SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853

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A SURVEY OF FUNCTIONAL LAWS OF THE ITERATED LOGARITHM FOR SELF-SIMILAR PROCESSES

by

Murad S. Taqqu and Claudia Czado[1]

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Murad S. Taqqu[1] and Claudia Czado[2]

School of Operations Research
and Industrial Engineering
Cornell University
Ithaca, NY 14853

ABSTRACT

A process $X(t)$ is self-similar with index $H > 0$ if the finite-dimensional distributions of $X(at)$ are identical to those of $a^H X(t)$ for all $a > 0$. Consider self-similar processes $X(t)$ that are Gaussian or that can be represented through Wiener-Itô integrals. The paper surveys functional laws of the iterated logarithm for such processes $X(t)$ and for sequences whose normalized sums converge weakly to $X(t)$. The goal is to motivate the results by including outline of proofs and by highlighting relationships between the various assumptions.

The paper starts with a general discussion of functional laws of the iterated logarithm, states some of their formulations and sketches the reproducing kernel Hilbert space set-up.

1. INTRODUCTION

This paper surveys results of Taqqu, Stout, Lai, Kôno, Fox, Mori and Oodaira concerning those functional laws of the iterated logarithm which govern Gaussian and finite variance non-Gaussian sequences whose normalized sums converge weakly to a self-similar process $X(t)$. The paper also includes a discussion of upper and lower functional laws. We have attempted to motivate the results by including outline of proofs and by highlighting relationships between the various assumptions.

A process $X(t)$ is self-similar with index H if for all $a > 0$, the finite-dimensional distributions of $X(at)$ and $a^H X(t)$ are identical. When $X(t)$ has mean 0, satisfies $EX^2(1) = 1$ and has stationary increments, its covariance is

$$\Gamma(s,t) = EX(s)X(t) = \frac{1}{2}\{s^{2H} + t^{2H} - |s-t|^{2H}\}$$

where $0 < H < 1$. When $X(t)$ is Gaussian, it is known as Fractional Brownian motion. It becomes Brownian motion when $H = 1/2$. The process $X(t)$, however, need not be Gaussian. It can be for example an m-integral process (see Section 5).

Section 2 starts with an introduction to functional laws of the iterated logarithm and their various formulations. The reproducing kernel Hilbert space set-up is developed in Section 2.2. Section 2.3 states Kuelbs' Strong Convergence Theorem and Section 2.4 gives Kôno's one-sided laws for self-similar processes.

Section 3 covers the main results concerning functional laws for Gaussian self-similar processes. Two approaches are compared; that of Oodaira-Taqqu and the one of Kôno.

Section 4 deals with upper and lower functional laws for Gaussian self-similar processes and the corresponding result of Lai and Stout.

Section 5 concerns non-Gaussian m-integral processes. It states unpublished upper functional laws due to Fox. It also discusses the results of Mori and Oodaira concerning functional laws for a subclass of m-integral self-similar processes.

2. BASIC CONCEPTS

2.1 Functional Law of the Iterated Logarithm

Let $C[0,1]$ be the set of continuous functions on $[0,1]$ with the sup-norm topology; let

$$\|f\|_C = \sup_{0 \le t \le 1} |f(t)|,$$

let

$$d(f,g) = \|f-g\|_C$$

be the distance in $C[0,1]$ between the functions f and g, and let

$$d(f,K) = \inf_{g \in K} \|f-g\|_C.$$

A functional law of the iterated logarithm for a sequence of random functions $f_n(t,\omega)$, $n \geq 1$ in $C[0,1]$ is often expressed as follows:

$$\begin{cases} \{f_n \geq 1\} \text{ is relatively compact in } C[0,1] \text{ with probability 1.} & (2.1) \\ \text{The set of all possible limit points of } \{f_n, n \geq 1\} \text{ is a.s. equal to a given compact set } K \text{ in } C[0,1]. & (2.2) \end{cases}$$

Remarks

1. A sequence of functions in $C[0,1]$ can be viewed as a set S of points in $C[0,1]$. The sequence is relatively compact if the closure of that set S is compact in $C[0,1]$ with respect to the sup-norm topology. Since $C[0,1]$ is a complete metric space, relative compactness is equivalent to the property that every sequence in S contains a uniformly convergent subsequence. Note that in this case, the cluster set, i.e. the set of all possible limit points, cannot be empty.

2. Symbolically, (2.2) is written as

$$C\{f_n, n \geq 1\} = K \quad \text{a.s.} \tag{2.2}$$

where $C\{f_n, n \geq 1\}$ is the set of limit points (cluster set) of f_n.

3. (2.2) does not imply (2.1). Statement (2.1) may not hold because (2.2) does not ensure that every subsequence has a limit. Consider for example the sequence of functions

$$f_n(t) = \begin{cases} \sin nt & \text{if } n \equiv 0 \pmod 3 \\ -1 & \text{if } n \equiv 1 \pmod 3 \\ +1 & \text{if } n \equiv 2 \pmod 3 \end{cases} \qquad n \geq 1,\ t \in [0,1].$$

Statement (2.2) holds because

$$C\{f_n,\ n \geq 1\} = \{f(t) = -1,\ t \in [0,1]\} \cup \{f(t) = +1,\ t \in [0,1]\}$$

is obviously contained in a compact set in $C[0,1]$. However, statement (2.1) does not hold because the subsequence f_{n_k}, $n_k = 3k$, $k \geq 1$, has no convergent subsequence.

4. A second formulation of the law of the iterated logarithm is as follows:

$$\begin{cases} \text{For every } \ \varepsilon > 0, \ \ P(f_n \in K_\varepsilon \ \text{ eventually}) = 1. & (2.3) \\ C\{f_n,\ n \geq 1\} = K \ \text{ a.s.} & (2.2) \end{cases}$$

Here, K_ε is the set of functions distant from K by less than ε and "eventually" means "from a certain n on". We show, in Remark 7 below, that this second formulation of the functional law is equivalent to the previous one. This formulation expresses the fact that the random functions f_n are contained in an ε-neighborhood of K when n is large and that their limit points fill up the set K.

If (2.3) is put in the form (2.3') below, this second formulation of the functional law of the iterated logarithm becomes:

$$\begin{cases} P(\lim\limits_{n\to\infty} d(f_n,K) = 0) = 1 & (2.3') \\ C\{f_n,\ n \geq 1\} = K \ \text{ a.s.} & (2.2) \end{cases}$$

or symbolically

$$f_n \rightleftarrows K \ \text{ a.s.}$$

5. A third formulation of the functional law of the iterated logarithm, also justified in Remark 7 below, is as follows:

$$\begin{cases} \text{For every given } \ \varepsilon > 0, \ \ P(f_n \in K_\varepsilon \ \text{ eventually}) = 1 & (2.3) \\ P(\|f_n - x\|_C < \varepsilon \ \text{ i.o. for every } \ x \in K) = 1 & (2.4) \end{cases}$$

where i.o. means "for infinitely many n". Relation (2.4) expresses the fact that every $x \in K$ is a limit point of f_n but it does not exclude the existence of other limit points. Relation

(2.4) can also be stated as

$$P(\liminf_{n\to\infty} \|f_n - x\|_C = 0 \quad \text{for every} \quad x \in K) = 1. \tag{2.4'}$$

6. If (2.2) is replaced by the following weaker statement

$$C\{f_n, n \geq 1\} \subset K \quad \text{a.s.} \tag{2.5}$$

with K compact in $C[0,1]$, then (2.1) and (2.5) are said to form an upper functional law of the iterated logarithm for f_n, $n \geq 1$.

7. The following relations

$$(*) \left\{\begin{matrix}(2.1)\\ \\(2.5)\end{matrix}\right. \Leftrightarrow (2.3),$$

$$(**) \left\{\begin{matrix}(2.1)\\ \\(2.2)\end{matrix}\right. \Rightarrow (2.4),$$

$$(***) \left\{\begin{matrix}(2.4)\\ \\(2.5)\end{matrix}\right. \Rightarrow (2.2),$$

when combined as follows,

$$\left\{\begin{matrix}(2.1)\\(2.2)\end{matrix}\right. \Leftrightarrow \begin{matrix}(2.1)\\(2.5)\\(2.2)\end{matrix} \begin{matrix}\swarrow\\ \\ \searrow\end{matrix} \begin{matrix}\left\{\begin{matrix}(2.3)\\(2.2)\end{matrix}\right.\\ \\ \left\{\begin{matrix}(2.3)\\(2.4)\end{matrix}\right.\end{matrix},$$

establish the equivalence between the various formulations of the functional law of the iterated logarithm.

We shall now verify Relations (*), (**) and (***). The sufficiency in (*) holds because "not (2.3)" and (2.1) contradict (2.5). Necessity holds because (2.3) yields (2.5) and it also yields (2.1) since for any ω outside a null set A, every subsequence $f_{n'}$ of f_n has a convergent subsequence. Indeed, fix $\omega \in A^c$. If $d(f_{n'}(\omega),K) < \varepsilon$ for large n', then $d(f_{n'}(\omega),x_{n'}) < \varepsilon$ for $x_{n'} \in K$. Since K is compact, $x_{n'}$ has a convergent subsequence $x_{n''}$ converging to x in K and therefore $d(f_{n''}(\omega),x) \to 0$ as $n'' \to \infty$. This establishes Relation (*). To verify Relation

(**), note that "not (2.4)" and (2.1) contradict (2.2). Relation (***) is obvious.

2.2 Reproducing Kernel Hilbert Space

In order to identify the set K that enters in the statement of the functional law of the iterated logarithm, it is convenient to introduce the notion of reproducing kernel Hilbert space (see also Jain and Marcus [9], p. 118).

