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EXTREMES OF MOVING AVERAGES OF RANDOM VARIABLES FROM THE DOMAIN OF THE DOUBLE EXPONENTIAL DISTRIBUTION

by

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EXTREMES OF MOVING AVERAGES OF RANDOM VARIABLES FROM THE DOMAIN OF ATTRACTION OF THE DOUBLE EXPONENTIAL DISTRIBUTION

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Abstract

Let $\{Z_n\}$ be an iid sequence of random variables with common distribution F which belongs to the domain of attraction of $\exp\{-e^{-x}\}$. If in addition, $F \in S_r(\gamma)$ (i.e., $\lim_{x\to\infty} P[Z_1 + Z_2 > x]/P[Z_1 > x] = d \in (0,\infty)$ and $\lim_{x\to\infty}(1 - F(x - y))/(1 - F(x)) = e^{\gamma y}$ for every $y \in \mathbb{R}$), then it is shown that a point process based on the moving average process $\{X_n := \sum_{j=-\infty}^{\infty} c_j Z_{n-j}\}$ converges weakly. A host of complementary results concerning extremal properties of $\{X_n\}$ can then be derived from this convergence result. These include the convergence of maxima to extremal processes, the limit point process of exceedances, the joint limit distribution of the largest and second largest and the joint limit distribution of the largest and smallest. Convergence of a sequence of point processes based on the max–moving average process $\{\bigvee_{j=-\infty}^{\infty} c_j Z_{n-j}\}$ is also considered.

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Running Head: Extremes of Moving Averages

1. INTRODUCTION

Consider a sequence of iid random variables $\{Z_k, -\infty < k < \infty\}$ whose common distribution F is both in the domain of attraction of $\Lambda(x) := \exp\{-e^{-x}\}$, $x \in \mathbb{R}$ and in $S_r(\gamma)$, $\gamma \geq 0$. We now explain these dual requirements.

A distribution function F is in the domain of attraction of the extreme value distribution $\Lambda(x) = \exp\{-e^{-x}\}$, $x \in \mathbb{R}$ if there exist $a_n > 0$, $b_n \in \mathbb{R}$ $(n \geq 1)$ and

$$(1.1) \qquad \lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x), \quad x \in \mathbb{R}$$

or equivalently

$$\lim_{n\to\infty} n\bar{F}(a_n x + b_n) = e^{-x}$$

where $\bar{F}(x) = 1 - F(x)$. If $\{Z_n, n \geq 1\}$ are iid with distribution F this can be rephrased as

$$\lim_{n\to\infty} P[\bigvee_{i=1}^{n} Z_i \leq a_n x + b_n] = \Lambda(x), \quad x \in \mathbb{R}.$$

We assume $x_0 := \sup\{x: F(x) < 1) = \infty$. When (1.1) holds we write $F \in D(\Lambda)$. Cf. de Haan (1970), Resnick (1987).

A distribution F is in the class $S_r(\gamma)$ for $\gamma \geq 0$ if $F(x) < 1$ for all x and

$$(1.2) \qquad \lim_{x\to\infty} \bar{F}(x-y)/\bar{F}(x) = e^{\gamma y}, \text{ for each } y \in \mathbb{R},$$

and

$$\text{(1.3)} \qquad \lim_{x\to\infty} (1 - F*F(x))/\bar{F}(x) = \lim_{x\to\infty} P[Z_1 + Z_2 > x]/P[Z_1 > x] = d < \infty .$$

The constant d is known to equal $2Ee^{\gamma X_1}$ which was proved for the case $F(0) = 0$ by Chover et al (1973) (see also Cline (1986b); Embrechts (1984)) and extended to the case that F concentrates on $\mathbb{R}$ by Willekens (1986). When $F(0) = 0$ and $d = 2$, (1.3) implies (1.2) with $\gamma = 0$ (Chistyakov (1964)), and in this case the class $S_r(0)$ is called in the literature S, the subexponential class. For our purposes it is not natural to restrict attention to distributions concentrating on $[0,\infty)$.

The standing assumption in this paper is that the distribution F satisfies

$$F \in D(\Lambda) \cap S_r(\gamma).$$

The condition $F \in D(\Lambda)$ is necessary for a sequence of point processes based on $\{(Z_k - b_n)/a_n,\ k \geq 1\}$ to converge. The condition $F \in S_r(\gamma)$ is necessary for the tail of $Z_1 + Z_2$ to be comparable to $1 - F$ and means that sequences of point processes based on linear combinations of the $\{Z_n,\ -\infty < n < \infty\}$ will have interesting convergence properties. The case where $F \notin S_r(\gamma)$ will be discussed elsewhere. See Rootzén (1986).

The condition $F \in D(\Lambda) \cap S_r(\gamma)$ was discussed in Goldie and Resnick (1988a) where various sufficient conditions on F were reviewed and examples discussed of $F \in D(\Lambda) \cap S_r(\gamma)$. See also the important paper by Cline (1986a).

The class $S_r(0) \cap D(\Lambda)$ includes the lognormal distribution as well as the distribution with tail

$$\bar{F}(x) = \exp\{-x/(\log x)^{\alpha}\},\ \ x \geq 1,\ \ \alpha > 0.$$

Rootzén (1983, 1986) considers the class $F(x;\ \alpha,\ p)$ defined for $x \geq 1$, $K > 0$ by

$$\bar{F}(x;\ \alpha,\ p) = Kx^{\alpha}\exp\{-x^{p}\}.$$

If $0 < p < 1$ then $F(x;\ \alpha,\ p) \in S_r(0) \cap D(\Lambda)$ while if $p = 1$ and $\alpha < -1$, $F(x;\ \alpha,\ p) \in S_r(1) \cap D(\Lambda)$ (Rootzén (1986)). Embrechts (1983) considers the class of generalized inverse Gaussian distributions $N^{-1}(x;\ \lambda,\ \chi,\ \psi)$ whose density $n^{-1}(x;\ \lambda,\ \chi,\ \psi)$ is defined by

$$n^{-1}(x;\ \lambda,\ \chi,\ \psi) = Kx^{\lambda-1}\exp\{-(\chi x^{-1} + \psi x)/2\}$$

for $x > 0$ where $(\lambda,\ \chi,\ \psi) \in ((0,\ \infty) \times [0,\ \infty) \times (0,\ \infty)) \cup (\{0\} \times (0,\ \infty)^2) \cup ((-\infty,\ 0) \times (0,\ \infty) \times [0,\ \infty))$. Embrechts (1983) shows that if $(\lambda,\ \chi,\ \psi) \in (-\infty,\ 0) \times (0,\ \infty) \times [0,\ \infty)$ then $N^{-1}(x;\ \lambda,\ \chi,\ \psi) \in S_r(\psi/2)$ and one readily verifies, for example using the von Mises criterion (Resnick (1987), page 40), that in this parameter region we also have membership in $D(\Lambda)$.

Under mild conditions on a real sequence $\{c_j\}$ the series $\sum_{j=-\infty}^{\infty} c_j Z_{-j}$ converges and we may define the main objects of this study, namely the stationary sequence of moving averages

$$\{X_n,\ -\infty < n < \infty\} = \{\sum_{j=-\infty}^{\infty} c_j Z_{n-j},\ -\infty < n < \infty\}. \tag{1.4}$$

We study the extreme value behavior of this sequence by means of a point process technique which looks at a sequence of point processes based on $\{X_n\}$.

Related work on extremes of moving averages is Finster (1982), Rootzén (1978, 1983, 1986), Davis and Resnick (1985a,b, 1986). See also Goldie and Resnick (1988b) for point process results relevant to the present setting.

In Section 2 we review some background on point processes and prove a convergence result about a sequence of point processes based on stationary variables which is flexible enough for our needs. Section 3 builds on this treatment to prove a convergence result about a sequence of point processes based on our moving averages (1.4). We give some remarks and some complementary extreme value results in Section 4. Included are discussions about convergence of maxima to extremal processes, the extremal index, exceedances, joint limit distributions of the largest and second largest as well as the largest and smallest among $\{X_1,\dots,X_n\}$. In Section 5 we discuss max–moving averages based on $\{Z_i\}$.

We end this introductory section with a discussion of the conditions needed for convergence of the series $\sum_{j=-\infty}^{\infty} c_j Z_{-j}$.

PROPOSITION 1.1: Suppose $F \in D(\Lambda)$ so the Balkema and de Haan (1972) representation holds (Balkema and de Haan (1972); Resnick (1987)), viz

$$\bar{F}(x) = \theta(x) \exp\{-\int_{z_0}^{x} (1/f(u))du\} \tag{1.5}$$

for some z_0 and $x > z_0$ where $\theta(x) \to \theta \in (0, \infty)$ as $x \to \infty$, $f > 0$ is absolutely continuous on (z_0, ∞) with density f' and $\lim_{u\to\infty} f'(u) = 0$. Given $\epsilon > 0$, there exists $x_0 = x_0(\epsilon)$ such that for $x \geq x_0$

$$\frac{\bar{F}(c^{-1}x)}{\bar{F}(x)} \leq (1 + \epsilon)\left(\frac{f(x)}{x}\right)^{1/\epsilon}\left(\frac{c}{\epsilon(1 - c)}\right)^{1/\epsilon} \tag{1.6}$$

for any $0 < c < 1$.

Furthermore if for some $\delta > 0$, $\sum_{j=-\infty}^{\infty} |c_j|^{1-\delta} < \infty$ and $E|Z_1| < \infty$, then $\sum_{j=-\infty}^{\infty} c_j Z_{-j}$ converges absolutely almost surely and if additionally $c_j \geq 0$ for all j then for m such that $\sum_{|j|>m} c_j^{1-\delta} < 1$ we have

(1.7) $$\lim_{x\to\infty} P[\sum_{|j|\geq m} c_j Z_{-j} > x]/\bar{F}(x) = 0.$$

PROOF: Since f is absolutely continuous we have for $u \geq 1$

$$f(xu) - f(x) = \int_x^{xu} f'(s)ds$$

and since $f' \to 0$ we have for sufficiently large x that the above is less than $\epsilon x(u-1)$ whence

$$1/f(xu) \geq 1/(f(x) + \epsilon x(u-1))$$

and thus

$$\begin{aligned}
\frac{\bar{F}(c^{-1}x)}{\bar{F}(x)} &= (1 + o(1))\exp\{-\int_x^{c^{-1}x} (1/f(u))du\} \\
&= (1 + o(1))\exp\{-\int_1^{c^{-1}} x(1/f(xu))du\} \\
&\leq (1 + o(1))\exp\{-\int_1^{c^{-1}} (f(x) + \epsilon x(u-1))^{-1}xdu\} \\
&= (1 + o(1))\exp\{-\log(1 + \epsilon x(c^{-1} - 1)/f(x)))^{\epsilon^{-1}}\} \\
&= (1 + o(1))(1 + \frac{\epsilon(c^{-1} - 1)}{x^{-1}f(x)})^{-\epsilon^{-1}} \\
&\leq (1 + \epsilon)(\frac{f(x)}{x})^{1/\epsilon}(\frac{c}{\epsilon(1 - c)})^{1/\epsilon}
\end{aligned}$$

for large x and (1.6) follows.

