is a local martingale measure for V.*

Proof. By the definition of Q each X^n is a martingale. Define the stopping times $\{T_n\}_{n=1}^{\infty}$ by

$$T_n = \begin{cases} 1 & \text{if } \omega_1 > n \\ \infty & \text{if } \omega_1 \leq n \end{cases}$$

It's clear that the T_n are increasing and that $T_n \to \infty$ a.s.

Under Q, $Z(1)$ is a geometric random variable with mean 2, so $E_Q Z(1) = 2 = Z(0)$. Since $Z(2 \wedge T_k) \leq 2^k$, it's obvious that $Z(2 \wedge T_k)$ is integrable. On the event $\{T_k > 1\}$

$$E_Q[Z(2 \wedge T_k) \mid \mathcal{F}_1] = Z_1 + \frac{1}{2}(2^{Z_1} - 2^{Z_1}) = Z_1$$

Thus for each k, Z^{T_k} is a martingale. Hence, Q is a local martingale measure for Z.

□

Since Q is a local martingale measure for V, it follows by Theorem 2.3.2, that V does not admit any arbitrage opportunities in the sense of the results in Chapter 2.

5.2 Unbounded Winnings

Each of the results of Stricker and Delbaen for continuous processes involved closing the set of obtainable outcomes, K, under some topological operation. In Stricker's result for discontinuous processes, he takes the closure of the set $K - B_+$. The appearance of B_+ is due to this wildness of the set of outcomes associated with a discontinuous process. We will give an example which illustrates this wildness. One can win in a reasonable sense by betting on this process, but the only way to win involves the possibility of arbitrarily large winnings.

Let $X_1, X_2, X_3, \ldots$ be independent and have the following distributions:

$$X_n = \begin{cases} -1 & \text{with probability} \quad 1/3 \\ 1 & \text{with probability} \quad 2/3 - 2^{-n} \\ r_n & \text{with probability} \quad 2^{-n} \end{cases}$$

Where $r_n = 2^{2n}(\prod_{k=1}^{n} 2^k) = 2^{\frac{n(n+5)}{2}}$.

We will let $q_n = 1/r_n$.

Let $Z_0 = 0$, and $Z_n = X_1 + X_2 + \cdots + X_n$.

Let F be the σ-fields generated by the process. One can view Z as a continuous time process that only changes values at the times $1, 2, \ldots$

Define the set K as Stricker does, ie.

$$K = \left\{ \sum_{i=1}^{n} \lambda_i X_i \mid \lambda_i \in \mathcal{F}_{i-1} \right\}$$

It is not hard to check that there is no equivalent martingale measure for this process. Thus by Stricker's Theorem we must have that

$$\overline{K - B_+} \cap L^1_+ \neq \{0\}.$$

We will show that for this process:

$$\overline{K} \cap L^1_+ = 0.$$

That is, if one is not allowed to throw money away, the "winning strategies" will not converge in L^1. Intuitively, all of the winning strategies for this process are so good that they win too much money to converge in L^1.

Suppose that $Y_1, Y_2, \ldots \in K$ and $Y_n \to Y$ in L^1. We will show that

$$Y \geq 0 \Longrightarrow Y = 0.$$

Each Y_n has the form:

$$Y_n = \sum_{i=1}^{m_n} \lambda_i^n X_i, \quad \lambda_i^n \in \mathcal{F}_{i-1}.$$

By taking subsequences if necessary, we may assume that $m_1 \leq m_2 \leq m_3 \leq \ldots$ Several times we will use the fact that if $A \in \mathcal{F}_n$, then

$$P(A) = 0 \quad \text{or} \quad P(A) \geq \prod_{k=1}^{n} 2^{-k} = 2^{\frac{-n(n+1)}{2}} \tag{5.1}$$

The first step is to show that in order for the Y_n to converge in L^1, it is necessary that λ_k^n be small for large k.

Claim *There exists M such that for each n,*

$$|\lambda_k^n| \leq 2^{-k} \quad \forall k \geq M$$

Proof. By enumerating the possible cases, one can check that for any numbers a and b

$$E \mid a + bX_n \mid \geq \max\{a\left(1 - 2^{-(n-1)}\right), 2^{-(n-1)} r_n b\} \tag{5.2}$$

Set:

$$\epsilon = 2 \prod_{n=1}^{\infty} (1 - 2^{-n}) > 0$$

Since they converge to Y, the Y_n are a Cauchy sequence in L^1. Thus there exists N such that

$$E \mid Y_k - Y_l \mid \leq \epsilon \qquad\qquad \forall\, k,l \geq N$$

Now let $M = m_N$.

If $n \leq N$ then, by the assumption that the m_k are increasing, $\lambda_k^n = 0$ for $k \geq M$.

So choose $n > N$ such that $m_n \geq M$.

Let $X = \sum_{i=1}^{M}(\lambda_i^N - \lambda_i^n)X_i$.