Let $\Gamma(s,t)$ be a covariance kernel, continuous in $0 \le s,t \le 1$. Consider the linear space

$$L = \{\textstyle\sum_{j=1}^{n} a_j \Gamma(t_j,\cdot),\ a_i \text{ real},\ t_i \in [0,1],\ i = 1,\ldots,n,\ n \ge 1\}$$

and define an inner product on L by

$$\langle \textstyle\sum_{j=1}^{n} a_j \Gamma(t_j,\cdot),\ \sum_{k=1}^{m} b_k \Gamma(s_k,\cdot)\rangle := \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k \Gamma(t_j,s_k). \quad (2.6)$$

Let $H(\Gamma)$ be the completion of L under the norm given by the inner product. $H(\Gamma)$ is called the reproducing kernel Hilbert space (RKHS) corresponding to Γ with norm $\|\ \|_{H(\Gamma)}$.

This Hilbert space has the so-called "reproducing kernel property":

$$f(t) = \langle f,\Gamma(t,\cdot)\rangle,\ \forall\ f \in H(\Gamma). \quad (2.7)$$

(2.7) follows directly from (2.6) for $f \in L$ and immediately extends to $f \in H(\Gamma)$.

Using (2.7) and the Cauchy-Schwartz inequality, we get for $f \in H(\Gamma)$

$$\begin{aligned} |f(s) - f(t)| &= |\langle f,\Gamma(s,\cdot) - \Gamma(t,\cdot)\rangle| \\ &\le \|f\|_{H(\Gamma)}\ \|\Gamma(s,\cdot) - \Gamma(t,\cdot)\|_{H(\Gamma)}. \end{aligned}$$

Therefore $H(\Gamma)$ consists of continuous functions. (The Hilbert space $H(\Gamma)$ can thus be realized as a subset of the Banach space $C[0,1]$.) Since

$$\|f\|_C \le \Big(\sup_{0\le t\le 1} \|\Gamma(t,\cdot)\|_{H(\Gamma)}\Big)\ \|f\|_{H(\Gamma)},\ \forall\ f \in H(\Gamma), \quad (2.8)$$

convergence in $H(\Gamma)$ implies convergence in $C[0,1]$.

Furthermore, the unit ball

$$\{f: \|f\|_{H(\Gamma)} \leq 1\}$$

of $H(\Gamma)$ is compact in the $C[0,1]$ sup-norm topology. (The compactness of the unit ball can be established by showing that it is closed and relatively compact. Relative compactness is shown by using Relation (2.8) to verify the conditions of the Arzela-Ascoli Theorem. For details see Kuelbs ([13], Lemma 2.1(iv)). For an alternative proof see Oodaira ([21], Lemma 3).)

The compact set K that appears in (2.2) can often be identified with the unit ball $\{f: \|f\|_{H(\Gamma)} \leq 1\}$ of $H(\Gamma)$ for some suitable Γ. In the Gaussian case, the right Γ is often the covariance kernel of the Gaussian law that appears in a weak convergence result involving the f_n, suitably normalized.

For instance, if $Y_n(t)$ is the linear interpolation of $\sum_{i=1}^{k} Z_i$, $0 \leq k \leq n$, that is $Y_n(t) = \sum_{i=1}^{[nt]} Z_i + (nt-[nt])Z_{[nt]+1}$, $0 \leq t \leq 1$, where Z_i are i.i.d. random variables with mean 0 and variance 1, then $(1/\sqrt{n})Y_n(t)$ converges weakly in $C[0,1]$ to Brownian motion $B(t)$, whose covariance kernel is $\Gamma(s,t) = \min(s,t)$. Then (Strassen [25]), the following functional law of the iterated logarithm holds:

$$f_n(t) = \frac{Y_n(t)}{\sqrt{2n \log \log n}} \twoheadrightarrow K \quad \text{a.s.}$$

where

$$K = \text{unit ball of RKHS } H(\Gamma) \text{ with } \Gamma(s,t) \text{ equal to } \min(s,t).$$

In fact (see Jain and Marcus [9], p. 121).

$$K = \{f \in C[0,1],\ f(0) = 0,\ f \text{ is absolutely continuous}, \int_0^1 \left(\frac{df(t)}{dt}\right)^2 dt \leq 1\}.$$

Further examples will appear in Section 3.

Thus, the functional law of the iterated logarithm is a statement which involves two norms, namely the sup-norm $\|\ \|_C$ and the Hilbert space norm $\|\ \|_{H(\Gamma)}$. The paths $f_n(t,\omega)$ are regarded as continuous functions of t and their distances to the set K are measured using the norm $\| \|_C$. The non-random continuous functions $f(t)$ forming the set K are also measured with the Hilbert space norm $\| \|_{H(\Gamma)}$ and are required to belong to the unit ball of that Hilbert space.

2.3 A Strong Convergence Theorem for Banach Space Valued Random Variables

The RKHS $H(\Gamma)$ and its unit ball K can also be defined when $C[0,1]$ is replaced by a real separable Banach space B (see Kuelbs [13] pp. 749-750). In this case $H(\Gamma)$ can still be realized as a subset of B. A basic tool for establishing the functional law of the iterated logarithm is the strong convergence theorem for Banach space valued random variables, due to Kuelbs [13]. We shall apply that theorem in the case $B = C[0,1]$.

Theorem 2.1 (Kuelbs [13], Theorem 3.1, p. 753). Let K denote the unit ball of the RKHS $H(\Gamma)$ realized as a subset of B. Let $\{Y_n, n \geq 1\}$ be a sequence of B-valued random variables such that for some sequence of positive constants $\{\phi_n\}$ we have

$$P\{\omega: \limsup_{n\to\infty} g(\frac{Y_n(\omega)}{\phi_n}) \leq \sup_{x \in K} g(x)\} = 1 \quad \text{for} \quad g \in B^*, \tag{2.9}$$

where B^* is the dual space of B.

Then:

I. We have

$$P\{\omega: C(\frac{Y_n(\omega)}{\phi_n}\}) \subset K\} = 1 \tag{2.10}$$

and thus

$$P\{\omega: \{\frac{Y_n(\omega)}{\phi_n}, n \geq 1\} \text{ is relatively compact in } B\} = 1 \tag{2.11}$$

if and only if

$$P\{\omega: \lim_{n\to\infty} d(\frac{Y_n(\omega)}{\phi_n}, K) = 0\} = 1 \tag{2.12}$$

where $d(x,K) = \inf_{x \in K} \|x-y\|$ and $\| \|$ is the norm in B.

II. Suppose that

$$P\{\omega: \limsup_{n\to\infty} g\left(\frac{Y_n(\omega)}{\phi_n}\right) = \sup_{x\in K} g(x)\} = 1 \quad \text{for } g \text{ in } B^* \tag{2.13}$$

and

$$P\{\omega: \{\frac{Y_n(\omega)}{\phi_n}, n \geq 1\} \text{ is relatively compact in } B\} = 1 \tag{2.14}$$

hold. If the RKHS $H(\Gamma)$ is infinite dimensional, then

$$P\{\omega: C\left(\frac{Y_n(\omega)}{\phi_n} : n \geq 1\right) = K\} = 1. \tag{2.15}$$

Remark. From Part I of this theorem, we conclude that (2.9) and (2.11) are sufficient conditions for an upper functional law of the iterated logarithm. From Part II of the theorem we see that (2.13) and (2.14) are sufficient conditions for a functional law of the iterated logarithm.

2.4 One-sided Laws for Self-Similar Processes

Let $\{X(t), 0 \leq t < \infty\}$ be a self-similar process of order $H > 0$. The proof of the following result makes full use of the self-similarity of X and is straightforward in that it uses only Chebyshev's inequality and the Borel-Cantelli lemma. X need not be Gaussian nor possess stationary increments. Stronger conclusions will be obtained when we shall focus on special X's.

Theorem 2.2 (Kôno [12]). Let $f(x)$ and $h(x)$ be positive, continuous functions defined on the positive half line such that

$$Ef\left(\sup_{0\leq t\leq 1} |X(t)|\right) < \infty \tag{2.16}$$

and

$$\int_1^\infty \frac{dx}{xf(h(x))} < \infty. \tag{2.17}$$

a) If both f and h are non-increasing, then

$$\liminf_{s\to+\infty} \frac{\sup_{0\le t\le s} |X(t)|}{s^H h(s)} \ge 1 \quad \text{a.s.} \tag{2.18}$$

b) If both f and h are non-decreasing and

$$\lim_{x\downarrow 1} \limsup_{n\ge 1} \frac{h(x^n)}{h(x^{n-1})} = a < \infty, \tag{2.19}$$

then

$$\limsup_{s\to+\infty} \frac{|X(s)|}{s^H h(s)} \le a \quad \text{a.s.} \tag{2.20}$$

<u>Proof</u>. The process $Y(s) = \sup_{0\le t\le s} |X(t)|$ is also self-similar with index H. Suppose that f and h are both non-increasing. Then

$$P\{Y(s) \le y\} = P\{Y(1) \le s^{-H}y\} \le P\{f(Y(1)) \ge f(s^{-H}y)\} \le \frac{A}{f(s^{-H}y)}$$

by Chebyshev where $A = Ef(Y(1)) < \infty$. At geometric times c^n, $c > 1$, $n = 1,2,\ldots$

$$Y(c^n) \ge c^{nH}h(c^n)$$

a.s., for all large n, because by Borel-Cantelli,

$$\sum_{n=1}^{\infty} P\{Y(c^n) \le c^{nH}h(c^n)\} \le A \sum_{n=1}^{\infty} \frac{1}{f(h(c^n))} \le A' \int_1^{\infty} \frac{dx}{xf(h(x))} < \infty.$$

Thus, a.s. for all large n and $c^n \le s \le c^{n+1}$,

$$\frac{Y(s)}{s^H h(s)} \ge \frac{Y(c^n)}{c^{(n+1)H}h(c^n)} \ge c^{-H}$$

since h is non-increasing. Letting $s \to \infty$ and $c \downarrow 1$ yields the result (2.18).

The proof of (2.20) is similar. Assume f and h non-decreasing. Then $P\{Y(s) \ge y\} \le \dfrac{A}{f(s^{-H}y)}$, and by Borel-Cantelli, $Y(c^n) \le c^{nH}h(c^n)$ a.s. for all large n. The conclusion (2.20) follows from the assumptions on h. □

Remarks

1. Lower laws such as (2.18) are typically harder to obtain than upper laws such as (2.20). The lower law (2.18) applies only to $Y(s) = \sup_{0<t<s} |X(t)|$ which is a monotonization of $|X(s)|$. The upper law (2.20) applies obviously to both $|X(s)|$ and $Y(s)$.