Next if $E|Z_1| < \infty$ and $\sum_{j=-\infty}^{\infty} |c_j|^{1-\delta} < \infty$ then

$E|\sum_{j=-\infty}^{\infty} c_j Z_{-j}| \le E|Z_1| \sum_{j=-\infty}^{\infty} |c_j| < \infty$. Furthermore following Rootzén (1986) we have if $\sum_{|j|>m} c_j^{1-\delta} < 1$ and $x > 0$

$$\frac{P[\sum_{|j|>m} c_j Z_{-j} > x]}{\bar{F}(x)} \le \frac{P[\sum_{|j|>m} c_j Z_{-j} > \sum_{|j|>m} c_j c_j^{-\delta} x]}{\bar{F}(x)}$$

$$\le \frac{P\{\bigcup_{|j|>m} [Z_{-j} > c_j^{-\delta} x]\}}{\bar{F}(x)}$$

$$\le \sum_{|j|>m} \left(\frac{\bar{F}(c_j^{-\delta} x)}{\bar{F}(x)}\right).$$

Applying (1.6) with $c = c_j^{\delta}$ we get an upper bound of

$$K\left(\sum_{|j|>m} c_j^{\delta/\epsilon}\right)(x^{-1}f(x))^{1/\epsilon}.$$

Provided $\delta/\epsilon \ge 1 - \delta$; i.e., $\frac{\delta}{1-\delta} \ge \epsilon$ we have $\sum_{|j|>m} c_j^{\delta/\epsilon} < \infty$ and since $\lim_{x\to\infty} x^{-1}f(x) = \lim_{x\to\infty} f'(x) = 0$ the result follows. □

Under the additional assumption that $F \in S_r(\gamma)$, we may prove a stronger result than (1.7), namely that (1.7) holds with $\sum_j c_j Z_{-j}$ in place of $\sum_{|j|\ge m} c_j Z_{-j}$ provided $0 \le c_j < 1$ and $\sum_j c_j^{1-\delta} < \infty$ for some $\delta > 0$. To prove this we need the following preparatory proposition which is a minor variant of Theorem 2.7 in Embrechts and Goldie (1982). See Theorem 1 of Cline (1986a) for the following formulation when $m = 2$.

PROPOSITION 1.2: Let $\{Y_i,\ 1 \le i \le m\}$ be independent random variables and suppose $F \in S_r(\gamma)$. If

$$\lim_{t\to\infty} P[Y_i > t]/\bar{F}(t) = \alpha_i \in [0, \infty)$$

for $i = 1,\dots,m$ then

$$\begin{aligned}(1.8)\qquad \lim_{t\to\infty} P[\sum_{i=1}^{m} Y_i > t]/\bar{F}(t) &= \sum_{i=1}^{m} \alpha_i\, E\exp\{\gamma \sum_{j\ne i} Y_j\} \\ &= \sum_{i=1}^{m} (\alpha_i/E\exp\{\gamma Y_i\}) \cdot E\exp\{\gamma \sum_{j=1}^{m} Y_j\}.\end{aligned}$$

PROPOSITION 1.3: Let $\{Z_t\}$ be an iid sequence of random variables with common distribution $F \in D(\Lambda) \cap S_r(\gamma)$. Let $\{c_j\}$ be a sequence of constants such that $0 \le c_j \le 1$ for all j and $\sum_{j=-\infty}^{\infty} c_j^{1-\delta} < \infty$ for some $\delta > 0$. Then

$$(1.9)\qquad \lim_{t\to\infty} P[\sum_j c_j Z_{-j} > t]/\bar{F}(t) = k^+ E\exp\{\gamma \sum_j c_j Z_{-j}\}/Ee^{\gamma Z_1}$$

where $k^+ = \#\{j\colon c_j = 1\}$.

PROOF: Choose m so large that $\sum_{|j|>m} c_j^{1-\delta} < 1$ and set

$$X = \sum_{|j|\le m} c_j Z_{-j},\quad Y = \sum_{|j|>m} c_j Z_{-j}\,.$$

Observe that $F \in D(\Lambda)$ implies $1-F$ is rapidly varying whence

$$P[c_j Z_{-j} > t]/\bar{F}(t) \to \begin{cases} 1 & \text{if } c_j = 1 \\ 0 & \text{if } c_j < 1 \end{cases}.$$

It follows from Proposition 1.2 that $P[X > t]/\bar{F}(t) \to k^+ Ee^{\gamma X}/Ee^{\gamma Z_1}$. Moreover, by Proposition 1.1, $P[Y > t]/\bar{F}(t) \to 0$, and now applying Proposition 1.2 once again, we obtain (1.9). □

Remark: Suppose $F \in D(\Lambda) \cap S_r(\gamma)$ and satisfies the tail balancing condition,

$$\frac{\bar{F}(x)}{\bar{F}(x) + F(-x)} \to p, \quad \frac{F(-x)}{\bar{F}(x) + F(-x)} \to 1-p \tag{1.10}$$

as $x \to \infty$ where $0 < p \le 1$. Now assume the coefficients $\{c_j\}$ satisfy $\sum_j |c_j|^{1-\delta} < \infty$ for some $\delta > 0$ and $|c_j| \le 1$ for all j. Define $k^+ = \#\{j\colon c_j = 1\}$, $k^- = \#\{j\colon c_j = -1\}$, $c^+ = \max\{c,0\}$, $c^- = -\min\{c,0\}$ and consider the two independent random variables $X_+ = \sum_j c_j^+ Z_{-j}$ and $X_- = \sum_j c_j^- Z_{-j}$. By Proposition 1.3 and (1.10) we have

$$P[-X_- > t]/F(-t) = P[\sum_j c_j^-(-Z_{-j}) > t]/F(-t) \to k^- E\exp\{-\gamma X_-\}/E\exp\{-\gamma Z_1\}.$$

Since $F(-t)/\bar{F}(t) \to (1-p)/p$ we may apply Proposition 1.2 to the independent sum $X_+ + (-X_-) = \sum_j c_j Z_{-j}$ and conclude

$$\begin{aligned} &\lim_{t\to\infty} P[\sum_j c_j Z_{-j} > t]/\bar{F}(t) \\ &= (k^+/E\exp\{\gamma Z_1\} + k^-(1-p)/(pE\exp\{-\gamma Z_1\}))E\exp\{\gamma \sum_j c_j Z_{-j}\}. \end{aligned} \tag{1.11}$$

2. POINT PROCESSES AND STATIONARY MIXING SEQUENCES

Our results in the next and succeeding sections are based on point processes and we first review relevant notation and background and then give a convergence to Poisson result general enough for our needs.

Weak convergence notation and usage are as in Billingsley (1968) except that "=>" is used to indicate weak convergence. For point processes we follow Neveu (1976); see also Kallenberg (1976), and Resnick (1986, 1987). Let E be a state space which for our purposes is a subset of a compactified Euclidean space. Let $\mathcal{E}$ be the σ–algebra generated by open sets. For $x \in E$, $F \in \mathcal{E}$, $\epsilon_x(F) = 1$ if $x \in F$, 0 otherwise. A point measure m is defined to be a measure of the form $\sum_{i \in I} \epsilon_{x_i}$ which is non–negative integer valued and finite on relatively compact subsets of E. The class of such measures is $M_p(E)$ and $\mathcal{M}_p(E)$ is the smallest σ–algebra making the evaluation maps $m \to m(F)$ measurable where $m \in M_p(E)$ and $F \in \mathcal{E}$. A point process on E is a measurable map from a probability space $(\Omega, \mathcal{A}, P)$ to $(M_p(E), \mathcal{M}_p(E))$. Let $C_K^+(E)$ be the continuous functions $E \to R_+$ with compact support. A useful topology for $M_p(E)$ is the vague topology which renders $M_p(E)$ a complete separable metric space. If $\mu_n \in M_p(E)$, $n \geq 0$ then μ converges vaguely to μ_o (written $\mu_n \xrightarrow{v} \mu_0$) if $\mu_n(f) \to \mu_o(f)$ for all $f \in C_K^+(E)$ where remember $\mu(f) = \int f d\mu$.

A Poisson process on $(E, \mathcal{E})$ with mean measure μ is a point processs ξ satisfying for all $F \in \mathcal{E}$:

$$P[\xi(F) = k] = \begin{cases} e^{-\mu(F)}(\mu(F))^k/k! & \text{if } \mu(F) < \infty \\ 0 & \text{if } \mu(F) = \infty \end{cases}$$

and if $F_1,\dots,F_n \in \mathcal{E}$ are mutually disjoint, then $\xi(F_1),\dots,\xi(F_n)$ are independent. We assume μ is Radon. We will call ξ PRM (Poisson random measure) with mean measure μ on $(E, \mathscr{E})$, or PRM(μ) for short.

In Theorem 2.1 below, we generalize the point process convergence result of Adler (1978). See also Hsing (1985) and Leadbetter, Lindgren and Rootzén (1983). First we define a mixing condition which is similar to condition D in Adler (1978).