By using equation 5.2 with $b = \lambda_{m_n}^n$ and $a = X + \sum_{i=M+1}^{m_n-1} \lambda_i^n X_i$, we have

$$\begin{aligned} E \mid Y_N - Y_n \mid &= E \mid X + \sum_{i=M+1}^{m_n} \lambda_i^n X_i \mid \\ &= E\left[E[\mid X + \sum_{i=M+1}^{m_n-1} \lambda_i^n X_i + \lambda_{m_n}^n X_{m_n} \mid \mid \mathcal{F}_{m_n-1}]\right] \\ &\geq (1 - 2^{-(m_n-1)})E \mid X + \sum_{i=M+1}^{m_n-1} \lambda_i^n X_i \mid \end{aligned} \tag{5.3}$$

By using equations 5.2 and 5.3 repeatedly we get that

$$\begin{aligned} E \mid Y_N - Y_n \mid &\geq \left(\prod_{i=k}^{m_n-1} (1-2^{-i}) \right) E \mid X + \sum_{i=M+1}^{k} \lambda_i^n X_i \mid \\ &\geq 2\epsilon E \mid X + \sum_{i=M+1}^{k-1} \lambda_i^n X_i + \lambda_k^n X_k \mid \end{aligned} \tag{5.4}$$

By using equations 5.2 and 5.4 together and conditioning we have that

$$\begin{aligned} E \mid Y_N - Y_n \mid &\geq \epsilon E \mid r_k \lambda_k^n \mid 2^{-k} \\ &\geq \epsilon\, r_k 2^{\frac{k(k+3)}{2}} \|\lambda_k^n\|_\infty 2^{-k} \\ &= \epsilon\, 2^k \|\lambda_k^n\|_\infty \end{aligned}$$

Since $E \mid Y_N - Y_n \mid \leq \epsilon$ we have that $\mid \lambda_k^n \mid \leq 2^{-k}$ a.s.

$\square$

The second step is to show that because λ_i^n are small for large i, $Y \geq 0$ implies that $Y_n \xrightarrow{P} 0$.

Claim *For any* k,

$$\left(\sum_{i=1}^{k} \lambda_i^n X_i\right)^- \xrightarrow{P} 0 \text{ as } n \to \infty$$

Proof. Suppose not, then there exists k and $\delta > 0$ such that

$$P\left(\left(\sum_{i=1}^{k} \lambda_i^n X_i\right)^- > \delta\right) > \delta$$

for every n along some subsequence. Now for $k \geq M$, since $|\lambda_k^n| \leq 2^{-k}$ we have for any n that:

$$\begin{aligned} P(\,|\sum_{i=k}^{m_n} \lambda_i^n X_i \,| < 2^{-i}\,) \quad &\geq P(\, X_i \in \{-1,1\};\ k \leq i \leq m_n\,) \\ &\geq \prod_{i=k}^{\infty}(1-2^{-i}) \end{aligned} \tag{5.5}$$

Let m be such that $m \geq M$ and $2^m < \delta/2$. We then have that:

$$\begin{aligned} P(Y_n < -\delta/2) \quad &\geq P(\textstyle\sum_{i=1}^{k} \lambda_i^n X_i \leq -\delta;\ \sum_{i=k+1}^{m} \lambda_i^n X_i \leq 0;\ \sum_{i=m+1}^{m_n} \lambda_i^n \leq \delta/2) \\ &\geq P(\textstyle\sum_{i=1}^{k} \lambda_i^n X_i \leq -\delta;\ \sum_{i=k+1}^{m} \lambda_i^n X_i \leq 0;\ X_i \neq r_k\ \forall\, i > m) \\ &\geq P(\textstyle\sum_{i=1}^{k} \lambda_i^n X_i \leq -\delta;\ \sum_{i=k+1}^{m} \lambda_i^n X_i \leq 0)\ \left(\prod_{i=m+1}^{\infty}(1-2^{-i})\right) \\ &\geq P(\textstyle\sum_{i=1}^{k} \lambda_i^n X_i \leq -\delta\,)\left(\prod_{i=k+1}^{m} 3^{-i}\right)\left(\prod_{i=m+1}^{\infty}(1-q_i)\right) \\ &> \delta\left(\textstyle\prod_{i=k+1}^{m} 3^{-i}\right)\left(\prod_{i=m+1}^{\infty}(1-2^{-i})\right) \end{aligned}$$

But, since the last line is independent of n and $Y_n \xrightarrow{P} Y$, this contradicts $Y \geq 0$. Thus we have that,

$$\left(\sum_{i=1}^{k} \lambda_i^n X_i\right)^- \xrightarrow{P} 0$$

This implies that:

$$\left(\sum_{i=1}^{k} \lambda_i^n X_i\right)^+ \xrightarrow{P} 0$$

$\square$

Let $\epsilon > 0$, and let k be such that $2^{-k} < \epsilon/2$. By equation 5.5, we have that

$$\begin{aligned} P(\mid Y_n \mid > \epsilon) &\leq P\left(\mid \sum_{i=1}^k \lambda_i^n X_i \mid > \epsilon/2\right) + P\left(\mid \sum_{i=k+1}^{m_n} \lambda_i^n X_i \mid > \epsilon/2\right) \\ &\leq P\left(\mid \sum_{i=1}^k \lambda_i^n X_i \mid > \epsilon/2\right) + \left(1 - \prod_{i=k+1}^{\infty}(1-2^{-i})\right) \end{aligned}$$