2. The function f does not enter into the conclusions (2.18) and (2.20). However, it limits the applicability of the theorem through (2.16) and it provides a check on the growth of $h(x)$ through (2.17). Condition (2.17) will be analyzed further in Section 3.1 below.

3. FUNCTIONAL LAWS OF THE ITERATED LOGARITHM FOR GAUSSIAN SELF-SIMILAR PROCESSES

3.1 Statement of the Results and Discussion

Kôno [10] makes the following assumption:

Assumption (K). Let $\{Y(t),\ 0 \le t < \infty\}$ be a centered, path continuous Gaussian process with $Y(0) = 0$ and variance $v^2(t) = E(Y(t)^2)$, which satisfies the following conditions:

i) $\lim_{t\to\infty} v(t) = \infty$

ii) There exists a nondegenerate, path continuous Gaussian process $\{X(t),\ 0 \le t \le 1\}$ such that

$$\frac{Y(nt)}{v(n)} \overset{L}{\to} X(t) \quad \text{as} \quad n \to \infty \quad \text{in} \quad C[0,1]$$

where "L" means weak convergence.

Remarks

1. Lamperti [16] has shown that the resulting process $X(t)$ is necessarily self-similar (=semistable) with index $H > 0$ and that the normalization function $v(t)$ is regularly varying, i.e. $v(t) = t^H L(t)$ with $L(t)$ slowly varying.

2. Two crucial parts of ii) in assumption (K) are that $\frac{Y(nt)}{v(n)}$ converges weakly

a) to a Gaussian process

b) to a continuous process.

<u>Examples of processes which satisfy assumption (K)</u>

1. Suppose that $Y(nt)$ is the linear interpolation of $\sum_{i=1}^{[nt]} Z_i$, where Z_i are i.i.d. random variables with mean zero and finite variance. Then $X(t)$ is Brownian motion.

2. In Example 1, suppose that Z_i is a sequence of real stationary Gaussian random variables with mean 0 and covariances $r(k) = EZ_iZ_{i+k}$, $k \geq 1$, satisfying either

$$\text{(I)} \begin{cases} 1/2 < H < 1 \\ r(k) \sim k^{2H-2}L(k) \quad \text{as} \quad k \to \infty \end{cases}$$

or

$$\text{(II)} \begin{cases} 0 < H < 1/2 \\ r(k) \sim -k^{2H-2}L(k) \quad \text{as} \quad k \to \infty \\ r(0) + 2\sum_{k=1}^{\infty} r(k) = 0 \end{cases}$$

where $L(k)$ is a slowly varying function. Then Assumption (K) is satisfied with $v(n) = (H|2H-1|)^{-1/2}n^H L^{1/2}(n)$ (see Taqqu [27] p. 236.) In fact, the limiting process $X(t)$ is Fractional Brownian motion, a Gaussian process with mean 0 and covariances

$$EX(t)X(s) = \frac{1}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}.$$

<u>The additional normalization function $h(t)$</u>. A functional law of the iterated logarithm for $Y(\cdot)$ is a statement about the behavior of $\frac{Y(nt)}{v(n)h(n)}$ as $n \to \infty$. Kôno [10] assumes that the additional normalization function $h(t)$ satisfies the following conditions:

$$\begin{cases} h(t) \text{ is a positive, nondecreasing and continuous} \\ \text{function defined on the positive half line, such that} \\ \displaystyle\int^{\infty} \frac{e^{-\frac{1}{2}(1+\varepsilon)h^2(t)}}{t}\,dt \quad \begin{matrix} < \infty & \text{if } \varepsilon > 0 \\ = \infty & \text{if } \varepsilon < 0. \end{matrix} \end{cases} \tag{3.1}$$

<u>Remarks</u>

1. The usual normalization function $h(t) = (2 \log\log t)^{1/2}$ satisfies (3.1) because

$$\int \frac{e^{-\frac{1}{2}(1+\varepsilon)h^2(t)}}{t}\,dt = \int \frac{1}{t(\log t)^{1+\varepsilon}}\,dt = \frac{1}{-\varepsilon(\log t)^{\varepsilon}} \quad \text{for } \varepsilon \neq 0.$$

2. Condition (3.1) with $\varepsilon > 0$ is identical to condition (2.17) of Theorem 2.2 if one sets there $f(t) = e^{(1/2)(1+\varepsilon)t^2}$.

3. Let us examine Theorem 2.2 when $\varepsilon > 0$, $f(t) = e^{(1/2)(1+\varepsilon)t^2}$, and $h(t) = (2 \log \log t)^{1/2}$. We have remarked that condition (3.1) holds and is identical to (2.17). However, this $h(t)$ also satisfies condition (2.19) of Theorem 2.2 with $a = 1$. Thus, if $X(t)$ is a self-similar process of index $H > 0$ satisfying

$$Ee^{\frac{1}{2}(1+\varepsilon)(\sup_{0\le t\le 1}|X(t)|)^2} < \infty \tag{3.2}$$

then

$$\limsup_{t\to+\infty} \frac{|X(t)|}{t^H(2\log\log t)^{1/2}} \le 1 \quad \text{a.s.}$$

Condition (3.2) is satisfied when $X(t)$ is a Gaussian process with stationary increments (use for instance Fernique's lemma 3.1 below). The conclusion holds in that case, but a stronger one, involving a full functional law of the iterated logarithm, can be established for such an $X(t)$ (see Theorem 3.3 below).

4. If (3.1) is satisfied, then for every fixed $\delta > 0$ we have

$$\int_2^{\infty} \frac{h(t)^{\delta} e^{-\frac{1}{2}(1+\varepsilon)h^2(t)}}{t}\,dt \quad \begin{aligned} &< \infty \quad \text{if } \varepsilon > 0 \\ &= \infty \quad \text{if } \varepsilon < 0. \end{aligned}$$

That last relation is used by Lai and Stout ([14], p. 732) to characterize an upper-lower class test (see Section 4).

<u>d-dimensional space</u>. Results for the functional law of the iterated logarithm can be formulated in the space of d-dimensional continuous functions

$$C^d[0,1] = \{f^d: [0,1] \to R^d \ \text{ continuous}\}$$

endowed with the norm

$$\|f^d(\cdot)\|_C = \sup_{0\le t\le 1} \|f^d(t)\|_d = \sup_{0\le t\le 1} \left(\sum_{i=1}^d f_i^2(t)\right)^{1/2}$$

if $f^d = (f_1,\ldots,f_d)$. We use $\| \ \|_d$ to denote the usual Euclidean norm in R^d. Let

$$Y^d(t,\omega) = (Y_1(t,\omega),\ldots,Y_d(t,\omega)),$$

where $Y_1,\ldots,Y_d$ are independent copies of the process Y, and let

$$f_n^d(t,w) = \frac{Y^d(nt,\omega)}{v(n)h(n)}, \quad 0 \le t \le 1,$$

where the function $h(\cdot)$ is defined as in (3.1). Let K^d be the unit ball of the Hilbert space $H(\Gamma)^d = H(\Gamma) \oplus \cdots \oplus H(\Gamma)$ (d summands), where $H(\Gamma)$ is the reproducing kernel Hilbert space corresponding to the covariance kernel Γ of the limit process $X(t)$, i.e.

$$\Gamma(s,t) = E(X(s)X(t)).$$

The norm of $f^d = (f_1,\ldots,f_d) \in H(\Gamma)^d$ is

$$\|f^d\|_{H(\Gamma)^d} = \left(\sum_{i=1}^d \|f_i\|^2_{H(\Gamma)}\right)^{1/2}.$$

The following upper functional law of the iterated logarithm holds for $Y^d = (Y_1,\ldots,Y_d)$.

Theorem 3.1 (Kôno [10], p. 14). Under the assumption (K) we have

$$\begin{cases} P(\{f_n^d(t,\omega),\ n \ge 1\} \text{ is relatively compact in } C^d[0,1]) = 1 & (3.3) \\ P(C\{f_n^d(t,\omega)\} \subset K^d) = 1. & (3.4) \end{cases}$$

The full functional law of the iterated logarithm, i.e. (3.3) and

$$P(C\{f_n^d(t,\omega)\} = K^d) = 1 \qquad (3.4')$$

requires an additional condition.

Theorem 3.2 (Kôno [10], p. 14). Suppose that assumption (K) and also

$$\lim_{\substack{m\to\infty \\ n/m\to\infty}} \frac{E(Y(ms)Y(nt))}{v(m)v(n)} = 0 \quad \text{for every} \quad 0 \le s,t \le 1 \tag{3.5}$$

hold, then we have (3.3) and (3.4'), that is, $f_n^d \Rightarrow K^d$.

The proof of Theorem 3.1 and Theorem 3.2 will be sketched in Section 3.2.

In practice, to verify assumption (K), one may have to use the following, slightly more restrictive condition, used by Oodaira [21] and Taqqu ([27], p. 230).

Assumption (T). Let $\Gamma(s,t)$, $0 \le s,t \le 1$, be a strictly positive definite covariance kernel with $\Gamma(t,t)$ strictly increasing to $\Gamma(1,1) = 1$. Suppose $\{Y(t),\ t \ge 0\}$ is Gaussian and has continuous covariance kernel, satisfies $Y(0) = 0$ and also

i) $\lim_{n\to\infty} \sup_{0\le s,t\le 1} \left| \frac{E(Y(ns)Y(nt))}{v^2(n)} - \Gamma(s,t) \right| = 0$

where $v(n) \uparrow \infty$ as $n \to \infty$.

ii) There exists a non-negative, strictly increasing and continuous function ϕ on R^+ satisfying $\int_1^\infty \phi(e^{-u^2})du < \infty$ such that

$$E(Y(ns)-Y(nt))^2 \le \phi^2(|s-t|)v^2(n)$$

for every $0 \le s,t \le 1$ and $n \ge 0$.