For each $n \geq 1$, let $\{X_{n,i}, i \geq 1\}$ be a stationary sequence of random elements of E. In order to define our mixing condition, let $T > 0$ be fixed and let $\mathscr{C}$ be the finite collection of functions

$$\mathscr{C} = \{h_0, h_1, \ldots, h_m\}$$

where $h_0 \equiv 1$, $h_i \in C_k^+(E)$, $h_i \leq 1$, $i = 1,\ldots,m$. We then say that the array $\{X_{n,j}, j \geq 1, n \geq 1\}$ satisfies condition D* if for any two disjoint intervals of integers I_1 and I_2 which are contained in $1,2,\ldots,[nT]$ and separated by ℓ, we have

$$\Big| E \prod_{i=1}^{2} \prod_{i \in I_j} g_i(X_{n,i}) - \prod_{i=1}^{2} E \prod_{i \in I_j} g_i(X_{n,i}) E \prod_{i=p+\ell}^{p+k} g_i(X_{n,i}) \Big| \leq \alpha_{n,\ell}$$

where $1 - g_i \in \mathscr{C}$ and $\alpha_{n,\ell(n)} \to 0$ as $n \to \infty$ for some subsequence $\ell(n) \to \infty$ with $\ell(n) = o(n)$. The function $\alpha_{n,\ell(n)}$ may depend on both $\mathscr{C}$ and T.

Following Lemma 3.3.1 of Leadbetter, Lindgren and Rootzén (1983) we observe that condition D* has the following straightforward generalization. Let $I_1,\ldots,I_k$ be disjoint collections of integers which are separated by at least ℓ and such that $\bigcup_{j=1}^{k} I_j \subset [1, nT]$. Then

$$\Big| E \prod_{j=1}^{k} \prod_{i \in I_j} g_i(X_{n,i}) - \prod_{j=1}^{k} E \prod_{i \in I_j} g_i(X_{n,i}) \Big| \leq (k-1)\alpha_{n,\ell}$$

whenever $1 - g_i \in \mathscr{C}$.

THEOREM 2.1: Suppose for each $n \geq 1$, $\{X_{n,i}, i \geq 1\}$ is a stationary sequence of random elements of E and that the array $\{X_{n,i}, i \geq 1, n \geq 1\}$ satisfies condition D^*. Further assume that there exists a Radon measure ν on E such that

$$nP[X_{n,1} \in \cdot\,] \xrightarrow{v} \nu \tag{2.1}$$

($\xrightarrow{v}$ denotes vague convergence) and for any $g \in C_K^+(E)$, $g \leq 1$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{i=2}^{[n/k]} Eg(X_{n,1})g(X_{n,i}) = 0. \tag{2.2}$$

Then in $M_p([0,\infty) \times E)$,

$$\sum_{k=1}^{\infty} \epsilon_{(kn^{-1}, X_{n,k})} \Rightarrow \sum_{k=1}^{\infty} \epsilon_{(t_k, j_k)}$$

where the limit is $PRM(dt \times d\nu)$.

PROOF: We demonstrate weak convergence by showing Laplace functionals converge (Neveu (1976)) and so we must show for any $f \in C_K^+([0,\infty) \times E)$,

$$E \exp\{-\sum_{i=1}^{\infty} f(i/n, X_{n,i})\} \to \exp\{-\iint_{[0,\infty)\times E} (1 - e^{-f(t,x)}) dt\,\nu(dx), \tag{2.3}$$

the latter expression being the Laplace functional of $PRM(dt \times d\nu)$.

We first show for $T > 0$ fixed and $f \in C_K^+(E)$ that

$$\text{(2.4)} \qquad E\exp\{-\sum_{i=1}^{[nT]} f(X_{n,i})\} \to \exp\{-T\int_E (1-e^{-f(x)})\nu(dx).$$

For each n and k fixed partition the integers $1, 2, \ldots, [nT]$ into $2k$ consecutive blocks of size $[[nT]/k] - \ell(n)$ and $\ell(n)$, i.e.,

$$I_j = ((j-1)r_n + 1, \ldots, jr_n - \ell(n)),\ I_j^* = (jr_n - \ell(n) + 1, \ldots, jr_n),\ j = 1, \ldots, k-1$$

and

$$I_k = ((k-1)r_n + 1, \ldots, kr_n - \ell(n)),\ I_k^* = (kr_n - \ell(n) + 1, \ldots, [nT])$$

where $r_n = [[nT]/k]$. Then as in the proof of Lemma 2.3 of Hsing, Hüsler and Leadbetter (1986),

$$\begin{aligned}
\text{(2.5)} \qquad & |E\exp\{-\sum_{i=1}^{[nT]} f(X_{n,i})\} - (E\exp\{-\sum_{i=1}^{r_n} f(x_{n,i})\})^k| \\
& \le |E\exp\{-\sum_{i=1}^{[nT]} f(X_{n,i})\} - E\exp\{-\sum_{j=1}^{k}\sum_{i\in I_j} f(X_{n,i})\}| \\
& + |E\exp\{-\sum_{j=1}^{k}\sum_{i\in I_j} f(X_{n,i})\} - (E\exp\{-\sum_{i\in I_1} f(X_{n,i})\})^k| \\
& + |(E\exp\{-\sum_{i\in I_1} f(X_{n,i})\})^k - (E\exp\{-\sum_{i=1}^{r_n} f(X_{n,i})\})^k| \\
& \le \sum_{j=1}^{k} E(1-\exp\{-\sum_{i\in I_j^*} f(X_{n,i})\}) + (k-1)\,\alpha_{n,\ell(n)} + kE(1-\exp\{-\sum_{i\in I_1^*} f(X_{n,i})\})
\end{aligned}$$

$$\leq ((k-1)\ell(n) + (k+\ell(n))E(1-e^{-f(X_{n,1})}) + (k-1)\alpha_{n,\ell(n)}$$
$$+ k\ell(n)E(1-e^{-f(X_{n,1})})$$

where we have used the inequality

$$\left|\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i\right| \leq \sum_{i=1}^{n} |y_i - x_i|, \quad 0 \leq y_i, \ x_i \leq 1, \ i = 1,\ldots,n.$$

By (2.1) $nE(1-e^{-f(X_{n,1})}) \to \int_E (1-e^{-f(x)})\nu(dx)$ and since $\ell(n) = o(n)$, the final bound in (2.5) converges to zero as $n \to \infty$.

Applying the inequality

$$1 - \sum_{i \leq r_n} y_i \leq \prod_{i \leq r_n} (1-y_i) \leq 1 - \sum_{i \leq r_n} y_i + \sum_{1 \leq i < j \leq r_n} y_i y_j, \quad 0 \leq y_i \leq 1$$

with $y_i = 1 - \exp\{-f(X_{n,i})\}$ and taking expectations we obtain

$$\text{(2.6)} \qquad 1 - r_n E(1-e^{-f(X_{n,1})}) \leq E\exp\{-\sum_{i=1}^{r_n} f(X_{n,i})\} \leq 1 - r_n E(1-e^{-f(X_{n,1})})$$
$$+ r_n \sum_{j=2}^{r_n} E(1-e^{-f(X_{n,1})})(1-e^{-f(X_{n,j})}).$$

From (2.1) and (2.2), we have

$$kr_n E(1-e^{-f(X_{n,1})}) \to T \int_E (1-e^{-f(x)})\nu(dx)$$

and

$$\limsup_{n\to\infty} r_n \sum_{j=2}^{r_n} E(1-e^{-f(X_{n,1})})(1-e^{-f(X_{n,j})}) = o(k^{-1}) \quad \text{as } k \to \infty.$$

Therefore, after raising the ouside two terms of (2.6) to the k^{th} power and taking limits as $n \to \infty$ and then as $k \to \infty$, the outside two terms converge to

$$\exp\{-T \int_E (1-e^{-f(x)})\nu(dx)\}.$$

This combined with the result in the preceding paragraph proves (2.4).

Now let $f \in C_K^+([0, \infty) \times E)$ and suppose the suppport of f is contained in $[0, T] \times K$, K a compact subset of E with $\nu(\partial K) = 0$. By the uniform continuity of f, given $\epsilon > 0$, there exists a partition

$$0 = a_1 < b_1 < a_2 < b_2 < \ldots < a_m < b_m = T$$

such that

$$\sum_{j=1}^{m-1} (a_{j+1} - b_j) < \epsilon/\nu(K)$$

and

$$\sup_{\substack{t \in (a_j, b_j] \\ x \in E}} |f(b_j, x) - f(t, x)| < \epsilon\, T^{-1}/\nu(K), \quad j = 1,\ldots,m$$

Then writing $\Sigma_{(j)}$ for $\sum_{i/n \in (a_j, b_j]}$, we have

$$
\begin{aligned}
(2.7) \qquad & |E \exp\{-\sum_i f(i/n, X_{n,i})\} - E \exp\{-\sum_{j=1}^{m} \Sigma_{(j)} f(i/n, X_{n,i})\}| \\
& \leq E(1 - \exp\{-\sum_{j=1}^{m-1} \sum_{i/n \in (b_j, a_{j+1})} f(i/n, X_{n,i})\} \\
& \leq P[\bigcup_{j=1}^{m-1} \bigcup_{i/n \in (b_j, a_{j+1}]} \{X_{n,i} \epsilon K\}] \\
& \leq \sum_{j=1}^{m-1} (na_{j+1} - [nb_j]) P[X_{n,1} \in K] \\
& \to (\epsilon/\nu(K))\nu(K) = \epsilon.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(2.8) \qquad & E|\exp\{-\sum_{j=1}^{m} \Sigma_{(j)} f(i/n, X_{n,i})\} - \exp\{-\sum_{j=1}^{m} \Sigma_{(j)} f(b_j, X_{n,i})\}| \\
& \leq \sum_{j=1}^{m} \Sigma_{(j)} E(|f(i/n, X_{n,i}) - f(b_j, X_{n,i})| I_{[X_{n,i} \epsilon K]}) \\
& \leq n \,\epsilon\, P[X_{n,1} \in K]/\nu(K) \\
& \to \epsilon.
\end{aligned}
$$

Now, since $[na_j] - [nb_{j-1}] > \ell(n)$ for large n,

$$
(2.9) \qquad |E \exp\{-\sum_{j=1}^{m} \Sigma_{(j)} f(b_j, X_{n,i})\} - \prod_{j=1}^{m} E \exp\{-\Sigma_{(j)} f(b_j, X_{n,i})\}| < (m-1)\alpha_{n,\ell(n)}
$$

and by stationarity and (2.4),

$$
(2.10) \quad E \exp\{-\Sigma_{(j)} f(b_j, X_{n,i})\} \to \exp\{-(b_j - a_j) \int_E (1 - e^{-f(b_j, x)}) \nu(dx)\}, \quad j = 1, \ldots, m.
$$

Also,

$$
\begin{aligned}
&\Big|\exp\{-\sum_{j=1}^{m}(b_j - a_j)\int_E(1-e^{-f(b_j,x)})\nu(dx)\} - \exp\{-\iint_{[0,\infty)\times E}(1-e^{-f(t,x)})dt\nu(dx)\}\Big| \\
&\le \sum_{j=1}^{m}\Big|\int_{(a_j,b_j]\times E}((1-e^{-f(b_j,x)}) - (1-e^{-f(t,x)}))dt\nu(dx)\Big| \\
&+ \Big|\sum_{j=1}^{m}\int_{(a_j,b_j]\times E}(1-e^{-f(t,x)})dt\nu(dx) - \int_{[0,\infty)\times E}(1-e^{-f(t,x)})dt\nu(dx)\Big| \\
&\le (\epsilon T^{-1}/\nu(K))\sum_{j=1}^{m}(b_j - a_j)\nu(K) + \nu(K)\sum_{j=1}^{m-1}(a_{j+1} - b_j) \\
&= \epsilon + \epsilon = 2\epsilon.
\end{aligned}
$$

This plus (2.7) – (2.10) give

$$
\limsup_{n\to\infty}\Big|E\exp\{-\sum_i f(i/n, X_{n,i})\} - \exp\{-\iint_{[0,\infty)\times E}(1-e^{-f(t,x)})dt\nu(dx)\}\Big| < 4\epsilon
$$

and since $\epsilon > 0$ is arbitrary, (2.3) follows as desired. $\square$

In many applications, the variables $\{X_{n,i}, i \ge 1\}$ are m–dependent and in such cases conditions D^*, (2.1) and (2.2) are fairly easy to verify. See Proposition 3.1.

