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REAL-TIME OPTIMIZATION FOR AIRCRAFT AVAILABILITY

by

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1. Real Time Execution Systems

Real time decision-making focuses on answering the question, "What do I do next?" The answer to this question in a military resupply and repair system requires determining what repairs to initiate now, at each location, what parts and in what quantities should now be shipped via what modes to each location from each stocking location. These decisions are constrained by repair capacities, transport capacities, and carcasses of reparable parts that are available at each location over time. Furthermore, availability of weapons systems over a planning horizon will depend on current and subsequent resource allocation decisions. It is likely that constraints on minimum expected performance over time may be stipulated so as to affect current and future repair and transport decisions.

The goal of this note is to describe a set of integrated dynamic inventory, repair and transportation allocation models. These models are designed to answer the questions we have raised, that is, to determine what to repair at each location, what transportation means to employ to ship serviceable inventory among locations, and which locations to resupply throughout the planning horizon so as to minimize the total repair and transportation costs over the planning horizon while satisfying availability, repair, transport, and inventory constraints in each period.

The remainder of this note is organized as follows. In the next section, we describe the basic operating environment we will represent in our model, followed by the nomenclature that we will employ throughout this note. We then construct the function that measures operational availability at a location and investigate its mathematical properties. After describing this function, we present the basic model and show how to represent it as a linear program.

We then show how to extend the basic model to include maintenance at operating bases and the possibility of lateral resupply among bases. Again, we show how this environment can be represented as a large scale linear optimization problem.

We conclude by considering a different type of planning model. The time horizon considered in the first resource allocation models is short, perhaps two weeks in length or so. The model described in the final section has a planning horizon of 30 to 60 days. Rather than considering the exact timing of events and actions, as we do in our earlier models, this model is a single period model. Actions occurring throughout the horizon are detemined without consideration of their timing. The goal is to meet expected availability targets at each location at the end of the horizon while incurring the minimum total transportation, repair, and procurement costs over the horizon.

2. Environment

The evironment we will examine is a two echelon system consisting of J bases and a depot. Furthermore, we assume that the system operates as follows.

Flying activity occurs in each period of a finite planning horizon at the bases. Demands correspond to part failures that arise due to flying activity at the bases. These parts are designated as line replaceable units, or LRUs. Hence, demand for replacement LRUs occurs at the the bases as a consequence of LRU failures on the aircraft. Each failure of an LRU also generates a request for resupply. If the unit is to be repaired at the depot, then the source of resupply is the depot's stock. Otherwise, it is the base's repair system. When the depot is notified of the failure of the LRU at a base that requires depot repair (or is condemned), the depot will plan to ship a replacement item at a time that matches system requirements for available aircraft at each base. That is, resupply of a base is not automatically made even when stock is on-hand at the depot because there may be a greater need for that stock at another base. That need may arise in the near future due to increased flying activity at another base, which will result in future LRU expected removals. Thus the depot may not just react to requests for resupply based on past removals of LRUs at a base, but may also act proactively to avoid shortages of LRUs in the future at a base. Thus the notion of a constant target stock level is inappropriate in environments where demands for LRUs occur according to non-stationary (dynamic) demand processes.

3. Development of the Availability Function

The key constraint in our model will represent the expected aircraft availability at each base. This type of constraint is expressed as a function of the base's LRU stock levels in each period. We now define additional nomenclature required to construct the expected availability functions. Let

S_{ijt} = cumulative available stock of LRU i at base j at time t,

$\mathcal{B}_{ijt}(S_{ijt})$ = expected backorders at time t for LRU i at base j given S_{ijt}, and

$p_{ijt}(\cdot)$ = probability distribution for the number of units of LRU i in resupply at base j at time t.

Now, there are two resupply sources for a base: the depot warehouse and the base's repair operation. Let

$p^d_{ijt}(\cdot)$ = probability distribution for the number of removals of LRUs of type i at base j that require depot repair over the interval from period 0 through period t,

$p^b_{ijt}(\cdot)$ = probability distribution for the number of removals of LRUs of type i at base j that are in the base's repair system at time period t,

The probability distribution $p_{ijt}(\cdot)$ is the convolution of the two distributions $p^d_{ijt}(\cdot)$ and $p^b_{ijt}(\cdot)$.