Remarks

1. Let $\{X(t),\ 0 \le t \le 1\}$ in Assumption (K) be the Gaussian process with covariance Γ. Since Y is also Gaussian, (T) implies (K). Indeed, i) implies that the finite-dimensional distributions of $\frac{Y(nt)}{v(n)}$ converge to those of $X(t)$ as $n \to \infty$ and ii) implies tightness. If the sup in i) is suppressed then (T) still implies (K).

2. (K) implies condition i) of (T) without the sup. It does not imply ii). Condition ii) is merely a sufficient condition for a Gaussian process to be continuous.

3. The proof of Theorem 3.2 uses results of Carmona and Kôno [3] which remain valid if (T) is replaced by (K). (See the proof of Corollary 4.1, Remark 3.4 and Theorem 4.1 of Carmona and Kono [3].)

3.2 Sketch of the Proof of Theorem 3.1 and 3.2

We start with Theorem 3.1. The typical way to establish the relative compactness statement (3.3), is to use the Arzela-Ascoli theorem. One must show that $\{f_n^d(t,\omega),\ n \geq 1\}$ is a.s. equicontinuous and uniformly bounded, that is

$$\lim_{\delta\to 0} \sup_{n\geq 1} \sup_{\substack{0\leq s,t\leq 1\\ |s-t|<\delta}} \| f_n^d(t,\omega) - f_n^d(s,\omega)\|_d = 0 \quad \text{a.s.}$$

and

$$\sup_n \sup_{0\leq t\leq 1} \| f_n^d(t,\omega)\|_d < \infty \quad \text{a.s.}$$

A.s. equicontinuity is shown by subdividing the real line into intervals of the form $[c^k, c^{k+1}]$, where $c > 1$ and applying the Borel-Cantelli lemma on sets of the form

$$A_k = \left\{\omega:\ \sup_{c^k\leq n\leq c^{k+1}} \sup_{\substack{0\leq s,t\leq 1\\ 0\leq |s-t|\leq\delta}} \left|\left|\frac{Y^d(ns,\omega) - Y^d(nt,\omega)}{v(n)}\right|\right|_d \geq \varepsilon h(c^k)\right\}$$

which are subsets of

$$B_k = \left\{\omega:\ \sup_{\substack{0\leq s,t\leq 1\\ 0\leq |s-t|\leq\delta}} \|\Delta_k(\omega)\|_d \geq \nu\varepsilon h(c^k)\right\}.$$

Here,

$$\Delta_k(\omega) = \frac{Y^d(c^{k+1}s,\omega) - Y^d(c^{k+1}t,\omega)}{v(c^{k+1})}$$

and $\nu = \inf_{1\leq t/s\leq 2} v(s)/v(t)$. To bound the probability of B_k, apply the following lemma of Fernique [6].

<u>Lemma 3.1</u> (Fernique [6]). If Y is a nontrivial Gaussian vector in a real separable Banach space B with norm $\| \ \|$, then there exists an $\alpha > 0$ so that

(i) $E(\exp(\alpha\|Y\|)) < \infty$.

In particular,

(ii) $E(\|Y\|^2) < \infty$

and for every $t \geq 2$

(iii) $P(\|Y\| \geq t\, E(\|Y\|^2)^{1/2}) \leq \exp[-\frac{t^2}{96} \log 3]$.

[Proof: for (i) see Fernique [6], Theorem 1.3.2, p. 11 and for (iii) see Carmona and Kôno [3], Remark 2.1.]

Lemma 3.1 is applied to the Gaussian vector Δ_k, viewed as an element of the Banach space B of d-dimensional continuous functions on the set $\{0 \leq s,t \leq 1,\ 0 \leq |s-t| \leq \delta\}$. The norm on B is the sup norm. If $a_{k,\delta} = E(\sup\|\Delta_k\|_d^2)$, then

$$P(A_k) \leq P(B_k) = P\Big(\sup_{0\leq s,t\leq 1} \|\Delta_k\|_d \geq \frac{\nu\varepsilon h(c^k)}{a_{k,\delta}^{1/2}} \cdot a_{k,\delta}^{1/2}\Big)$$

$$\leq \exp\Big(-\frac{\nu^2\varepsilon^2 h^2(c^k)}{96 a_{k,\delta}} \log 3\Big).$$

Let $a_\delta = E(\sup_{\substack{0\leq s,t\leq 1\\ 0\leq|s-t|\leq\delta}} \|x^d(s)-x^d(t)\|_d^2)$. The weak convergence assumption (ii) of (K) entails $a_{k,\delta} \to a_\delta$ as $k \to \infty$ (Carmona and Kôno [3], Lemma 3.1, p. 245), so that $a_{k,\delta} \leq Ca_\delta$ for some constant C. Furthermore by continuity of x^d and Fernique's lemma 3.1 (ii), we have $a_\delta \to 0$ as $\delta \to 0$ and thus $a_\delta \leq \frac{2\nu^2\varepsilon^2}{96(1+\varepsilon)C} \log 3$ for small enough δ. Then

$$\sum_{k=1}^\infty P(A_k) \leq \sum_{k=1}^\infty \exp\left(-\frac{\nu^2\varepsilon^2 h^2(c^k)}{96Ca_\delta}\right) \leq \sum_{k=1}^\infty e^{-\frac{(1+\varepsilon)h^2(c^k)}{2}}$$

$$= \frac{c}{c-1} \sum_{k=1}^\infty \frac{e^{-\frac{(1+\varepsilon)h^2(c^k)}{2}}}{c^k} (c^k - c^{k-1})$$

which is finite by condition (3.1) on h. A.s. uniform boundedness follows from a.s. equicontinuity and $Y^d(0) = 0$.

We now sketch the proof of $P(C\{f_n^d(t,\omega)\} \subset K^d) = 1$. Because of a.s. equicontinuity, it is sufficient to show that $P(C\{f^d_{c^k}, k \geq 1\} \subset K^d) = 1$. This is done by applying the Strong Convergence Theorem 2.1, with $B = C^d[0,1]$. To verify

$$P\{\omega: \limsup_{n\to\infty} g(\frac{Y^d(c^k t,\omega)}{v(c^k)h(c^k)}) \leq \sup_{x^d \in K^d} g(x^d)\} = 1 \qquad (2.9)$$

for all $g \in (C^d[0,1])^*$, use the fact (Kuelbs [13], Lemma 2.1(iv), p. 750)

$$\sup_{x^d \in K^d} g(x^d) = \int_{C^d[0,1]} [g(y)]^2 d\mu(y)$$

where μ is the probability distribution on $C^d[0,1]$ of the process X^d. Then set

$$C_k = \{\omega: g(f^d_{c^k}(\cdot,\omega) \geq [(1+\varepsilon) \int_{C^d[0,1]} g^2(y)d\mu(y)]^{1/2}\}$$

and proceed as before: use the weak convergence assumption and Fernique's Lemma 3.1 (iii) to conclude $\sum_{k=1}^{\infty} P(C_k) < \infty$. Then (2.9) holds by the Borel-Cantelli Lemma. □

Remark. If (K) is replaced by (T), then Theorem 3.1 still holds, since (T) => (K). Oodaira [21] provides a different proof of Theorem 3.1, when (T) holds. Relative compactness is proved by using Lemma 3.2 given below instead of Lemma 3.1. Lemma 3.2, which is also due to Fernique [6] merely uses the tail behavior of the Gaussian distribution and as such can be extended to cover the non-Gaussian Hermite processes, as we will see in Section 5. This is not the case for Lemma 3.1. Lemma 3.1 is based on Theorem 1.3.2 of Fernique ([6] p. 11), which uses properties of the Gaussian distribution that have no counterparts in other distributions.

Lemma 3.2 (Fernique [6], pp. 48-51). Let $\{Y(t), 0 < t < 1\}$ be a real separable Gaussian process with mean 0 and let

$$\phi_Y(u) = \sup_{\substack{0\le s,t\le 1 \\ |s-t|\le u}} [E(Y(s)-Y(t))^2]^{1/2}.$$

If $\int_1^\infty \phi_Y(e^{-x^2})dx < \infty$, then $Y(t)$ is a.s. sample continuous, and for all $p \ge 2$, p integer, and all $x \ge \sqrt{1 + 4 \log p}$ we have

$$P\left(\sup_{t\in[0,1]} |Y(t)| \ge x[\sup_{s,t\in[0,1]} (E(Y(s)Y(t)))^{1/2} + 4 \int_1^\infty \phi_Y(p^{-u^2})du]\right)$$

$$\le 4p^2 \int_x^\infty e^{-u^2/2}du.$$

Incidentally, Oodaira [21] completes the proof of Theorem 3.1 under the assumption (T) without using the high powered Strong Convergence Theorem 2.1.

Proof of Theorem 3.2. Suppose that $h(t) = (2 \log \log t)^{1/2}$. Carmona and Kôno ([3], Theorem 4.1), using a lemma of Nisio [20], show that the conditions of the second part of the Strong Convergence Theorem 2.1 of Kuelbs are satisfied and therefore the conclusion of Theorem 3.2 holds for $f^d_{n_k}$, where $n_k = [c^k]$, $c > 1$. Now proceed as in Taqqu ([27], p. 231): Since relation (3.4') holds for f_{n_k}, it holds for f_n as well, because

$$K^d = C\{f^d_{n_k}\} \subset C\{f^d_n\} \subset K^d$$

where the last inclusion is a consequence of Theorem 3.1. Relation (3.3), i.e. relative compactness, holds by Theorem 3.1.

When $h(t)$ merely satisfies the condition (3.1) one needs to check that an analogue of Carmona and Kôno's Theorem 4.1 [3] still holds. □

3.3 The Case of Weak Convergence to Fractional Brownian Motion

Set

$$\Gamma(s,t) = \frac{1}{2} \{s^{2H} + t^{2H} - |s-t|^{2H}\}.$$

This is the covariance kernel of the mean 0 Gaussian process $B_H(t)$ known as Fractional Brownian Motion. Let K be the unit ball of the RKHS $H(\Gamma)$ and let K^d be the unit ball of $H(\Gamma)^d$. Taqqu ([27] Corollaries A1 and A2) has shown that a functional law of the iterated logarithm holds for $Y(nt) = B_H(nt)$ and a similar one holds also for $Y(nt)$ when defined as in Example 2 of Section 3.1. The proof is based on the fact that both conditions (T) and (3.5) are satisfied. Kôno obtains the following more general result.