We close this section with a modification of Proposition 1.1 in Resnick (1986) which will be used in Section 3.

Proposition 2.2: (a) Let E and E′ be two LCCB spaces with E compact and suppose $T: E \to E'$ is continuous on an open subset S of E. Then if $m \in M_p(E)$ is a point measure with support contained in S, the mapping $\hat{T}: M_p(E) \to M_p(E')$ defined by

$$
\hat{T}(\sum_i \epsilon_{x_i}) = \sum_i \epsilon_{Tx_i}
$$

is continuous at m.

(b) Suppose E_1, E_2, E_2' are LCCB spaces with E_2 compact and $T\colon E_2 \to E_2'$ is continuous on an open subset S_2 of E_2. If $m \in M_p(E_1 \times E_2)$ has the property

$$m(E_1 \times S_2^c) = 0$$

then
$$\hat{T}\colon M_p(E_1 \times E_2) \to M_p(E_1 \times E_2')$$

defined by

$$\hat{T}(\sum_i \epsilon_{(t_i, x_i)}) = \sum_i \epsilon_{(t_i,\, Tx_i)}$$

is continuous at m.

Proof: (a) Suppose $m_n \overset{v}{\to} m$. Then for large enough n and some k; $1 \leq k < \infty$,

$$m_n(S) = m(S) = k.$$

By Lemma 1.14 in Neveu (1976), there is an enumeration of the points of m_n and m such that

$$m_n = \sum_{j=1}^{k} \epsilon_{x_j^{(n)}}, \quad m = \sum_{j=1}^{k} \epsilon_{x_j}$$

and $x_j^{(n)} \to x_j$ as $n \to \infty$ for $j = 1,\ldots,k$. Thus, by the continuity of T on S

$$\hat{T}m_n = \sum_{j=1}^{k} \epsilon_{Tx_j^{(n)}} \to \sum_{j=1}^{k} \epsilon_{Tx_j} = \hat{T}m.$$

(b) Suppose $m_n = \Sigma\epsilon_{(t_i^{(n)},\, x_i^{(n)})} \in M_p(E_1 \times E_2)$ converges vaguely to $m = \Sigma\epsilon_{(t_i,\, x_i)}$. For any $f \in C_K^+(E_1 \times E_2')$ we need to show

$$\sum_i f(t_i^{(n)}, Tx_i^{(n)}) \to \sum_i f(t_i, Tx_i)$$

as $n \to \infty$. Suppose K_1 is compact in E_1 and K_2' is compact in E_2' and let $K_1 \times K_2'$ contain the support of f. Take $G_1 \subset E_1$ open, relatively compact such that $K_1 \subset G_1$ and $m(\partial(G_1 \times E_2)) = 0$. Then

$$\sum_i \epsilon_{(t_i,\, Tx_i)} 1_{[t_i \in G_1]} \overset{v}{\to} \Sigma\epsilon_{(t_i,\, Tx_i)} 1_{[t_i \in G_1]}$$

and from the convergence result in (a)

$$\sum_i f(t_i^{(n)}, Tx_i^{(n)}) = \sum_i f(t_i^{(n)}, Tx_i^{(n)}) 1_{[t_i^{(n)} \in G_1]} \to \sum_i f(t_i, Tx_i) 1_{[t_i \in G_1]} = \sum_i f(t_i, Tx_i). \quad \square$$

3. MOVING AVERAGES

In this section we consider the weak convergence of a sequence of point processes based on the moving average process

$$X_t = \sum_{i=-\infty}^{\infty} c_i Z_{t-i}, \quad t = 0, \pm 1, \ldots$$

where for some $\delta > 0$,

(3.1) $$\sum_{i=-\infty}^{\infty} |c_i|^{1-\delta} < \infty .$$

Without loss of generality, we shall assume

(3.2) $$\max\{|c_i|;\ i = 0, \pm 1, \ldots\} = 1$$

since otherwise we may consider the rescaled process $\{X_t/\max\{|c_i|\}$. Note (3.2) implies that one or more of the c_i's has absolute value equal to 1. Set

(3.3) $$I^+ = \{i:\ c_i = 1\},\ I^- = \{i:\ c_i = -1\}$$

and let k^+ and k^- denote the cardinality of the sets I^+ and I^- respectively.

Proposition 3.1: Let $\{Z_t\}$ be an iid sequence of random variables with common distribution F satisfying (1.1), with $a_n \to \gamma^{-1} \in (0, \infty]$ and the tail balancing conditions (1.10). Further assume $\{c_i\}$ is a sequence of constants satisfying (3.1) and (3.2). For a fixed integer m define the random vectors $X_{n,k}$ for $k \geq 1$ by

(3.4) $$X_{n,k} = a_n^{-1}(Z_k - b_n, -Z_k - b_n, Z_{k-i},\ 0 < |i| \leq 2m).$$

Then

(3.5)
$$N_n := \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, X_{n,k})}$$

$$\Rightarrow \qquad N := N_1 + N_2$$
$$= \sum_{k=1}^{\infty} \epsilon_{(t_{k1},\, j_{k1},\, -\infty,\, \gamma Y_{k1}(i),\ 0 < |i| \le 2m)}$$
$$+ \sum_{k=1}^{\infty} \epsilon_{(t_{k2},\, -\infty,\, j_{k2},\, \gamma Y_{k2}(i),\ 0 < |i| \le 2m)}$$

in $M_p([0,\infty) \times E)$, $E = ([-\infty,\infty]^2 \backslash \{(-\infty,-\infty)\}) \times [-\infty,\infty]^{4m}$, where N is the superposition of 2 independent PRM's N_1 and N_2 with mean measures

$$dte^{-x}dx\,\epsilon_{-\infty}(dy) \prod_{0<|i|\le 2m} F(\gamma^{-1}dz_i)$$

and

$$dt\,\epsilon_{-\infty}(dx)p^{-1}(1-p)e^{-y}dy \prod_{0<|i|\le 2m} F(\gamma^{-1}dz_i).$$

In the representation for N_1, $\sum_{k=1}^{\infty} \epsilon_{(t_{k1},\, j_{k1})}$ is $\mathrm{PRM}(dte^{-x}dx)$ independent of the iid sequence $\{Y_{k1}(i),\ k \ge 1,\ i \ne 0\}$ which has common distribution F. The points of N_2 have a similar representation.

Proof: First observe that N is a $\mathrm{PRM}(dt \times d\nu)$ where

$$\nu(dx, dy, dz_i, 0 < |i| \le 2m)$$
$$= [(e^{-x}dx)\,\epsilon_{-\infty}(dy) + \epsilon_{-\infty}(dx)(1-p)/p(e^{-y}dy)] \prod_{0<|i|\le 2m} F(\gamma^{-1}dz_i).$$

In view of Theorem 2.1, it suffices to check D^*, (2.1) and (2.2). For $x > -\infty$ or $y > -\infty$, we have by (1.1) and (1.10)

$$\begin{aligned}
&nP[a_n^{-1}(Z_1 - b_n, -Z_1 - b_n) \in [x, \infty] \times [y, \infty]] \\
&= P[Z_1 \geq a_n x + b_n, Z_1 \leq -(a_n y + b_n)] \to e^{-x}\epsilon_{-\infty}(\{y\}) + \frac{1-p}{p}\epsilon_{-\infty}(\{x\})e^{-y}
\end{aligned}$$

(cf. Goldie and Resnick, 1988b). Since $a_n \to \gamma^{-1}$ and Z_{k-i} is independent of Z_k for $i \neq 0$ it is now straightforward to show

$$nP[X_{n1} \in \cdot] \overset{v}{\to} \nu \tag{3.6}$$

on E which verifies (2.1). Moreover since the sequence $\{X_{n,k}, k = 1,2,\ldots\}$ is $4m$ dependent, the mixing condition D^* is automatically satisfied.

Finally let $g \in C_K^+(E)$, $g \leq 1$ with support contained in the set $A \times [-\infty, \infty]^{4m}$, where A is a compact subset of $[-\infty, \infty]^2 \backslash \{(\infty, -\infty)\}$. Then

$$\begin{aligned}
&\limsup_{n\to\infty} n \sum_{i=2}^{[n/k]} Eg(X_{n,1})g(X_{n,i}) \\
&\leq \limsup_{n\to\infty} n \sum_{i=2}^{[n/k]} P[a_n^{-1}(Z_1 - b_n, -Z_1 - b_n, Z_i - b_n, -Z_i - b_n) \in (A \times A) \\
&\leq \limsup_{n\to\infty} n^2 k^{-1}(P[a_n^{-1}(Z_1 - b_n, -Z_1 - b_n) \in A])^2 \\
&\leq k^{-1}(\nu(A \times [-\infty, \infty]^{4m}))^2
\end{aligned}$$

by (3.6) which, upon letting $k \to \infty$, implies (2.3). Applying Theorem 2.1, we have (3.5). □

PROPOSITION 3.2: In addition to the hypotheses of Proposition 3.1, assume $F \in S_r(\gamma)$. In $M_p([0,\infty) \times (-\infty,\infty] \times [-\infty,\infty])$, we have

$$\eta_n := \sum_{i \in I^+} \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(Z_{k-i} - b_n),\, a_n^{-1} \sum_{\ell \neq i} c_\ell Z_{k-\ell})}$$

$$+ \sum_{i \in I^-} \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(-Z_{k-i} - b_n),\, a_n^{-1} \sum_{\ell \neq i} c_\ell Z_{k-\ell})}$$

$$=> \qquad \eta := \sum_{i \in I^+} \sum_{k=1}^{\infty} \epsilon_{(t_{k1}, j_{k1}, \gamma \sum_{\ell \neq i} c_\ell Y_{k1}(\ell - i))}$$

$$+ \sum_{i \in I^-} \sum_{k=1}^{\infty} \epsilon_{(t_{k2}, j_{k2}, \gamma \sum_{\ell \neq i} c_\ell Y_{k2}(\ell - i))}$$

where the points of the limit have the description given in the previous proposition.