Let N_j be the number of aircraft at base j and q_i be the quantity of LRU i per aircraft. Then, the maximum number of backorders of LRU i at location j is given by $N_j q_i$. If the backorders equal $N_j q_i$ then it must be the case that all aircraft at location j are grounded and no more removals can have occurred. Conditioning on this fact, we approximate the expected number of backorders by

$$\mathcal{B}_{ijt}\left(S_{ijt}\right) \simeq \frac{\sum_{\substack{S_{ijt}<x \\ x\leq S_{ijt}+N_j q_i}} \left(x - S_{ijt}\right) p_{ijt}\left(x\right)}{\sum_{x\leq S_{ijt}+N_j q_i} p_{ijt}\left(x\right)}.$$

Observe that a sufficient condition for $\mathcal{B}_{ijt}\left(S_{ijt}\right) < N_j q_i$ is $p_{ijt}\left(x\right) > 0$ for all $0 < x < S_{ijt} + N_j q_i$. This is reasonable so we will assume henceforth that $\mathcal{B}_{ijt}\left(S_{ijt}\right) < N_j q_i$.

Assuming independence, we compute the approximate expected availability as follows.

The approximate probability that a particular aircraft has a "hole" for LRU i (i.e. a backorder) is given by

$$\frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}.$$

If we assume all LRU failures are independent, then

$$\prod_{i=1}^{I}\left[1 - \frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}\right]^{q_i}$$

is the probability that a random aircraft is operational. Then,

$$N_j \prod_{i=1}^{I}\left[1 - \frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}\right]^{q_i}$$

measures the expected number of operational aircraft and

$$N_j \prod_{i=1}^{I}\left[1 - \frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}\right]^{q_i}\left[1 - \prod_{i=1}^{I}\left[1 - \frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}\right]^{q_i}\right]$$

is the variance of the number of operational aircraft.

Now suppose a_{jt} measures the minimum target probability that a random aircraft is available at base j at time t. Then we have the constraint

$$\prod_{i=1}^{I}\left[1 - \frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}\right]^{q_i} \geq a_{jt}$$

or

$$-\sum_{i=1}^{I} q_i \ln\left[1 - \frac{\mathcal{B}_{ijt}\left(S_{ijt}\right)}{N_j q_i}\right] \leq -\ln a_{jt} > 0.$$

The left hand side is well-defined because we assume $\mathcal{B}_{ijt}\left(S_{ijt}\right) < N_j q_i$.

Let us temporarily suppress the subscripts and focus on the properties of the function

$$\ln\left[1-\frac{\mathcal{B}(S)}{Nq}\right].$$

Note that $\mathcal{B}(S)$ is a convex, strictly decreasing function of S. Assuming $\mathcal{B}(S) < Nq$, $1-\frac{\mathcal{B}(S)}{Nq}$ is a concave function that approaches the value of 1 as $S \to \infty$. Suppose we approximate $\mathcal{B}(S)$ by its continuous approximation, which we assume is differentiable. Call this $\overline{\mathcal{B}}(S)$. Observe that $\frac{d\overline{\mathcal{B}}(S)}{dS} < 0$ and $\frac{d^2\overline{\mathcal{B}}(S)}{dS^2} > 0$.

Let $f(S) = \ln\left[1-\frac{\overline{\mathcal{B}}(S)}{Nq}\right]$. Then, assuming $\overline{\mathcal{B}}(S) < Nq$,

$$f'(S) = \frac{1}{1-\frac{\overline{\mathcal{B}}(S)}{Nq}} \frac{-d\overline{\mathcal{B}}(S)}{dS} > 0$$

and

$$\begin{aligned} f''(S) &= \frac{1}{1-\frac{\overline{\mathcal{B}}(S)}{Nq}} \frac{-d^2\overline{\mathcal{B}}(S)}{dS^2} + (-1)^2 \frac{d\overline{\mathcal{B}}(S)}{dS} \left[\frac{1}{1-\frac{\overline{\mathcal{B}}(S)}{Nq}}\right]^2 \frac{-d\overline{\mathcal{B}}(S)}{dS} \\ &= -\left[\frac{1}{1-\frac{\overline{\mathcal{B}}(S)}{Nq}}\right] \left\{\frac{d^2\overline{\mathcal{B}}(S)}{dS^2} + \left[\frac{1}{1-\frac{\overline{\mathcal{B}}(S)}{Nq}}\right]\left(\frac{d\overline{\mathcal{B}}(S)}{dS}\right)^2\right\}. \end{aligned}$$

Since the term in the brackets is positive, $f''(S) < 0$, and therefore $f(S)$ is a strictly increasing, concave function of S.