Theorem 3.3 (Kôno [10], p. 14). Suppose that $Y(t)$ satisfies Assumption (K), has stationary increments and that the limit in $C[0,1]$ of $\frac{Y(nt)}{v(n)}$ is the Fractional Brownian Motion process $X(t) = B_H(t)$. Then the functional law of the iterated logarithm holds, that is

$$P(\{f_n^d(t,\omega),\ n \geq 1\} \text{ is relatively compact in } C^d[0,1]) = 1$$

and

$$P(C\{f_n^d(t,\omega)\} = K^d = 1.$$

Theorem 3.1 implies the relative compactness part so that the result follows once condition (3.5) is established. Condition (3.5) is easy to check for the Fractional Brownian Motion process X but it is hard to prove for a general process Y. Kôno introduces a smoothing function $\delta(a)$, such that condition (3.5) can be shown to hold for the processes

$$\overline{X}(t,\omega) = c \int_0^2 X(at,\omega)\delta(a)da, \quad c = \text{constant}$$

$$\overline{Y}(t,\omega) = \int_0^2 Y(at,\omega)\delta(a)da.$$

Since $\overline{Y}(nt)$ suitably normalized converges weakly to $\overline{X}$, Theorem 3.2 holds for $\overline{Y}$. Its conclusion will hold also for Y by choosing δ appropriately.

3.4 Applications to Sums of Strongly Dependent Gaussian Random Variables

Let $Y(nt)$ be the linear interpolation of $\sum_{i=1}^{[nt]} Z_i$. Suppose now that $Z_i = G(U_i)$, where U_i is a stationary Gaussian sequence with mean zero and unit variance, and G is a function such that $EG(U) = 0$ and $EG^2(U) < \infty$, where U is an $N(0,1)$ random variable.

Let $m \geq 1$ be the Hermite rank of G, that is, the first non-zero term in the expansion of G in Hermite polynomials. The Hermite polynomials are given by

$$H_m(u) = (-1)^m e^{u^2/2} \frac{d^m}{du^m} e^{-u^2/2}, \quad m = 0,1,2,\ldots, \tag{3.6}$$

thus, $H_0(u) = 1$, $H_1(u) = u$, $H_2(u) = u^2-1$, $H_3(u) = u^3-3u$. Suppose further that the U_i have covariance $EU_iU_{i+k} \sim k^{(2H-2)/m}g(k)$ where $1/2 < H < 1$ and $g(k)$ a slowly varying function at infinity and bounded on bounded intervals. Then (Taqqu [28]; Dobrushin and Major [5]) letting $v^2(n) = E(\sum_{i=1}^n G(U_i))^2$ and $J(m) = E(G(U)H_m(U))$, we have

$$\frac{Y(nt)}{v(n)} \overset{L}{\to} \frac{J(m)}{m!} X_m(t) \quad \text{in} \quad C[0,1] \quad \text{as} \quad n \to \infty$$

where $X_m(t)$ is a so-called "Hermite process". (See Section 5.)

When $m = 1$, the limiting process $X_1(t)$ is Fractional Brownian motion with index $H \in (1/2,1)$. It is a Gaussian process. If we choose $G(u) = H_1(u) = u$, then $Y(nt)$ is Gaussian as well and we are in the case of Example 2(I) of Section 3. Assumption (K) is satisfied and the conclusions of Theorem 3.1 and Theorem 3.3 hold. However, this continues to be true for more general functions G that have Hermite rank $m = 1$. This is because of the following Strong Reduction Theorem.

<u>Theorem 3.4</u> (Taqqu [27], p. 206). Let $m \geq 1$ and $1/2 < H < 1$. Then

$$\lim_{N\to\infty} \frac{1}{v(N)} \max_{1\leq n\leq N} \left|\sum_{i=1}^n G(U_i) - \frac{J(m)}{m!} H_m(U_i)\right| = 0 \quad \text{a.s.}$$

if

$$EG^p(U) < \infty, \quad p > 2 \max\left(\frac{1}{2-2H}, \frac{1}{2H-1}\right). \tag{3.7}$$

Kôno [11] mentions that the additional condition (3.7) can be dropped if instead of Serfling's inequality (see [24]) which was used in the original proof, one applies the following real variable lemma.

Lemma 3.3 (Kôno [11]). Let $\{a_k, k = 1,2,\ldots,2^{n+1}-1\}$ be a sequence of real numbers and set

$$S(0,k) = S(k) = \sum_{j=1}^{k} a_j$$

and

$$S(a,k) = \sum_{j=a+1}^{a+k} a_j.$$

Then we have

$$\max_{1\leq k<2^{n+1}} |S(k)|^2 \leq (n+1) \sum_{j=0}^{n} \sum_{k=0}^{2^{n-j}-1} |S(k2^{j+1}, 2^j)|^2.$$

This lemma is a slight modification of that of Rademacher ([23], p. 118) and Alexits ([1], p. 82).

4. UPPER AND LOWER CLASS RESULTS FOR GAUSSIAN SELF-SIMILAR PROCESSES

From the functional law of the iterated logarithm for $f_n^d(t)$ one can get the usual law of the iterated logarithm by setting $d = 1$ and applying the projection map $\phi(x) = x(1)$ to $x \in C[0,1]$ (see Stout [26], p. 290). Thus Theorem 3.2 yields the usual law of the iterated logarithm

$$\limsup_{n\to\infty} \frac{Y(n)}{h(n)v(n)} = 1 \quad \text{a.s.} \tag{4.1}$$

with $h(\cdot)$ as in (3.1).

Statement (4.1) is equivalent to the following two statements which hold for any $\varepsilon > 0$:

The "upper class result"

$$P\left(\frac{Y(n)}{v(n)} > (1+\varepsilon)h(n) \quad \text{i.o.}\right) = 0 \tag{4.2}$$

and the "lower class result"

$$P\left(\frac{Y(n)}{v(n)} > (1-\varepsilon)h(n) \quad \text{i.o.}\right) = 1 \tag{4.3}$$

where, as usual, "i.o." means "for infinitely many n".

The function $h(n)$ (typically $(2 \log\log n)^{1/2}$) which appears in (4.1) has the property that (4.2) holds for the is the reproducing kernel Hilbert space attached to the covariance kernel of Brownian motion (see Section 2.2).

The set K_f belongs to a Banach space of functions $C_\nu(R^+)$ which we now define. Let

$$\gamma(t) = |t|(1 + |\log|t||)^{1/2}, \quad t \neq 0$$

with $\gamma(0) = 0$, and let

$$\nu(t) = t^{H-(m/2)}\gamma(t)^{m/2} = t^H(1 + |\log t|)^{m/2}, \quad t > 0.$$

The Banach space of functions is defined by

$$C_\nu(R^+) = \{y: R^+ \to R^+, \text{ continuous, with } \lim_{t\to\infty} \frac{y(t)}{\nu(t)} = \lim_{t\to 0} \frac{y(t)}{\nu(t)} = 0\}$$

and it is endowed with the norm $\|y\|_\nu = \sup_{t>0} \frac{|y(t)|}{\nu(t)}$. We have $K_f \subset C_\nu(R^+)$ because by Schwarz inequality and the scaling properties of f_t,

$$|y(t)|^2 \leq \|f_t\|^2\|\dot{z}\|_2^{2m} = t^{2H-m}(\int_{R^m} f_1^2(\frac{x_1}{t}, \ldots, \frac{x_m}{t})dx_1\ldots dx_m)\|\dot{z}\|_2^{2m}$$

$$= t^{2H}\|f_1\|^2_{L^2(R^m)} \|\dot{z}\|_2^{2m}.$$

The main result is

<u>Theorem 5.2</u> (Mori and Oodaira [19], Theorem 3.1). Let $Y(t)$ be an m-integral process whose kernels f_t satisfy assumption (M0) and let

$$Y_n(t) = \frac{Y(nt)}{n^H(2 \log \log n)^{m/2}} \quad t \geq 0, \ n \geq 3.$$

Then

1. $Y_n \in C_\nu(R^+)$ a.s. $\forall$ $n \geq 3$
2. $\{Y_n, n \geq 3\}$ is a.s. relatively compact in $C_\nu(R^+)$
3. $C\{Y_n, n \geq 3\} = K_f$ a.s. and K_f is compact in $C_\nu(R^+)$.

function $(1+\varepsilon)h(n)$ and (4.3) holds for the function $(1-\varepsilon)h(n)$. One may dissociate relation (4.2) from (4.3) and formulate the following more general question. For which positive nondecreasing function $s(n)$ does one have

$$P\left(\frac{Y(n)}{v(n)} > s(n) \quad i.o.\right) = 0, \tag{4.4}$$

and for which ones does one have

$$P\left(\frac{Y(n)}{v(n)} > s(n) \quad i.o.\right) = 1? \tag{4.5}$$

The function $s(n)$ is called an upper class function of $\frac{Y(n)}{v(n)}$ if (4.4) holds and a lower class function of $\frac{Y(n)}{v(n)}$ if (4.5) holds.

When Y can be represented as

$$Y(n) = \sum_{i=1}^{n} Z_i, \tag{4.6}$$

where $\{Z_i, i \geq 1\}$ is a zero-mean stationary Gaussian sequence, then Lai and Stout [14] provide the following answer to the question.