PROOF: Choose $m > 0$ so large that $|c_i| < 1$ for $|i| > m$ which implies $I^+ \cup I^- \subset [-m, m]$. Define

$$\eta_n^{(m)} = \sum_{i \in I^+} \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(Z_{k-i} - b_n),\, a_n^{-1} \sum_{\ell \neq i}^{m} c_\ell Z_{k-\ell})}$$

$$+ \sum_{i \in I^-} \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(-Z_{k-i} - b_n),\, a_n^{-1} \sum_{\ell \neq i}^{m} c_\ell Z_{k-\ell})}$$

where $\sum_{\ell \neq i}^{m}$ denotes the sum over ℓ, $0 < |\ell| \le m$, $\ell \neq i$. Let $\eta^{(m)}$ have the analogous relationship to η, (i.e., the sum $\sum_{\ell \neq i} c_\ell Y_{k,r}(\ell - i)$ is replaced by $\sum_{\ell \neq i}^{m} c_\ell Y_{k,r}(\ell - i)$, $r = 1, 2$. We first show that for each $i \neq 0$,

$$(3.7) \qquad \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(\pm Z_{k-i} - b_n),\, a_n^{-1} \sum_{\ell \neq i}^{m} c_\ell Z_{k-\ell})} - \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(\pm Z_k - b_n),\, a_n^{-1} \sum_{\ell \neq i}^{m} c_\ell Z_{k+i-\ell})} \xrightarrow{P} 0$$

in $M_p([0,\infty) \times (-\infty,\infty] \times [-\infty,\infty])$. Let $C = (\alpha_0, \beta_0] \times (\alpha_1 \beta_1] \times (\alpha_2, \beta_2]$ be a relatively compact subset of $[0,\infty) \times (-\infty,\infty] \times [-\infty,\infty]$. For $i > 0$, the difference in (3.7) evaluated at the set C is

$$(3.8) \qquad \Big(\sum_{\alpha_0 n - i < k \le \alpha_0 n} - \sum_{\beta_0 n - i < k \le \beta_0 n}\Big) \epsilon_{(kn^{-1},\, a_n^{-1}(\pm Z_k - b_n),\, a_n^{-1} \sum_{\ell \neq i}^{m} c_\ell Z_{k+i-\ell})}(C)$$

and the expectation of the absolute value of this expression is bounded by

$$(2i+1) P[a_n^{-1}(\pm Z_1 - b_n) > \alpha_1] \to 0$$

as $n \to \infty$ by (1.1) and (1.10). Thus the difference in (3.7) evaluated at the set C, converges in probability to zero. Since relatively compact sets of the form $(\alpha_0, \beta_0] \times (\alpha_1, \beta_1] \times (\alpha_2, \beta_2]$ constitute a DC–semiring (cf. Kallenberg (1983)), (3.7) now follows. The case $i < 0$ is dealt with in a similar fashion.

Consider the map $T_{i1}: [0,\infty] \times E \to [0,\infty) \times ([-\infty,\infty]^2 \backslash \{(-\infty,-\infty)\}) \times [-\infty,\infty]$ defined by

$$T_{i1}(t, x, y, z_\ell, 0 < |\ell| \le 2m) = \begin{cases} (t, x, y, \sum_{\ell \neq i}^{m} c_\ell z_{\ell-i}) & \text{if } z_{\ell-i} \in \mathbb{R},\ 0 < |\ell| \le m,\ \ell \neq i \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.2b ($E_2 = [-\infty, \infty]^{4m}$ is compact) this function induces a map

$$\hat{T}_{i1}: M_p([0, \infty) \times E) \to M_p([0, \infty) \times ([-\infty, \infty]^2 \backslash \{(-\infty, \infty)\}) \times [-\infty, \infty])$$

which is a.s. continuous relative to the limit point process N in Proposition 3.1. Furthermore the maps T_{i2}: $[0, \infty) \times ([-\infty, \infty]^2 \backslash \{(-\infty, -\infty)\}) \times [-\infty, \infty]$ defined by

$$\begin{aligned} T_{i2}\,(t, x, y, z) &= (t, x, z) \ \text{ if } i \in I^+ \\ &= (t, y, z) \ \text{ if } i \in I^- \end{aligned}$$

induce continuous maps
$\hat{T}_{i2}$: $M_p([0, \infty) \times ([-\infty, \infty]^2 \backslash \{(-\infty, -\infty)\}) \times [-\infty, \infty]) \to M_p([0, \infty) \times (-\infty, \infty] \times [-\infty, \infty])$ by Proposition 1.1 of Resnick (1986) or Proposition 3.18 of Resnick (1987). Thus from (3.7), the continuous mapping theorem and the fact that addition is vaguely continuous (Kallenberg (1983)) we get

$$\begin{aligned} \eta_n^{(m)} &= \sum_{i \in I^+ \cup I^-} \hat{T}_{i2} \circ \hat{T}_{i1}(N_n) + o_p(1) \\ &\Rightarrow \sum_{i \in I^+ \cup I^-} \hat{T}_{i2} \circ \hat{T}_{i1}(N) = \eta^{(m)}. \end{aligned}$$

as asserted.

To complete the proof, it suffices to show by Theorem 4.2 in Billingsley (1968) that

$$\eta^{(m)} \to \eta \ \text{ a.s. as } \ m \to \infty \tag{3.10}$$

and for every $\delta > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[\rho(\eta_n^{(m)}, \eta_n) > \delta] = 0 \tag{3.11}$$

where ρ is the vague metric. Clearly $\eta^{(m)} \to \eta$ as since $\gamma \sum_{\ell \neq i}^m c_\ell Y_{k,r}(\ell - i) \to \gamma \sum_{\ell \neq i} c_\ell Y_{k,r}(\ell - i)$ a.s., $r = 1, 2$.

As for (3.11), by definition of the vague metric it is enough to show that for every $\delta > 0$ and $h \in C_K^+([0, \infty) \times (-\infty, \infty] \times [-\infty, \infty])$ with $h \leq 1$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[|\int h d\eta_n^{(m)} - \int h d\eta_n| > \delta] = 0. \tag{3.12}$$

Suppose the support of h is contained in $[0, \alpha] \times [\beta, \infty] \times [-\infty, \infty]$. If $d(\cdot, \cdot)$ is the metric on $[-\infty, \infty]$, then by the uniform continuity of h, given $\epsilon > 0$, there exists $\theta > 0$ such that

$$\sup\{|h(t, x, y) - h(t, x, z)| : t \in [0, \infty),\ x \in (-\infty, \infty]\} < \epsilon$$

whenever $d(y, z) < \theta$. The probability in (3.12) is bounded by

$$\delta^{-1} E|\int h d\eta_n^{(m)} - \int h d\eta_n|$$

and this expectation is in turn bounded by

$$\epsilon \sum_{i \in I^+ \cup I^-} \sum_{k=1}^{[n\alpha]} P(A_{k-i,n}) + 2 \sum_{i \in I^+ \cup I^-} \sum_{k=1}^{[n\alpha]} P(A_{k-i,n} \cap B_{k,i}) \tag{3.13}$$

where

$$A_{k-i,n} = \begin{cases} [a_n^{-1}(Z_{k-i} - b_n) \geq \beta] & \text{if } i \in I^+ \\ \\ [a_n^{-1}(-Z_{k-i} - b_n) \geq \beta] & \text{if } i \in I^- \end{cases}$$

and

$$B_{k,i} = [d(a_n^{-1} \sum_{\ell \neq i} c_\ell Z_{k-\ell}, a_n^{-1} \sum_{\ell \neq i}^{m} c_\ell Z_{k-\ell}) \geq \theta].$$

Since $A_{k-i,n}$ and $B_{k,i}$ are independent, $a_n \to \gamma^{-1}$ and

$$nP(A_{k-i,n}) \to \begin{cases} e^{-\beta} & \text{if } i \in I^+ \\ \\ \frac{1-p}{p} e^{-\beta} & \text{if } i \in I^- \end{cases}$$

the $\limsup_{n\to\infty}$ of (3.13) is bounded by

$$e^{-\beta}(k^+ + k^-(1-p)/p)(\epsilon + \sum_{i \in I^+ \cup I^-} P[d(\gamma \sum_{\ell \neq i} c_\ell Z_{-\ell}, \gamma \sum_{\ell \neq i}^{m} c_\ell Z_{-\ell}) \geq \theta])$$

which upon letting $m \to \infty$ is equal to

$$\epsilon e^{-\beta}(k^+ + k^-(1-p)/p).$$

Since $\epsilon > 0$ is arbitrary, we conclude that the limit must be zero which proves (3.12). □

THEOREM 3.3: Let $\{X_t\}$ be the moving average process $X_t = \sum_{i=-\infty}^{\infty} c_i Z_{t-i}$, where $\{Z_t\}$ is an iid sequence of r.v.'s with common distribution $F \in D(\Lambda) \cap S_r(\gamma)$, $\gamma \geq 0$, and which satisfies the balancing condition (1.10) Assume the coefficients $\{c_i\}$ satisfy (3.1) and (3.2). Then in $M_p([0, \infty) \times (-\infty, \infty])$,

$$\xi_n := \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(X_k - b_n))}$$

$$\Rightarrow \quad \xi := \xi_1 + \xi_2 = \sum_k \sum_{i\in I^+} \epsilon_{(t_{k1}, j_{k1} + \gamma \sum_{\ell \neq i} c_\ell Y_{k,1}(\ell - i))}$$

$$+ \sum_k \sum_{i \in I^-} \epsilon_{(t_{k2}, j_{k2} + \gamma \sum_{\ell \neq i} c_\ell Y_{k,2}(\ell - i))}$$

where the points in the limit are transformations of the points described in Proposition 3.1.