Analogously, the function $-\ln\left[1-\frac{\mathcal{B}(S)}{Nq}\right]$ is a discrete, strictly decreasing convex function of S.

3.1 A Piecewise Linear Convex Approximation to the Availability Constraint

We use the observations about the convexity of $-\ln\left[1-\frac{\mathcal{B}(S)}{Nq}\right]$ to write a convex optimization problem. Let $G(S) = -\ln\left[1-\frac{\mathcal{B}(S)}{Nq}\right]$. $G(S)$ is a discrete, strictly decreasing convex function of S. Let $\{s_0, s_1, ..., s_k, ..., s_K\}$ denote a subset of $K+1$ distinct values chosen from $\{0, 1, 2, ..., \overline{s}\}$ with $s_0 = 0$ and $s_K = \overline{s}$, where $\overline{s}$ is some upper bound on the optimal value of cumulative stock allocated to this item-location. Let $G_k = G(s_k)$, for $k = 0, 1, ..., K$. Furthermore, let $\triangle_k = s_k - s_{k-1}$, and $g_k = (G_k - G_{k-1})/\triangle_k$ for $k = 1, 2, ..., K$. By the properties of $G(\cdot)$, we have $g_1 < g_2 < ... <$

$g_K \leq 0$. Let $\widehat{G} = \{G_0, K, (\triangle_k, g_k)_{k=1}^K\}$ summarize the data of the approximation.

Replacing S, the decision variables are a collection of K weights, w_k, $k = 1, 2, ..., K$ satisfying:

$$0 \leq w_k \leq \triangle_k, \; k = 1, 2, ..., K.$$

Then the function $G()$ can be approximated as:

$$G(\sum_{k=1}^{K} w_k) \simeq G_0 + \sum_{k=1}^{K} g_k w_k,$$

whenever $w_k(\triangle_{k-1} - w_{k-1}) = 0$, for all $k = 2, ..., K$. The monotonicity of the g_k's will ensure that an optimal basic feasible solution to the linear program (to be proposed below) will satisfy this condition.

We first construct the model's constraints in the next section.

4. Model Development

4.1 Lead Times and Planning Horizons

H_0 = depot repair planning horizon = maximum number of periods to consider repair of depot carcasses,

H_j = base j repair planning horizon = maximum number of periods to consider repair of base j carcasses. We assume $H_j \leq H_0$.

$E_{i0}(s)$ = depot repair time for LRU i including time to transport to the depot warehouse for a unit inducted into repair in period s. We assume $E_{i0}(s) + s < E_{i0}(t) + t$ whenever $s < t$.

$E_{ij}(s)$ = base j repair time for LRU i for a unit inducted into repair in period s. We assume $E_{ij}(s) + s < E_{ij}(t) + t$ whenever $s < t$.

H_{i0}^y = depot shipping planning horizon for LRU i : $H_{i0}^y = H_0 + E_{i0}(H_0)$,

H_{ij}^z = base j transshipment planning horizon for LRU i = max $\{H_j + E_{ij}, H_{i0}^y + L_{ij}^T\}$. Note this horizon is unnecessarily long, since transshipments are likely to be uneconomical past $H_j + E_{ij}$

but it simplifies the presentation.

$\overline{H}_j$ = base j availability planning horizon, $\overline{H}_j \geq H^z_{ij}$ for all $i \in I$,

$\underline{H}_j$ = first period to consider for base j availability target,

L^r_{ij} = regular transportation lead time from the depot to base j for LRU i,

L^e_{ij} = emergency transportation lead time from the depot to base j, $L^e_{ij} < L^r_{ij}$,

$L^b_{jj'}$ = the lateral transfer time from base j to base j' (we could make this depend on the LRU type)

4.2 Known Units in Queue, in Repair, or in Transit

Let

R_{i0t} = cumulative number of LRU i carcasses available at the depot for repair through time t (only units that are planned for subsequent repair),

R_{ijt} = cumulative number of carcasses of LRU i awaiting repair at base j through time t.