Theorem 4.1 (Lai and Stout [14], p. 741). Let Y have the representation (4.6) and assume that Assumption (K) holds with X self-similar of index $0 < H \leq 1$. Assume also that $v(n) = n^H L(n)$ satisfies the following conditions:

i) There exist constants $M \geq 1$, $\delta > 0$ and $\gamma > H$ such that

$$\left|\left(\frac{v(n+m)}{v(m)}\right)^2 - 1\right| \leq M\left(\frac{m}{n}\right)^{\gamma} \quad \text{if} \quad \delta n \geq m \geq M.$$

ii) There exists a constant $\beta > 0$ such that

$$\limsup_{n\to\infty} \left\{\max_{n(\log\log n)^{-\beta} \leq j \leq n} \frac{L(j)}{L(n)}\right\} < \infty$$

and

$$\liminf_{n\to\infty} \left\{\min_{n(\log\log n)^{-\beta} \leq j \leq n} \frac{L(j)}{L(n)}\right\} > 0.$$

Let $s(t)$ be a positive nondecreasing function on $[1,\infty)$. Then

$$P\left(\frac{Y(n)}{v(n)} > s(n) \quad \text{i.o.}\right) = 0 \quad \text{or} \quad 1 \quad \text{according to} \tag{4.7}$$

$$\int_1^\infty \frac{1}{t}\, s(t)^{(1/H)-1} \exp\left(-\frac{1}{2}\, s^2(t)\right) dt < \infty \quad \text{or} \quad = \infty. \tag{4.8}$$

Remarks

1. Any Gaussian mean zero sequence $Y(n)$ with stationary increments admits the representation (4.6). Note also that the limiting process X in Theorem 4.1 is necesssarily fractional Brownian motion because fractional Brownian motion is the only Gaussian self-similar process with stationary increments.

2. Suppose that the summands Z_i are as in Example 2 of Section 3.1 and suppose in addition that the slowly varying function $L(n)$ satisfies condition ii) of Theorem 4.1. Then the upper-lower class test (4.7) and (4.8) holds. (Lai and Stout [14], Corollary 3).

3. If $h(t)$ satisfies (3.1) then it clearly satisfies (4.8) and therefore $(1+\varepsilon)h(t)$ is an upper-class function for $Y(n)/v(n)$ and $(1-\varepsilon)h(t)$ is a lower-class function for $Y(n)/v(n)$ whenever the remaining assumptions of Theorem 4.1 are satisfied.

4. The term $s(t)^{(1/H)-1}$ is negligible when the function is $s(t) = (1\pm\varepsilon)\times(2 \log \log t)^{1/2}$; however, it acquires importance when ε is replaced by $\varepsilon(t)$, e.g. when $s(t)$ is the lower class function $(2 \log \log t + \log \log \log t)^{1/2}$.

5. Not every regularly varying function $v(n)$ satisfies i) of Theorem 4.1. For example $v(n) = n^H(\log(n+3) + n^\beta \sin(n))$ is a regularly varying function, but does not satisfy i) when $\beta > -\gamma/2$ (see Taqqu [27], p. 232).

6. Condition i) implies (3.5), which is a hypothesis of Theorem 3.2. In Theorem 3.3 however, condition (3.5) is not assumed. The result then follows from the assumed weak convergence to Fractional Brownian Motion. This leads to the following question: does Theorem 4.1 hold without assuming condition i)?

An answer cannot be obtained by adopting the method of proof of Theorem 3.3. Indeed that method uses a smoothed process $\overline{Y}$. It can be shown that condition i) holds for $\overline{Y}$ and therefore the conclusion of Theorem 4.1 applies to $\overline{Y}$. However, the passage from $\overline{Y}$ to the original process Y introduces a factor of $(1 \pm \varepsilon)$ in condition (4.8) which affects the divergence or convergence of the integral.

5. FUNCTIONAL LAWS OF THE ITERATED LOGARITHM FOR PROCESSES REPRESENTED BY MULTIPLE-WIENER ITÔ INTEGRALS

5.1 m-integral Processes

We start by defining the processes that we shall consider in this section.

A random process $\{Y(t), t \geq 0\}$ is said to be an m-integral if there exist kernels $f_t \in L^2(R^m)$, $t \geq 0$ such that

$$Y(t) = \int \underset{R^m}{\cdots} \int f_t(x_1,\ldots,x_m)dB(x_1)\ldots dB(x_m) \tag{5.1}$$

where the integral is a multiple Wiener-Itô integral with respect to Brownian motion (see Itô [8]).

Relation (5.1) characterizes the finite-dimensional distributions of the process $Y(t)$. For fixed t, (5.1) defines an m-integral random variable. The process $Y(t)$ is non-Gaussian when $m > 1$.

Examples of m-integrals: The Hermite processes

These processes were briefly introduced in Section 3.4. The m-th Hermite processes $\{X_m(t), t \geq 0\}$ are self-similar with index $1/2 < H < 1$ and have stationary increments. They admit the following m-integral representation (Taqqu [28], p. 77)

$$X_m(t) = K \int \underset{R^m}{\cdots} \int [\int_0^t \Pi_{i=1}^m ((s-x_i)^+)^{H_0-(3/2)} ds]dB(x_1)\ldots dB(x_m) \tag{5.2}$$

where

$$H_0 = H_0(m,H) = \frac{(H-1)}{m} + 1 \in (1 - \frac{1}{2m}, 1)$$

and K a normalizing factor ensuring that $EX_m^2(1) = 1$. $X_m(t)$ has finite moments and its covariance is given by

$$\Gamma(s,t) = EX_m(t)X_m(s) = \frac{1}{2}\{s^{2H} + t^{2H} - |s-t|^{2H}\}. \tag{5.3}$$

It is the same as that of Fractional Brownian Motion. $X_m(t)$ is non-Gaussian for $m > 1$ but it is the Gaussian process Fractional Brownian Motion when $m = 1$.

Two approaches. There are presently two approaches that yield results related to functional laws of the iterated logarithm. One is due to Fox [7] and the other to Mori and Oodaira [19].

Fox's method is an extension to m-integral processes of the Oodaira-Taqqu approach. It does not require the process to be self-similar, but it has yielded at this point only upper functional laws of the iterated logarithm.

Mori and Oodaira's method yields functional laws of the iterated logarithm, but it has only been applied to certain types of self-similar m-integral processes, e.g. the Hermite processes. The method uses results available for Brownian motion and extends them to m-integral processes through the application of continuous mappings. It requires a judicious smoothing of the kernels f_t.

5.2 Upper Functional Laws of the Iterated Logarithm for m-Integral Processes

To formulate an upper functional law of the iterated logarithm for m-integrals $Y(t)$, we need to impose the following assumption (T'), which extends assumption (T) to the cases $m \geq 1$.

Assumption (T'). A process $\{Y(t), t \geq 0\}$ satisfies assumption (T'), if $v^2(n) = E(Y^2(n)) \to \infty$ as $n \to \infty$ and there exist a continuous covariance kernel $\Gamma(s,t)$, $s,t \in [0,1]$ and a positive nondecreasing function $g(x)$, $x \geq 0$ such that

i) $\lim\limits_{n\to\infty} \sup\limits_{0\leq s,t\leq 1} \left[\dfrac{E(Y(ns)Y(nt))}{v^2(n)} - \Gamma(s,t)\right] = 0$

ii) $E(Y(ns) - Y(nt))^2 \leq v^2(n)g^2(|s-t|)$ for all $n \geq 0$, $0 \leq s,t \leq 1$ and $\int_1^\infty g(e^{-u^{2/m}})du < \infty$

iii) $\Gamma(s,t)$ is strictly positive definite and $\Gamma(t,t)$ strictly monotone increasing to $\Gamma(1,1) = 1$.

Remarks

1. When $m = 1$ and $\{Y(t), t \geq 0\}$ is a Gaussian process then $(T') = (T)$. The difference between (T') and (T) resides in condition ii).

2. Condition (T') imposes only conditions on the second moments. These do not determine the finite dimensional distributions in the non-Gaussian cases $m \geq 2$.

The following upper functional law of the iterated logarithm is an extension to functional spaces of a result of Lai and Stout ([15], Corollary 5). It holds for m-integrals and involves the unit ball K of the reproducing kernel Hilbert space $H(\Gamma)$, where Γ is the "limiting" covariance, which appears in (T').

Theorem 5.1 (Fox [7], Theorem 1.1). Let $m > 0$ be an integer. Suppose that $\{Y(t), t \geq 0\}$ is a separable, m-integral process satisfying condition (T'). Then $Y(t)$ has a.s. continuous paths. Furthermore, there exists a constant $\theta = \theta(m)$ such that the set of the functions

$$f_n(t) := \frac{Y(nt)}{v(n)(\theta \log \log n)^{m/2}} \qquad 0 \leq t \leq 1,\ n \geq 3$$

is a.s. relatively compact in $C[0,1]$ and its set of limit points is a.s. contained in the unit ball K of the reproducing kernel Hilbert space $H(\Gamma)$.

Remark. It is important to note, that the constant θ depends on the Hermite rank m but not on the kernel f which appears in the m-integral representation (5.1) of the process because $Y(nt)/v(n)$ has unit variance at $t = 1$.

Remark on the proof of Theorem 5.1. The proof is similar to that of Theorem 3.1 under condition (T). We shall need a generalization to m-integrals of Fernique's Lemma 3.2. Such a generalization can be obtained because Lemma 3.2 merely depends on the tail behavior of Gaussian random variables; the tail behavior of m-integrals can be characterized as follows:

Lemma 5.1 (Major [17], p. 68, McKean [18]). For every $m \geq 1$ there exists an $N_0 = N_0(m) > 0$ and $\theta = \theta(m)$ such that for all m-integral random variables Y with unit variance, we have:

$$P(|Y| > x) \leq \exp\left(-\frac{x^{2/m}}{\theta}\right), \quad \forall\, x \geq N_0.$$

To obtain a substitute for Fernique's Lemma 3.2, which is only valid in the Gaussian case $m = 1$, use the modulus of continuity of $\{Y(t),\ t \in [0,1]\}$

$$\Phi_Y(h) := \sup_{\substack{|s-t| \leq h \\ s,t \in [0,1]}} (E[Y(s) - Y(t)]^2)^{1/2},$$

let $\| \ \|$ denote the sup norm and set $a = a(m) = 2^{m/2}/(2^{m/2}-1)$ and $b = \sum_{n=0}^{\infty} 2^k \exp(\frac{1}{2} - 2^{k-1}) < \infty$. With this notation we have the following substitute of Lemma 3.2.