PROOF: Let $M > 0$ be a large fixed constant and define the function T from $E_M := [0, \infty) \times [-M, \infty] \times [-\infty, \infty]$ into $[0, \infty) \times (-\infty, \infty]$ by

$$T(t, x, y) = \begin{cases} (t, (x + y)) & y \in \mathbb{R} \\ \\ (t, 0) & y \notin \mathbb{R} \end{cases}.$$

If we restrict the domain of the point processes η_n and η in Proposition 3.2 to the set E_M then by Proposition 2.2, the mapping $\hat{T}: M_p(E_M) \to M_p([0, \infty) \times (-\infty, \infty])$ defined by

$$\hat{T}(\sum_j \epsilon_{x_j}) = \sum_j \epsilon_{Tx_j}$$

is a.s. continuous at $\eta(\cdot \cap E_M)$. Thus,

$$\hat{T}\eta_n(\cdot \cap E_m) = \sum_{i\in I^+\cup I^-} \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(X_k - b_n))} 1_k^{(i)}$$

$$=> \quad \hat{T}\eta(\cdot \cap E_M)$$

where

$$1_k^{(i)} = \begin{cases} 1_{[a_n^{-1}(Z_{k-i} - b_n) \geq -M\,]} & \text{if } i \in I^+ \\ \\ 1_{[a_n^{-1}(-Z_{k-i} - b_n) \geq -M]} & \text{if } i \in I^-. \end{cases}$$

Now as $M \to \infty$,

$$\begin{aligned} \hat{T}\eta(\cdot \cap E_M) &= \sum_{i\in I^+} \sum_{k=1}^{\infty} \epsilon_{(t_{k1},\, j_{k1} + \gamma \sum_{\ell\neq i} c_\ell Y_{k,1}(\ell-i))} 1_{[j_{k1} \geq -M]} \\ &+ \sum_{i\in I^-} \sum_{k=1}^{\infty} \epsilon_{(t_{k2},\, j_{k2} + \gamma \sum_{\ell\neq i} c_\ell Y_{k,2}(\ell-i))} 1_{[j_{k2} \geq -M]} \\ &\to \xi \text{ a.s.} \end{aligned}$$

So to complete the proof of the theorem, it suffices to show, by Theorem 4.2 in Billingsley (1968) and the definition of the vague metric, that for all $\delta > 0$, and $h \in C_K^+([0, \infty) \times (-\infty, \infty])$, $h \leq 1$,

$$(3.14) \qquad \lim_{M\to\infty} \limsup_{n\to\infty} P[|\sum_{k=1}^{\infty} h(kn^{-1}, a_n^{-1}(X_k - b_n))(1 - \sum_{i\in I^+\cup I^-} 1_k^{(i)})| > \delta] = 0.$$

If h has support contained in $[0, \alpha] \times [\theta, \infty]$, then the probability in (3.14) is bounded by

$$P\Big[\bigcup_{k=1}^{[n\alpha]}\{a_n^{-1}(X_k-b_n)>\theta,\ 1-\sum_{i\in I^+\cup I^-}1_k^{(i)}\neq 0\}\Big]$$
$$\leq \alpha n P[a_n^{-1}(X_0-b_n)>\theta,\ 1-\sum_{i\in I^+\cup I^-}1_0^{(i)}\neq 0].$$

Now,

$$\{1-\sum_{i\in I^+\cup I^-}1_0^{(i)}\neq 0\}$$
$$\subseteq \bigcup_{\substack{i,j\in I^+\cup I^-\\ i\neq j}}\{1_0^{(i)}=1,\ 1_0^{(j)}=1\}\cup\{1_0^{(i)}=0 \text{ for all } i\in I^+\cup I^-\}$$

and

$$nP[1_0^{(i)}=1]\to\begin{cases} e^M & \text{if } i\in I^+\\ \\ \frac{(1-p)}{p}e^M & \text{if } i\in I^-.\end{cases}$$

Thus for $i\neq j$,

(3.15) $$nP[1_0^{(i)}=1,\ 1_0^{(j)}=1]=nP[1_0^{(i)}=1]P[1_0^{(j)}=1]\to 0$$

as $n\to\infty$ so to prove (3.14) we only need to consider

(3.16) $$nP[a_n^{-1}(X_0-b_n)>\theta,\ 1_0^{(i)}=0,\ i\in I^+\cup I^-].$$

Set $Y=\sum_i c_iZ_{-i}-\sum_{i\in I^+\cup I^-}c_iZ_{-i}$. Applying the remark in Section 1, (see (1.11))

$$(3.17) \qquad \frac{P[Y > t]}{\bar{F}(t)} \to 0$$

as $t \to \infty$. Now the probability in (3.16) is bounded above by

$$\sum_{i \in I^+ \cup I^-} nP[a_n^{-1}(X_0 - b_n) > \theta,\ a_n M < c_i Z_{-i} < a_n(-M) + b_n]$$
$$+ nP[a_n^{-1}(X_0 - b_n) > \theta,\ \max_{i \in I^+ \cup I^-} c_i Z_{-i} \le a_n M].$$

By (2.19) in Goldie and Resnick (1988b), the $\lim_{M\to\infty} \limsup_{n\to\infty}$ of each term in the first sum is zero. Moreover, the last term is bounded by

$$nP[a_n^{-1}(Y - b_n) > \theta - (k^+ + k^-)M] \to 0$$

as $n \to \infty$ by (3.17) and hence $\lim_{M\to\infty} \limsup_{n\to\infty}$ of (3.16) is zero. This combined with (3.15) proves (3.14) as claimed. □

In case the coefficients c_i are non–negative for all i, the balancing condition (1.10) is not required. The proof of the following theorem is essentially identical to the proof of Theorem 3.3 with the obvious modifications.

Theorem 3.4: Let $\{X_t\}$ be the moving average process $X_t = \sum_{i=-\infty}^{\infty} c_i Z_{t-i}$, where $\{Z_t\}$ is iid with common distribution $F \in D(\Lambda) \cap S_r(\gamma)$, $\gamma \ge 0$. If the coefficients $\{c_i\}$ are non–negative and satisfy (3.1) and (3.2), then

$$\xi_n = \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\ a_n^{-1}(X_k - b_n))} \Rightarrow \sum_k \sum_{i \in I^+} \epsilon_{(t_{k1},\ j_{k1} + \gamma \sum_{\ell \ne i} c_\ell Y_{k1}(\ell - i))}$$

where the points of the limit have a description analogous to those of Theorem 3.3.

4. REMARKS AND APPLICATIONS

Observe first in Theorem 3.3 that in the special case of subexponentiality where $\gamma = 0$, the limit point process is the sum of two independent compound Poisson point processes $\xi^+ + \xi^-$:

$$\xi = \sum_\ell k^+ \epsilon_{(t_{\ell 1}, j_{\ell 1})} + \sum_\ell k^- \epsilon_{(t_{\ell 2}, j_{\ell 2})}$$

where recall $k^\pm = \text{card } I^\pm$. Each point of $\xi^\pm$ has multiplicity $k^\pm$.

Once a sequence of point processes based on $\{X_k\}$ is shown to converge, there are standard techniques for gleaning the properties of extremes as corollaries and we briefly sumarize some of these applications. (Cf. Leadbetter, Lindgren and Rootzén (1983); Resnick (1986, 1987); Davis and Resnick (1985a,b).) We assume the hypotheses of Theorem 3.3 are in force.

a) Convergence of maxima to extremal processes: Define for $t > 0$

$$\begin{aligned} Y_n(t) &= a_n^{-1}(X_1 - b_n), \quad 0 < t < n^{-1} \\ &= a_n^{-1}\left(\bigvee_{i=1}^{[nt]} X_i - b_n\right), \quad t \geq n^{-1} \end{aligned}$$

and

$$\begin{aligned} Y(t) = &\left(\bigvee_{t_{k1} \leq t} \left(j_{k1} + \gamma \bigvee_{i \in I^+} \sum_{\ell \neq i} c_\ell Y_{k,1}(\ell - i)\right)\right) \\ &\vee \left(\bigvee_{t_{k2} \leq t} \left(j_{k2} + \gamma \bigvee_{i \in I^-} \sum_{\ell \neq i} c_\ell Y_{k,2}(\ell - i)\right)\right) = Y^+(t) \vee Y^-(t) \end{aligned}$$

and we have

$$Y_n \Rightarrow Y$$

in $D(0, \infty)$. (Here we are using the convention that the maximum taken over an empty index set is $-\infty$.) This is obtained by applying the maximum point functional to the convergence in Theorem 3.3 (cf. Resnick (1986, 1987)). Y^+ and Y^- are independent extremal processes and Y being the maxima of the two is thus also an extremal process. Note

$$\sum_k \epsilon_{(t_{k1},\, j_{k1} + \gamma \bigvee_{i \in I^+} \sum_{\ell \neq i} c_\ell Y_{k,1}(\ell - i))}$$

is PRM with mean measure of $[0, t] \times (x, \infty]$ equal to

$$te^{-x} E \exp\{\gamma \bigvee_{i \in I^+} (\sum_{\ell \neq i} c_\ell Y_{k,1}(\ell - i))\} =: te^{-x} m_+(\gamma)$$

and likewise

$$\sum_k \epsilon_{(t_{k2},\, j_{k2} + \gamma \bigvee_{i \in I^-} \sum_{\ell \neq i} c_\ell Y_{k,2}(\ell - i))}$$

is PRM with mean of $[0, t] \times (x, \infty]$ equal to

$$tp^{-1}(1-p)e^{-x} E \exp\{\gamma \bigvee_{i \in I^-} (\sum_{\ell \neq i} c_\ell Y_{k,2}(\ell - i))\} =: tp^{-1}(1-p)e^{-x} m_-(\gamma).$$

Note that $m_{\pm}(\gamma) = 0$ if $k^{\pm} = 0$. Thus $Y^{\pm}$ is extremal $- G_{\pm}$ where

$$G_{+}(x) = P[Y^{+}(1) \le x] = \Lambda(x)^{m_{+}(\gamma)}$$

$$G_{-}(x) = P[Y^{+}(1) \le x] = \Lambda(x)^{p^{-1}(1-p)m_{-}(\gamma)}.$$

Therefore $Y = Y^{+} \vee Y^{-}$ is extremal–G with

$$G(x) = G_{+}(x)G_{-}(x) = \Lambda(x)^{m_{+}(\gamma) + p^{-1}(1-p)m_{-}(\gamma)}.$$

Note that if $k^{+} = 1$ then

$$\begin{aligned} m_{+}(\gamma) &= E \exp\{\gamma \sum_{\ell \neq i} c_{\ell} Y_{k,1}(\ell - i)\} \\ &= \prod_{\ell \neq i} E \exp\{\gamma c_{\ell} Y_{1,1}(\ell)\} \end{aligned}$$

where i is the unique integer with $c_i = 1$. A similar calculation is possible if $k^{-} = 1$.