Thus for any sufficiently large k

$$\lim_{n\to\infty} P(\mid Y_n \mid > \epsilon) \leq 1 - \prod_{i=k+1}^{\infty} (1-2^{-i})$$

Since

$$\lim_{k\to\infty} \prod_{i=k}^{\infty} (1-2^{-i}) = 1$$

We have that $Y_n \xrightarrow{P} 0$. From which it follows that $Y = 0$. Thus

$$\overline{K} \cap B_+ = \{0\}$$

Remark 5.2.1 This also shows that if the Y_n are bounded elements of K and $Y_n \to Y$ a.s., then $Y \geq 0$ implies that $Y = 0$. Thus the condition of Delbaen's is not sufficient for discontinuous processes.

Chapter 6

Summary

The major topic of this thesis was the relationship between the financial condition of " no arbitrage " and the mathematical condition of the existence of an equivalent martingale measure for a stochastic process. The most notable differences between this work and the previous work in this area is that we study the existence of equivalent local martingale measures instead of just equivalent martingale measures (as defined by definition 1.0.6), and we propose a new definition of no arbitrage. In chapter 5 a family of processes was given for which an equivalent local martingale measure exists, but there is no equivalent martingale measure for this process. This family of processes shows that, at least for infinite families of assets, the existence of equivalent local martingale measures is the right question to be addressed. If one takes our approach, then Delbaen has completely solved this problem for continuous processes.

For discrete time processes, we have given a necessary condition for there to exist an equivalent local martingale measure and conjectured that this condition is also sufficient. In support of this conjecture we have shown that if the increments of a process are unbounded enough (that is if the process is what we call totally unbounded), then there always exists an equivalent local martingale measure. We have also shown that our conjecture holds for a class of processes which includes the example studied by Back and Pliska in the literature.

6.1 Future Research

We will now give several areas of future research suggested by this work. The most fundamental open question is conjecture 2.2.12. Most results relating no arbitrage to equivalent martingale measures are based on the Hahn-Banach theorem. One difficulty with this approach, is that to apply the Hahn-Banach Theorem (or a theorem based on it) one must introduce some sort of L^p assumption; this is unnatural since such a condition is not preserved under a change to an equivalent measure. Delbaen, using some results from functional analysis, was able to get around the problem of an L^p assumption. An approach like Delbaen's seems to hold some promise towards proving conjecture 2.2.12.

Because it involves $*$closed sets, conjecture 2.2.12 is somewhat messy. Proving conjecture 2.2.13 would be useful in that it simplifies our principle conjecture.

A final question we will address is the idea of uniqueness of an equivalent martingale measure. In several contexts, it is known that the martingale measure is unique if and only if the market model is *complete*. The idea of completeness was addressed by Harrison & Kreps [8] and Harrison & Pliska [9]. More recently completeness has been studied by Müller [14], Jarrow & Madan [12], Artzner & Heath [1], and Delbaen [5]. As is the case with the definition of arbitrage opportunities, in order to define completeness one must first define admissable trading strategies. Thus, there are several possible definitions for complete markets. In [5], Delbaen calls the market complete if the set of outcomes using very simple strategies (definition 1.2.6) is dense in L^1. Delbaen is able to characterize when the market is complete. It would be interesting to look at this question using $*$closure in the definition of completeness.

In [1], Artzner and Heath give an example of a market with discontinuous prices, in which the market is complete in the L^1 sense, but there are many equivalent martingale measures. Call a set A $*$dense in B if $A^* = B$. A possible definition of completeness is to say that the market is complete if the set of obtainable outcomes is $*$dense in the set of bounded random variables. With this definition of completeness, the model of Artzner and Heath does not have complete markets. It seems that it would be worthwhile to further study completeness under this definition.

Appendix A

Proof of Expected Value Lemma

Proof. Since $\epsilon = 2p - 1$,

$$\begin{aligned} E\log(1+\epsilon X_1) &= p\log(2p) + (1-p)\log(2(1-p)) \\ &= \log(2) + p\log(p) + (1-p)\log(1-p). \end{aligned} \tag{A.1}$$

Let $f(x) = 2x\log(x)$. Then $f(x)$ is a strictly convex function on $[0, \infty)$. If x, y are non-negative then

$$f(x/2 + y/2) \le 1/2f(x) + 1/2f(y) \tag{A.2}$$

with strict inequality unless $x = y$. Letting $x = p = 1 - y$ we have that

$$\log(\frac{1}{2}) < p\log(p) + (1-p)\log(1-p)$$

From which it follows that $E\log(1+\epsilon X_1) > 0$.

$\square$

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