Lemma 5.2 (Fox [7], Proposition 2.2). Let $m,p \geq 2$ be integers and $N(m,p) = \max\{N_0(m), [\frac{\theta(m)}{2}(1 + 4 \log p)]^{m/2}\}$. Furthermore, let $\{Y(t),\ 0 \leq t \leq 1\}$ be a separable m-integral process with covariance $R(s,t)$. Assume that

$$\int_1^{\infty} \phi_Y(e^{-u^{2/m}})du < \infty.$$

Then $Y(t)$ is a.s. continuous and for all $x \geq N(m,p)$

$$P(\|Y\| \geq x[\sqrt{\|R\|} + a \int_1^{\infty} \phi_Y(\tfrac{1}{2} p^{-u^{2/m}})du] \leq bp^2 \exp[-\frac{x^{2/m}}{\theta}].$$

The proof of this lemma is similar to Lemma 3.2. (See also Lemma 5.4 below.) We now apply Theorem 5.1 to the case where $Y(t)$ is a self-similar m-integral process with stationary increments. Then (T') holds with the process $Y(t)$ having covariance

$$\Gamma(s,t) = \frac{\sigma^2}{2}\{s^{2H} + t^{2H} - |s-t|^{2H}\}.$$

Applying Theorem 5.1, we get

Corollary 5.2: An m-integral self-similar process with stationary increments satisfies an upper functional law of the iterated logarithm: the conclusion of Theorem 5.1 holds.

This result, for instance, can be applied to the Hermite processes defined in Section 3.4. Note, however, that the Hermite processes do not exhaust the class of finite variance, stationary increments, self-similar processes (see Taqqu [29]).

Theorem 5.1 can also be applied to normalized partial sums that converge weakly to Hermite processes. For instance, consider the following quantities which have been defined in Section 3.4: the partial sums $\sum_{i=1}^{[nt]} G(U_i)$, their linear interpolation $Y(nt)$ and the normalization factor $v(n)$. Let m be the Hermite rank of G. Then, using the Strong Reduction Theorem 3.4 and the fact that $H_m(U_i)$ can be represented as an m-integral, we get:

Corollary 5.3 (Fox [7], Corollary 4.2). Suppose that the conditions that ensure weak convergence of $\frac{Y(nt)}{v(n)}$ to the Hermite processes hold (see Section 3.4). Then

$$f_n(t) = \frac{Y(nt)}{v(n)(\theta \log \log n)^{m/2}}$$

is a.s. relatively compact in $C[0,1]$ and its set of limit points is a.s. contained in the unit ball of the reproducing kernel Hilbert space corresponding to $\Gamma(s,t) = \frac{1}{2}\{s^{2H} + t^{2H} - |s-t|^{2H}\}$.

5.3 The Functional Law of the Iterated Logarithm for Some Self-Similar m-integral Processes

We now turn to the results for m-integral processes obtained by Mori and Oodaira [19]. Let $Y(t)$ be a m-integral process with kernels $f_t \in L^2(R^m)$, $t \geq 0$ and assume that f_t satisfies the following:

Assumption (M0). f_t, $t \geq 0$ can be expressed as

$$f_t(x_1,\dots,x_m) = \int_0^t q(v-x_1,\dots,v-x_m)dv$$

where $q: R^m \to R$ is a symmetric function with

$$q(cx_1,\dots,cx_m) = c^{-\lambda}q(x_1,\dots,x_m) \quad \text{where} \quad \lambda = \frac{m}{2} + 1 - H \quad (5.3)$$

$$\text{and} \quad \frac{1}{2} < H < 1,$$

$$\int_{R^m}\cdots\int \left|q(x_1,\dots,x_m)q(x_1+1,\dots,x_m+1)\right|dx_1\dots dx_m < \infty. \quad (5.4)$$

Condition (5.3) ensures that

$$f_t(x_1,\dots,x_m) = t^{H-(m/2)}f_1(\frac{x_1}{t},\dots,\frac{x_m}{t}) \quad \text{for} \quad t > 0$$

and $f_0 = 0$. Condition (5.3) and (5.4) ensure that $f_t \in L^2(R^m)$. Hence $Y(t)$ is self-similar with index H and has stationary increments. Since for every $r > 1$,

$$E\left|Y(s) - Y(t)\right|^r = E\left|Y(\left|s-t\right|)\right|^r = \left|s-t\right|^{rH}E(\left|Y(1)\right|^r),$$

we can assume that Y has continuous paths (Billingsley [2], p. 96).

Note that the kernels of Hermite processes satisfy condition (M0) but there are m-integral self-similar processes whose kernels do not fulfill assumption (M0).

To formulate Mori and Oodaira's result let

$$K_f = \{y(t) = \int_{R^m}\cdots\int f_t(x_1,\dots,x_m)\xi(x_1)\dots\xi(x_m)dx_1,\dots,dx_m: \|\xi\|_2^2 \le 1\}$$

where $\|\xi\|_2^2 = \int_{-\infty}^{+\infty}|\xi(x)|^2dx$. It is more enlightening to set $\xi(x) = \dot{z}(x) \equiv dz/dx$, and write

$$K_f = \{y(t) = \int_{R^m}\cdots\int f_t(x_1,\dots,x_m)\dot{z}(x_1)\dots\dot{z}(x_m)dx_1,\dots,dx_m: z \in \mathcal{H}, \|\dot{z}\|_2 < 1\}$$

where

$$\mathcal{H} = \{z: R \to R, \text{ absolutely continuous}, \ z(0) = 0, \ \dot{z} \in L^2(R)\}$$

Remarks

1. The function $\nu(t)$ is needed here because $Y_n(t)$ is a function on $[0,\infty)$ and not merely on $[0,1]$ with $Y_n(0) = 0$. For a given function $\nu(t)$, the faster $\nu(t)$ tends to infinity, the weaker the result; the faster $\nu(t)$ tends to zero, the stronger the result.

2. Let $C_0[0,1] = \{y \in C[0,1]: y(0) = 0\}$. The theorem remains valid if one replaces $C_\nu(R^+)$ by $C_0[0,1]$, since the change merely involves a continuous mapping.

3. Theorem 5.2 holds for $Y(t)$ that are Hermite processes (see (5.2)). In that case

$$f_t(x_1,\ldots,x_m) = \int_0^t \Pi_{i=1}^m \left((s-x_i)^+\right)^{H_0-(3/2)} dx_1,\ldots,dx_m$$

with $1 - 1/2m < H_0 < 1$, and

$$K_f = \{y(t) = \int_0^t [\int_{-\infty}^s((s-x)^+)^{H_0-(3/2)}\dot{z}(x)dx]^m ds\colon z \in H,\ \|\dot{z}\|_2 \le 1\}.$$

Observe that every function y in K_f is non-decreasing when m is even.

Corollaries from Theorem 5.2 can be obtained by using continuous mappings. For example, one obtains the following one-dimensional law of the iterated logarithm for Y.

Corollary 5.4 (Mori and Oodaira, [19], Corollary 3.2.). Under the assumptions of Theorem 5.2, we have

$$\limsup_{n\to\infty} \frac{Y(n)}{n^H(2 \log\log n)^{m/2}} = \ell_1 \quad \text{a.s.}$$

$$\liminf_{n\to\infty} \frac{Y(n)}{n^H(2 \log\log n)^{m/2}} = \ell_2 \quad \text{a.s.}$$

where $\ell_1 = \sup\{y(1), y \in K_f\}$ and $\ell_2 = \inf\{y(1), y \in K_f\}$. In particular, when $m = 2$, the constants ℓ_1 and ℓ_2 coincide with the supremum and infimum respectively of the eigenvalues of the integral operator on $L^2(R)$ with kernel f_1.

5.4. Sketch of the Proof of Theorem 5.2.

Step 1. Use Strassen's functional law of the iterated logarithm (see Section 2.2) to show that Theorem 5.2 holds when Y is Brownian motion $B(t)$, $-\infty < t < \infty$. In that case, $m = 1$, $H = 1/2$, $\nu(t)$ is replaced by $\gamma(t)$, the space of functions $C_\nu(R^+)$ is replaced by

$$C_\gamma(R) = \{z: R \to R, \text{ continuous, with } z(0) = 0 \text{ and } \lim_{t\to\pm\infty} \frac{z(t)}{\gamma(t)} = \lim_{t\to 0} \frac{z(t)}{\gamma(t)} = 0\}$$

endowed with the norm $\|z\|_\gamma = \sup_{t\neq 0} \frac{|z(t)|}{\gamma(t)}$, and finally the set K_f is replaced by

$$K = \{z: R \to R, \text{ absolutely continuous, } z(0) = 0, \|\dot{z}\| \leq 1\} \quad (5.5)$$

Remarks

1. Because $B(t)$ is unbounded at $t = \pm\infty$, we consider $C_\gamma(R)$ instead of $C(-\infty,+\infty)$ and $\|\ \|_\gamma$ instead of the sup-norm. We can do this because $|B(t)| \leq C\gamma(t)$ where $C > 0$.

2. The set $K \subset \mathcal{H}$ is the same as in Strassen's law of the iterated logarithm. Since $z \in \mathcal{H}$ satisfies $|z(t)| \leq |t|^{1/2}\|\dot{z}\|_2$, we have $K \subset \mathcal{H} \subset C_\gamma(R)$.