For other applications which depend only on the fact that Y is an extremal process see Resnick (1975, 1986, 1987), Davis and Resnick (1985a).

(b) Extremal index: From (1.11)

$$nP[X_1 > a_n x + b_n) \to re^{-x}$$

where $r = (k^{+}/E \exp\{\gamma Z_1\} + k^{-}(1 - p)/(pE \exp\{\gamma Z_1\}))E \exp\{\gamma X_1\}$. Now from (a)

$$P[a_n^{-1}(\bigvee_{i=1}^{n} X_i - b_n) \le x] \to \Lambda(x)^{m_+(\gamma) + p^{-1}(1-p)m_-(\gamma)}$$

so that the extremal index (cf. Leadbetter, Lindgren, and Rootzén (1983), page 67ff) is $\theta = (m_+(\gamma) + p^{-1}(1-p)m_-(\gamma))/r$.

(c) Exceedances: For $x \in \mathbb{R}$, we may consider the indices k such that $X_k > a_n x + b_n$ or equivalently the point process $\sum_{k=1}^{\infty} \epsilon_{kn^{-1}} 1_{[X_k > a_n x + b_n]}$. From Theorem 3.3 and the continuous mapping theorem we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(X_k - b_n))}(\cdot \times (x, \infty]) = \sum_{k=1}^{\infty} \epsilon_{kn^{-1}} 1_{[X_k > a_n x + b_n]} \\
&\Rightarrow \xi(\cdot \times (x, \infty]) \\
&= \sum_k \sum_{i \in I^+} \epsilon_{t_{k1}} 1_{[j_{k1} + \gamma \sum_{\ell \ne i} c_\ell Y_{k,1}(\ell - i) > x]} \\
&+ \sum_k \sum_{i \in I^-} \epsilon_{t_{k2}} 1_{[j_{k2}\, \gamma \sum_{\ell \ne i} c_\ell Y_{k,2}(\ell - i) > x]} \\
&= \sum_k \eta_k^+ \epsilon_{t_{k1}} + \sum_k \eta_k^- \epsilon_{t_{k2}}
\end{aligned}$$

where

$$\eta_k^+ = \sum_{i \in I^+} 1_{[j_{k1} + \gamma \sum_{\ell \ne i} c_\ell Y_{k1}(\ell - i) > x]}$$

with an analogous definition for η_k^-. So the limiting exceedence point process is compound Poisson.

If $\gamma = 0$

$$\eta_k^+ = k^+ 1_{[j_{k1} > x]}, \quad \eta_k^- = k^- 1_{[j_{k2} > x]}.$$

Suppose $\sum_k \epsilon_{T_{kr}}$, $r = 1, 2$ are two independent homogeneous Poisson processes on $[0, \infty)$ with rates e^{-x} and $p^{-1}(1-p)e^{-x}$. Then

$$\sum_k \epsilon_{(t_{kr}, j_{kr})}(\cdot \times (x, \infty)) \stackrel{d}{=} \sum_k \epsilon_{T_{kr}}, \quad r = 1, 2$$

and the exceedance point process

$$\sum_k \eta_k^+ \epsilon_{t_{k1}} + \sum_k \eta_k^- \epsilon_{t_{k2}} \stackrel{d}{=} k^+ \sum_k \epsilon_{T_{k1}} + k^- \sum_k \epsilon_{T_{k2}}.$$

(d) Joint distribution of the largest and 2^{nd} largest: Define $M_n = \max\{X_1, \ldots, X_n\}$ and $M_n^{(2)} = 2^{nd}$ largest among $\{X_1, \ldots, X_n\}$. Then

$$\begin{aligned} &P[a_n^{-1}(M_n - b_n) \leq x, a_n^{-1}(M_n^{(2)} - b_n) \leq y] \\ &= P[\xi_n([0, 1] \times (x, \infty] = 0, \xi_n([0, 1] \times (y, \infty]) \leq 1] \\ &\to P[\xi[0, 1] \times (x, \infty]) = 0, \xi([0, 1] \times (y, \infty]) \leq 1] \end{aligned}$$

In the special case where $\gamma = 0$, $k^+ \geq 2$ and $k^- \geq 2$, the set $\{\xi([0, 1] \times (y, x]) = 1\}$ has probability zero and hence the joint limit distribution of $a_n^{-1}(M_n - b_n, M_n^{(2)} - b_n)$ is

$$P[\xi([0, 1] \times [x \wedge y, \infty]) = 0] = \Lambda^{1/p}(x \wedge y).$$

Note if M has distribution $\Lambda^{1/p}$, then

$$\Lambda^{1/p}(x \wedge y) = P[M \le x, M \le y],$$

and hence

$$(a_n^{-1}(M_n - b_n),\ a_n^{-1}(M_n^{(2)} - b_n)) => (M, M).$$

(e) Upper and lower extremes: We first establish a weak convergence result for the sequence of point processes

$$\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\ a_n^{-1}(X_k - b_n),\ a_n^{-1}(X_k + b_n))}$$

from which joint limit behavior of the upper and lower extremes can be ascertained.

THEOREM 4.1: Under the assumptions of Theorem 3.3,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\ a_n^{-1}(X_k - b_n),\ a_n^{-1}(X_k + b_n))} \\
&=> \sum_{i \in I^+} \sum_{k=1}^{\infty} \Big(\epsilon_{(t_{k1},\ j_{k1} + \gamma \sum_{\ell \neq i} c_\ell Y_{k1}(\ell - i), \infty)} + \epsilon_{(t_{k2},\ -\infty,\ -j_{k2} + \gamma \sum_{\ell \neq i} c_\ell Y_{k2}(\ell - i))}\Big) \\
&+ \sum_{i \in I^-} \sum_{k=1}^{\infty} \Big(\epsilon_{(t_{k2}, j_{k2} + \gamma \sum_{\ell \neq i} c_\ell Y_{k2}(\ell - i),\ \infty)} \\
&+ \epsilon_{(t_{k1},\ -\infty, -j_{k1} + \gamma \sum_{\ell \neq i} c_\ell Y_{k1}(\ell - i))}\Big)
\end{aligned}$$

in $M_p([0,\infty) \times ([-\infty,\infty]^2 \backslash \{(-\infty,\infty)\}))$ where the points in the limit are obtained from those described in Proposition 3.1.

Sketch Of Proof: With E as defined in Proposition 3.1, let T be the continuous map from E into

$$F = ([-\infty,\infty]^2 \backslash \{(-\infty,-\infty)\}) \times ([-\infty,\infty]^2 \backslash \{(\infty,\infty)\}) \times [-\infty,\infty]^{4m}$$

given by

$$T(x, y, z_i, 0 < |i| \le 2m) = (x, y, -x, -y, z_i, 0 < |i| \le 2m).$$

Thus, by Proposition 1.1 of Resnick (1986) and Proposition 3.1,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(Z_k - b_n, -Z_k - b_n, -Z_k + b_n, Z_k + b_n, Z_{k-i}, 0 < |i| \le 2m))} \\
&=> \sum_{k=1}^{\infty} \epsilon_{(t_{k1}, j_{k1}, -\infty, -j_{k1}, \infty, \gamma Y_{k,1}(i), 0 < |i| \le 2m)} \\
&+ \sum_{k=1}^{\infty} \epsilon_{(t_{k2}, -\infty, j_{k2}, \infty, -j_{k2}, \gamma Y_{k,2}(i), 0 < |i| \le 2m)}
\end{aligned}$$

in $M_p([0,\infty) \times F)$. This result combined with the analogous proof given for Proposition 3.2 yields

$$\sum_{i\in I^+}\sum_{k=1}^{\infty}\epsilon_{(kn^{-1},\ a_n^{-1}(Z_{k-i}-b_n),\ a_n^{-1}(Z_{k-i}+b_n),\ a_n^{-1}\sum_{\ell\neq i}c_\ell Z_{k-\ell})}$$

$$+\sum_{i\in I^-}\sum_{k=1}^{\infty}\epsilon_{(kn^{-1},\ a_n^{-1}(-Z_{k-i}-b_n),\ a_n^{-1}(-Z_{k-i}+b_n),\ a_n^{-1}\sum_{\ell\neq i}c_\ell Z_{k-\ell})}$$

$$\Rightarrow \sum_{i\in I^+}\sum_{k=1}^{\infty}\epsilon_{(t_{k1},j_{k1},\ \infty,\ \gamma\sum_{\ell\neq i}c_\ell Y_{k,1}(\ell-i))}+\epsilon_{(t_{k2},\ -\infty,\ -j_{k2},\ \gamma\sum_{\ell\neq i}c_\ell Y_{k,2}(\ell-i))}$$

$$+\sum_{i\in I^-}\sum_{k=1}^{\infty}\epsilon_{(t_{k2},\ j_{k2},\ \infty,\ \gamma\sum_{\ell\neq i}c_\ell Y_{k,2}(\ell-i))}+\epsilon_{(t_{k1},\ -\infty,\ -j_{k1},\ \gamma\sum_{\ell\neq i}c_\ell Y_{k,2}(\ell-i))}$$

in $M_p([0,\infty)\times([-\infty,\infty]^2\setminus\{(-\infty,\infty)\})\times[-\infty,\infty])$. The conclusion of the Theorem now follows using the same type of argument as given for Theorem 3.3. $\square$