$\underline{S}_{i0t}$ = known cumulative supply of LRU i available at the depot through period t (includes units completing repair through period t), $t < H^y_{i0}$,

$\underline{s}_{ijt}$ = known quantity of LRU i arriving at base j during time period t, from repair or other sources (including, in the case $t = 0$, initial inventory and war readiness stocks),

4.3 Repair and Transport Capacities

C_{0t} = maximum number of carcasses (of any LRU type) that can be inducted into the depot repair process in period t,

C_{jt} = maximum number of carcasses (of any LRU type) that can be inducted into the repair process of base j in period t,

L_{jt} = maximum available regular transport capacity for period t for shipments made to base j,

l_i = per unit transport capacity required for LRU i. (We will assume $l_i = 1$ to simplify calculations.)

J_j^I is the set of bases that are eligible to laterally ship units to base j, J_j^O is the set of bases that base j can ship to,

$\overline{Z}_{ijj't}$ is the maximum amount of LRU i that can be shipped from j to j' in t.

4.4 Costs

e_{ij}^r = the unit cost to ship LRU i to base j from the depot via the regular transportation mode,

e_{ij}^e = the unit cost to ship LRU i to base j from the depot via the emergency transportation mode,

h_{i0} = the repair cost for LRU i at the depot,

h_{ij} = the repair cost for LRU i at base j,

M_j = a very large number for each base j to penalize violations of the availability constraint at that base,

$e_{ijj'}$ = the unit transshipment (lateral resupply cost) for LRU i for moving a unit from base j to base j'.

All cost parameters can be additionally subscripted by time period, t.

4.5 Availability Target Inputs

Let $G_{ijt}(s) = -q_i \ln\left(1 - \frac{\mathcal{B}_{ijt}(s)}{N_j q_i}\right)$, and let $\widehat{G}_{ijt} = \left\{G_{ijt0}, K_{ijt}, (\triangle_{ijtk}, g_{ijtk})_{k=1}^{K_{ijt}}\right\}$ summarize the data of a piecewise linear approximation to $G_{ijt}(\cdot)$.

a_{jt} = the availability target for base j at time t.

4.6 Decision Variables

v_{i0t} = number of LRUs of type i inducted into the repair process at the depot in period t, $t = 0, 1, ..., H_0$,

v_{ijt} = the quantity of LRU i inducted into the repair process at base j in period t, $t = 0, 1, ..., H_j$,

y_{ijt}^r = number of units of LRU i sent to base j from depot stock in period t using regular transport, for $t = 0, 1, ..., H_{i0}^{y}$

y_{ijt}^{e} = number of units of LRU i sent to base j from depot stock in period t using emergency transport, for $t = 0, 1, ..., H_{i0}^{y}$,

$z_{ijj't}$ = the amount of LRU i laterally transferred from base j to base j' in period t, , for $t = 0, 1, ..., H_{ij}^{z}$,

4.7 Supplementary Decision Variables

x_{ijt} = the quantity of LRU i arriving (from repair or other sources including, in the case $t = 0$, initial inventory and war readiness stock) at base j in period t, $t = 0, 1, ..., H_{ij}^{z}$

z_{ijt} = the quantity of LRU i arriving (from repair or other sources) at base j in period t that is reserved for lateral transfer in some period s, $s \geq t$, $t = 0, 1, ..., H_{ij}^{z}$,

S_{ijt} = cumulative stock of LRU i arriving at base j from all sources (including initial inventory), net of reserves for lateral transfer for periods 0 through to period t, $t = \underline{H}_j, ..., \overline{H}_j$. It is important to exclude reserves for lateral transfer since this stock is not available to satisfy demand for LRUs at the originating base.

κ = direct cost incurred from physical decisions,

b_{jt} = the shortfall in achieving the availability target at base j in period t, $t = \underline{H}_j, ..., \overline{H}_j$,

w_{ijtk} = weights on segments $k = 1, 2, ..., K_{ijt}$ of piecewise linear approximation $\widehat{G}_{ijt}$ such that $\sum_{k=1}^{K_{ijt}} w_{ijtk} = S_{ijt}$, the cumulative stock of LRU i at base j through period t, net of reserves.