Step 2. Establish Theorem 5.2 for m-integral processes with "smooth" kernels k_t, $t \geq 0$ defined as follows:

$$k_t(x_1,\ldots,x_m) = t^{H-(m/2)}\, k(\frac{x_1}{t}, \ldots, \frac{x_m}{t}) \quad t > 0 \quad (5.6)$$

with $k_0 = 0$, where k is an element of the following class $\mathcal{F}_m$:

i) $k: R^m \to R$ is continuous and symmetric

ii) $D_1 \ldots D_m k$ exists $(D_j = \frac{\partial}{\partial x_j})$ and satisfies

$$A(k) = \int \underset{R^m}{\cdots} \int |D_1 \ldots D_m k(x_1,\ldots,x_m)|\gamma(x_1)\ldots\gamma(x_m)dx_1\ldots dx_m < \infty$$

iii) There exists an even function $g \in L^2(R)$ such that for some $\alpha > 1/2$, $g(t) = o(|t|^{-\alpha})$ as $|t| \to \infty$ and

$$\left|k(x_1,\ldots,x_m)\right| < \Pi_{j=1}^m g(x_j).$$

Now introduce the mapping

$$\tau_k: C_\gamma(R) \to C_\nu(R^+)$$
$$z \mapsto y_z$$

where

$$y_z(t) = (-1)^m \int \underset{R^m}{\cdots} \int [D_1 \ldots D_m k_t(x_1,\ldots,x_m)] z(x_1)\ldots z(x_m) \cdot dx_1 \ldots dx_m.$$

It is easy to show, that

$$\|y_z\|_\nu \leq A(k)\|z\|_\gamma^m, \quad \forall\, z \in C_\gamma(R),$$

which implies that τ_k is a continuous mapping. For $z \in K \subset \mathcal{H} \subset C_\gamma(R)$, where K is as in (5.5), we can perform integration by parts and obtain

$$\tau_k(K) = \{y(t) = \int \underset{R^m}{\cdots} \int k_t(x_1,\ldots,x_m)\dot{z}(x_1)\ldots\dot{z}(x_m)dx_1\ldots dx_m : \|\dot{z}\|_2 \leq 1\}$$

Since τ_k is continuous, $\tau_k(K)$ is compact in $C_\nu(R)$.

Mori and Oodaira establish the following functional law for m-integrals with smooth kernels k_t:

Proposition 5.1 (Mori and Oodaira [19], Theorem 3.3). Let $\tilde{Y}(t) = \int \underset{R^m}{\cdots} \int k_t(x_1,\ldots,x_m)dB(x_1)\ldots dB(x_m)$, $t \geq 0$, where k_t is as in (5.6) and

$$\tilde{Y}_n(t) = \frac{\tilde{Y}(nt)}{n^H(2 \log\log n)^{m/2}} \quad t \geq 0,\ n \geq 3.$$

Then

1. $\tilde{Y}_n \in C_\nu(R^+)$ a.s. $\forall\, n$
2. $\{\tilde{Y}_n, n \geq 3\}$ is a.s. relatively compact in $C_\nu(R^+)$
3. $C\{\tilde{Y}_n, n \geq 3\} = \tau_k(K)$ a.s.

A main tool in the proof of Proposition 5.1 is the following integration by parts formula for multiple Wiener integrals with smooth kernels. First some notation. For $k \in \mathcal{F}_m$, let

$$I^{(m)}(k) = \int \underset{R^m}{\cdots} \int k(x_1,\ldots,x_m)dB(x_1)\ldots dB(x_m)$$

and

$$J^{(m)}(k) = (-1)^m \int \underset{R^m}{\cdots} \int [D_1\ldots D_m k(x_1,\ldots,x_m)]B(x_1)\ldots B(x_m) \cdot dx_1\ldots dx_m,$$

and also define the function $k^{[r]}$, $1 \leq r \leq m/2$ by

$$k^{[r]}(x_1,\ldots,x_{m-2r}) = \int \underset{R^r}{\cdots} \int k(x_1,\ldots,x_{m-2r},v_1,v_1,\ldots,v_r,v_r) \cdot dv_1\ldots dv_r.$$

Condition (iii) in the definition of F_m ensures that $I^{(m)}(k)$ is well-defined. Condition (ii) in the definition of F_m and the fact that $|B(x)| < C\gamma(x)$ ensure that $J^{(m)}(k)$ is well-defined. Note also that $k \in F_m$ implies $k^{[r]} \in F_{m-2r}$ and $(k_t)^{[r]} = (k^{[r]})_t$.

Lemma 5.3 (Mori and Oodaira [19], Lemma 5.2). For $k \in F_m$, we have

$$I^{(m)}(k) = J^{(m)}(k) + \sum_{r=1}^{[m/2]} (-1)^r \frac{m!}{2^r r!(m-2r)!} J^{(m-2r)}(k^{[r]})$$

where

$$J^{(0)}(k^{[r]}) = \int \underset{R^r}{\cdots} \int k(v_1,v_1,\ldots,v_r,v_r)dv_1\ldots dv_r.$$

This lemma is formally obtained by integration by parts and can be proved precisely by approximating the integrals by sums over finite intervals. The proof makes full use of the smoothness properties of the kernel k.

With this lemma it is easy to prove Proposition 5.1, since

$$\begin{aligned}\tilde{Y}_n(t) &= \frac{1}{n^{H-m/2}(2n \log\log n)^{m/2}} I^{(m)}(k_{nt}) \\ &= \frac{1}{n^{H-m/2}(2n \log\log n)^{m/2}} J^{(m)}(k_{nt}) \\ &\quad + \sum_{r=1}^{[m/2]} (-1)^r \frac{m!}{2^r r!(m-2r)!} \frac{J^{(m-2r)}(k_{nt}^{[r]})}{n^{H-m/2}(2n \log\log n)^{m/2}}.\end{aligned}$$

But

$$\frac{1}{n^{H-m/2}(2n \log \log n)^{m/2}} J^{(m)}(k_{nt})$$

$$= (-1)^m \int \underset{R^m}{\cdots} \int [D_1 \ldots D_m k_t(x_1,\ldots,x_m)] B_n(x_1)\ldots B_n(x_m) dx_1 \ldots dx_m$$

$$= \tau_k(B_n(t))$$

whereas, for $r > 1$,

$$\frac{J^{(m-2r)}(k_{nt}^{[r]})}{n^{H-m/2}(2n \log \log n)^{m/2}} = \frac{1}{(2 \log \log n)^r} \tau_{k^{[r]}}(B_n(t)) \to 0$$

as $n \to \infty$. Proposition 5.1 follows from Step 1 and the continuity of τ_k.

Step 3. The next step involves delicate approximations, whose details will be omitted. The goal is to approximate

$$Y(t) = \int \underset{R^m}{\cdots} \int f_t(x_1,\ldots,x_m) dB(x_1)\ldots dB(x_m)$$

by

$$Y^\varepsilon(t) = \int \underset{R^m}{\cdots} \int f_t^\varepsilon(x_1,\ldots,x_m) dB(x_1)\ldots dB(x_m)$$

such that

1) $f^\varepsilon \in F_m$

2) $f_t^\varepsilon(x_1,\ldots,x_m) = t^{H-m/2} f^\varepsilon(\frac{x_1}{t},\ldots,\frac{x_m}{t})$ for $t > 0$

3) $\int \underset{R^m}{\cdots} \int \left|f_{1+h}^\varepsilon - f_1^\varepsilon\right|^2 dx_1 \ldots dx_m < Ah^{2H}$ for $0 \leq h \leq 1$,

   where A is a constant independent of ε

4) $\int \underset{R^m}{\cdots} \int \left|f_1 - f^\varepsilon\right|^2 dx_1 \ldots dx_m < \varepsilon^2$

5) $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \left\|\frac{Y(nt)}{n^H(2 \log \log n)^{m/2}} - \frac{Y^\varepsilon(nt)}{n^H(2 \log \log n)^{m/2}}\right\|_\nu = 0$

   a.s.

We need 5). To prove it one uses 1), 2), 3) and 4) and the following lemma about tail probabilities of the sup of certain m-integral processes.

Lemma 5.4 (Mori and Oodairo, [19], Lemma 6.3). Let $u_t \in L^2(R^m)$, $t \geq 0$, be symmetric functions and let

$$W(t) = \int_{R^m} \cdots \int u_t(x_1,\dots,x_m)dB(x_1)\dots dB(x_m) \quad t \geq 0.$$

Further assume that there exists constants $A > 0$ and $0 < H < 1$, such that $W(t)$ satisfies

$$E(|W(t+h) - W(t)|^2)^{1/2} \leq Ah^H \quad t \geq 0, \quad 0 \leq h \leq 1$$

and

$$\sup_{0\leq t\leq 1} E(|W(t)|^2)^{1/2} < \frac{A}{2^{NH+1}} \quad \text{for some integer } N \geq 1.$$

Then there exist constants d and M dependent on A, H, m, N such that

$$P(\sup_{0\leq t\leq 1} |W(t)| > w) \leq 3 \exp(-M\, w^{2/m}), \ \forall\, w > d.$$

The proof of Lemma 5.4 is similar to that of Lemma 5.2 but it uses a sharper tail inequality for m-integral random variables obtained by Plikusas [22] (see also Mori-Oodaira [19], Lemma 6.1).

Step 4. Let $K \subset \mathcal{H} \subset C_\gamma(R)$ be as in (5.5). In order to avoid expressing the compact set as $\tau_{f^\varepsilon}(K)$, that is in terms of the approximating smooth kernels f_t^ε, introduce the mapping

$$\tilde{\tau}: \mathcal{H} \to C_\nu(R^+)$$
$$z \mapsto y_z$$

where

$$y_z(t) = \int_{R^m} \cdots \int f_t(x_1,\dots,x_m)\dot{z}(x_1)\dots\dot{z}(x_m)dx_1\dots dx_m.$$

By step 3, for $z \in K$,

$$\begin{aligned}\|\tilde{\tau}(z)-\tau_{f_\varepsilon}(z)\|_\nu &= \sup_{t>0} \frac{1}{\nu(t)} \int_{R^m} |f_t - f_t^\varepsilon|\,|\dot{z}(x_1)\dots\dot{z}(x_m)|dx_1\dots dx_m \\ &\leq \sup_{t>0} \frac{1}{\nu(t)}\, t^H \Big(\int_{R^m} |f_1 - f_1^\varepsilon|^2 dx_1\dots dx_m\Big)^{1/2} \|\dot{z}\|_2^{m/2} \\ &\leq \sup_{t>0} \frac{t^H}{\nu(t)}\, \varepsilon \leq \varepsilon.\end{aligned}$$

This implies that $\tilde{\tau}(K)$ is compact in $C_\nu(R^+)$ and can be used instead of $\tau_{f_\varepsilon}(K)$. Note finally that $\tilde{\tau}(K)$ is the K_f of Theorem 5.2.

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FOOTNOTES

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