The limit distribution of $(a_n^{-1}(M_n-b_n),\ a_n^{-1}(W_n+b_n))$ where $M_n=\bigvee_{i=1}^{n}X_i$ and $W_n=\bigwedge_{i=1}^{n}X_i$ can now be easily derived. For simplicity, assume $\gamma=0$. Then

$$\begin{aligned}&P[a_n^{-1}(M_n-b_n)\le x,\ a_n^{-1}(W_n+b_n)\ge y]\\ &=P[\textstyle\sum_k\epsilon_{(kn^{-1},\ a_n^{-1}(X_k-b_n),\ a_n^{-1}(X_k+b_n))}([0,1]\times[A\cup B])=0]\end{aligned}$$

where $A=[x,\infty]\times[-\infty,\infty]$ and $B=[-\infty,\infty]\times[-\infty,y]$. Assuming $k^+>0$ and $k^->0$, the limit of this probability reduces to

$$\begin{aligned}P[\textstyle\sum_k\epsilon_{(t_{k1},\ j_{k1})}([0,1]\times(x\wedge -y,\infty])=0,\ \sum_k\epsilon_{(t_{k2},\ j_{k2})}([0,1]\times(x\wedge -y,\infty])=0]&\\ =\exp\{-e^{-(x\wedge -y)}/p\}&\\ =\min\{\Lambda^{1/p}(x),\ \Lambda^{1/p}(-y)\}.&\end{aligned}$$

Not unexpectedly, this limit distribution belongs to the class of distributions described in Davis (1982). If $k^- = 0$, then

$$P[a_n^{-1}(M_n - b_n) \le x,\ a_n^{-1}(W_n + b_n) \ge y] \to \Lambda(x)\Lambda^{p^{-1}(1-p)}(-y),$$

with a similar result holding if $k^+ = 0$. Therefore, we conclude that the maximum and minimum are asymptotically independent if and only if $k^+ = 0$ or $k^- = 0$. This result differs from the corresponding result for the maximum and minimum of moving averages of random variables with regularly varying tail probabilities. In that setting, M_n and W_n are asymptotically independent if and only if the c_j's are all of the same sign (see Davis and Resnick, 1985a).

5. MAX–MOVING AVERAGES

Suppose $F \in D(\Lambda)$ (no other condition on F is needed) and consider the stationary sequence

$$\text{(5.1)} \qquad \{U_t,\ -\infty < t < \infty\} := \{\bigvee_{j=-\infty}^{\infty} c_j Z_{t-j},\ -\infty < t < \infty\}$$

where $0 \le c_j \le 1$ and $\{Z_j\}$ is iid with common distribution F. We assume without loss of generality that there is at least one i such that $c_i = 1$ and set $I = \{i\colon\ c_i = 1\}$ and $k^* = \text{card } I$. Since

$$P(U_1 \le x] = P[\bigvee_j c_j Z_{1-j} \le x\} = \prod_j F(c_j^{-1}x)$$

we need for all sufficiently large x

$$\prod_j F(c_j^{-1}x) > 0.$$

or what is equivalent

$$\sum_j \bar{F}(c_j^{-1}x) < \infty .$$

From Proposition 1.1

$$\sum_j \bar{F}(c_j^{-1}x) \leq (1 + \epsilon)\left(\frac{f(x)}{x}\right)^{1/\epsilon} \sum_j \left(\frac{c_j}{\epsilon(1 - c_j)}\right)^{1/\epsilon} \bar{F}(x)$$

so that provided $\sum_j c_j^{\delta} < \infty$ for some δ we have $\bigvee_j c_j Z_{-j} < \infty$ a.s. Assume $\sum_j c_j^{\delta} < \infty$.

We now investigate convergence of a sequence of point processes based on $\{U_j\}$. Define in $[-\infty, \infty]^{k^*} \backslash \{-\infty\}$ for $i \in I$

$$e_i = (\delta^*_{ji},\ j \in I)$$

where

$$\delta^*_{ji} = \begin{cases} 1 & \text{if } j = i \\ -\infty & \text{if } j \neq i. \end{cases}$$

We first have

(5.2)
$$\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},(a_n^{-1}(Z_{k-i} - b_n), i \in I))}$$
$$=> \sum_{i \in I} \sum_{k} \epsilon_{(t_k, j_k e_i)}$$

in $M_p([0, \infty) \times ([-\infty, \infty]^{k^*} \setminus \{-\infty\}))$ where $\sum_k \epsilon_{(t_k, j_k)}$ is $PRM(dte^{-x}dx)$. The proof of this statement can be based on the method of proof of Theorem 2.2 of Davis and Resnick (1985a) and is omitted. Note that Theorem 2.1 is not applicable as condition (2.2) fails.

Next observe for large x and fixed $\epsilon > 0$

$$P[\bigvee_{j \notin I} c_j Z_{-j} > x] \le \sum_{j \notin I} \bar{F}(c_j^{-1} x)$$
$$\le (1 + \epsilon)(\frac{f(x)}{x})^{1/\epsilon} \sum_j (\frac{c_j}{\epsilon(1 - c_j)})^{1/\epsilon} \bar{F}(x)$$

by Proposition 1.1 and since $f(x)/x \to 0$ we have

$$P[\bigvee_{j \notin I} c_j Z_{-j} > x]/\bar{F}(x) \to 0$$

whence

(5.3)
$$\lim_{n \to \infty} nP[a_n^{-1}(\bigvee_{j \notin I} c_j Z_{-j} - b_n) > x] = 0$$

and

$$nP[a_n^{-1}(\bigvee_{j \notin I} c_j Z_{-j} - b_n) \in \cdot] \stackrel{v}{\to} \epsilon_{-\infty}$$

on $[-\infty, \infty)$.

Now we may extend (5.2) to

$$(5.4)\qquad \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},(a_n^{-1}(Z_{k-i}-b_n),\, i\in I),\, a_n^{-1}(\bigvee_{j\notin I} c_j Z_{k-j}-b_n))}$$

$$\Rightarrow \sum_{i\in I} \sum_k \epsilon_{(t_{k1},\, j_k e_i,\, -\infty)}$$

in $M_p([0,\infty)\times([-\infty,\infty]^{k^*}\backslash\{-\infty\})\times[-\infty,\infty))$. To check this let

$$Z_{nk} = (a_n^{-1}(Z_{k-i}-b_n),\ i\in I)$$
$$m_{nk} = a_n^{-1}(\bigvee_{j\notin I} c_j Z_{k-j}-b_n)$$

and it suffices to show

$$(5.5)\qquad \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, Z_{nk},\, m_{nk})} - \sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, Z_{nk},\, -\infty)} \xrightarrow{P} 0$$

in $M_p([0,\infty)\times([-\infty,\infty]^{k^*}\backslash\{-\infty\})\times[-\infty,\infty])$ for then (5.5) and (5.2) immediately give (5.4). For (5.5) it suffices to show for any

$$f \in C_K^+([0,\infty)\times([-\infty,\infty]^{k^*}\backslash\{-\infty\})\times[-\infty,\infty]),$$

that for any $\zeta > 0$

$$(5.6)\qquad P[|\sum_k (f(kn^{-1}, Z_{nk}, m_{nk}) - f(kn^{-1}, Z_{nk}, -\infty))| > \zeta] \to 0.$$

Suppose the support of f is contained in $[0, T] \times K \times [-\infty, \infty]$ where K is compact. Since f is uniformly continuous, for any $\gamma > 0$ there exists M^* so small that

$$|f(t, z, m) - f(t, z, -\infty)| \le \gamma$$

for all t and z when $m \le M^*$. The probabiliity on the left side of (5.6) is

$$\begin{aligned}
&\le P[\bigvee_{k=1}^{[nT]} m_{nk} > M^*] \\
&+ P[|\sum_k f((kn^{-1}, Z_{nk}, m_{nk}) - f(kn^{-1}, Z_{nk}, -\infty))| > \zeta, \bigvee_{k=1}^{[nT]} m_{nk} \le M^*] \\
&= A + B.
\end{aligned}$$

Now $A \le nTP[\bigvee_{j \notin I} c_j Z_{-j} > a_n M^* + b_n] \to 0$ by (5.3) and

$$\begin{aligned}
\limsup_{n\to\infty} B &\le \limsup_{n\to\infty} P[\gamma \sum_k \epsilon_{(kn^{-1}, Z_{nk})}([0, T] \times K) > \zeta] \\
&\le P[\sum_{i \in I} \sum_k \epsilon_{(t_k, j_k)}([0, T] \times K) > \zeta\gamma^{-1}]
\end{aligned}$$

from (5.2). Since $\sum_{i\in I} \sum_k \epsilon_{(t_k, j_k)}([0, T] \times K)$ is a finite random variable, this bound can be made as small as we like by proper choice of γ. Hence (5.6) and (5.5) follow.

Now define

$$T: ([-\infty, \infty]^{k^*} \backslash \{-\infty\}) \times [-\infty, \infty] \to (-\infty, \infty]$$

by

$$T((z_i,\ i \le k^*), z) = \bigvee_{i=1}^{k} z_i \vee z$$

and applying Propositon 1.1 of Resnick (1986) to (5.4) we obtain in $M_p(0, \infty) \times (-\infty, \infty])$

(5.7)
$$\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\ a_n^{-1}(U_k - b_n))}$$
$$=> \sum_{i \in I} \sum_k \epsilon_{(t_k,\ j_k)} = k^* \sum_k \epsilon_{(t_k,\ j_k)}$$

where recall $\sum_k \epsilon_{(t_k,\ j_k)}$ is $\mathrm{PRM}(dte^{-x}dx)$.

From (5.7) it follows that

$$a_n^{-1}(\bigvee_{i=1}^{[nt]} U_i - b_n) => Y(t) = k^* \bigvee_{t_k \le t} j_k$$

so that

$$\lim_{n \to \infty} P[a_n^{-1}(\bigvee_{i=1}^{n} U_i - b_n) \le x] = \Lambda(x/k^*).$$

Furthermore the exceedance point process converges in $M_p([0, \infty))$:

$$\sum_{k=1}^{\infty} \epsilon_{(kn^{-1},\, a_n^{-1}(U_k - b_n))}(\cdot \times (x, \infty])$$
$$= \sum_{k=1}^{\infty} \epsilon_{kn^{-1}} 1_{[U_k > a_n x + b_n]}$$
$$=> k^* \sum_k \epsilon_{(t_k,\, j_k)}(\cdot \times (x, \infty])$$
$$= k^* \sum_k \epsilon_{t_k} 1_{[j_k > x]} \stackrel{d}{=} k^* \sum_k \epsilon_{T_k}$$

where $\sum_k \epsilon_{T_k}$ is $\mathrm{PRM}(e^{-x} dt)$.

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