4.8 Physical Constraints

$$\sum_{s=0}^{t} v_{i0s} \leq R_{i0t}, \ \forall i \in I, t = 0, 1, ..., H_0;$$

$$\sum_{s=0}^{t} v_{ijs} \leq R_{ijt}, \ \forall i \in I, \forall j \in J, t = 0, 1, ..., H_j;$$

$$\sum_i v_{i0t} \leq C_{0t},\ t = 0, 1, ..., H_0;$$
$$\sum_i v_{ijt} \leq C_{jt},\ t = 0, 1, ..., H_j;$$
$$\sum_j \sum_{s=0}^{t} \left(y^e_{ijs} + y^r_{ijs}\right) \leq \underline{S}_{i0t} + \sum_{s=0}^{\min\{t,H_j\}} 1_{\{s+E_{i0}(s)\leq t\}} v_{i0s},\ t = 0, \ldots, H^y_{i0};$$
$$L_{jt} \geq \sum_i y^r_{ijt} l_i,\ \forall j, t,\ t = 0, \ldots, H^y_{i0};$$
$$\begin{aligned} x_{ijt} &= \underline{s}_{ijt} + \sum_{s=0}^{\min\{t,H_j\}} 1_{\{s+E_{ij}(s)=t\}} v_{ijs} + \sum_{s=0}^{\min\left\{t,H^y_{i0}\right\}} 1_{\left\{s+L^e_{ij}=t\right\}} y^e_{ijs} \\ &+ \sum_{s=0}^{\min\left\{t,H^y_{i0}\right\}} 1_{\left\{s+L^r_{ij}=t\right\}} y^r_{ijs} + \sum_{j'\in J^I_j} \sum_{s=0}^{t} 1_{\left\{s+L^b_{j'j}=t\right\}} z_{ij'js}, \\ \forall i &\in I, \forall j \in J,\ t = 0, 1, ..., H^z_{ij}; \end{aligned}$$
$$z_{ijt} \leq x_{ijt},\ \forall i \in I, \forall j \in J,\ t = 0, 1, ..., H^z_{ij};$$
$$\sum_{s=0}^{t} \sum_{j'\in J^O_j} z_{ijj's} \leq \sum_{s=0}^{t} z_{ijs},\ t = 0, 1, ..., H^z_{ij};$$
$$z_{ijj't} \leq \overline{Z}_{ijj't},\ \forall i \in I, \forall j \in J,\ t = 0, 1, ..., H^z_{ij};$$
$$S_{ijt} \leq \sum_{s=0}^{t} (x_{ijs} - z_{ijs}),\ \forall i \in I,\ t = \underline{H}_j, ..., H^z_{ij};$$
$$S_{ijt} = S_{ijt-1}, \forall j \in J,\ t = H^z_{ij} + 1, ..., \overline{H}_j;$$
$$v_{i0t}, v_{ijt}, y^r_{ijt},\ y^e_{ijt}, z_{ijt}, z_{ijj't} \geq 0$$
$$\begin{aligned} \kappa &= \sum_{i=1}^{I} \left\{ \sum_{t=0}^{H_0} h_{i0} v_{i0t} + \sum_{j\in J} \sum_{t=0}^{H_{i0}} \left(e^r_{ij} y^r_{ijt} + e^e_{ij} y^e_{ijt}\right) + \sum_{j\in J} \sum_{t=0}^{H_j} h_{ij} v_{ijt} \right\} \\ &+ \sum_{i=1}^{I} \sum_{j\in J} \sum_{j'\in J^O_j} \sum_{t=0}^{H_{ijz}} e_{ijj't} z_{ijj't} \end{aligned}$$

4.9 Availability Target Constraints

$$\sum_{k=1}^{K_{ijt}} w_{ijtk} = S_{ijt},\ \forall i \in I, \forall j \in J,\ t = \underline{H}_j, ..., \overline{H}_j;$$

$$0 \leq w_{ijtk} \leq \triangle_{ijtk},\ \forall i \in I, \forall j \in J,\ t = \underline{H}_j, ..., \overline{H}_j, k = 1, ..., K_{ijt};$$

$$\sum_{i=1}^{I} G_{ijt0} + \sum_{i=1}^{I} \sum_{k=1}^{K_{ijt}} g_{ijtk} w_{ijtk} \leq -\ln a_{jt} - b_{jt},\ \forall j \in J,\ t = \underline{H}_j, ..., \overline{H}_j; \tag{1}$$

4.10 Objective Function

$$\min \kappa + \sum_{j \in J} \sum_{t=\underline{H}_j}^{\overline{H}_j} M_j b_{jt}$$