A NEW FAMILY OF VARIANCE BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO DIFFERENT REPLICATIONS AND TWO DIFFERENT BLOCK SIZES

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SUMMARY

This paper represents a contribution to the theory of balanced incomplete block designs. The existence and construction of a new family of binary variance balanced incomplete block designs with two different block sizes and two different replications is described. This family of designs is balanced in the sense that, under the usual homoscedastic linear model and the standard intrablock analysis of the responses, every normalized estimable linear function of the treatment effects can be estimated with the same variance.

1. INTRODUCTION AND BACKGROUND MATERIAL

Let $\Omega = \{a_1,\cdots,a_v\}$ be a set of v treatments. By a block design with parameters $v,b,k_1,k_2,\cdots,k_b;r_1,r_2,\cdots,r_v$ denoted by $BD(v,b,r_1,r_2,\cdots,r_v;k_1,k_2,\cdots,k_b)$ on Ω we shall mean an allocation of elements of Ω one on each of the $m = \sum_{j=1}^{b} k_j$ experimental units arranged in b blocks or groups of experimental units of size $k_1,k_2,\cdots,k_b$ such that a_i is assigned into r_i experimental units. Thus, $\sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$. A block design is said to be <u>proper</u> if $k_j=k$, $j=1,2,\cdots,b$. A block design in which $r_i=r$, $i=1,2,\cdots,v$, holds will be said to be an <u>equireplicate</u> design. A block design is said to be <u>connected</u> when any two treatments can be connected by a chain consisting ulternatively of treatments and blocks, so that if treatment a_i and block B_j are any two consecutive members of the chain, then treatment a_i occurs in block B_j. Connectedness is an important and desirable property, as we shall see later. A block design is said to be <u>incomplete</u> if there exists a block which does not contain all the elements of Ω.

If we can assume that the position of experimental units in the blocks bears no information whatsoever, then the usual v x b indicence matrix $N = (n_{ij})$, where n_{ij} scores the number of experimental units in the j^{th} block receiving the i^{th} treatment, completely characterizes the combinatorial arrangement of the design. A block design is said to be an <u>n-ary</u> block design if the entries of N constitute n distinct integers. A block design is said to be <u>pairwise balanced</u> if $NN' = T + \lambda J$, where N' is the transpose of N, T a diagonal matrix, λ a scalar and J a matrix with unit entries wverywhere.

A binary (n_{ij}=0,1) incomplete block design with v treatments each replicated r times and with b blocks each of size k such that $NN' = (r-\lambda)I + \lambda J$, I the identity matrix of order v, will be called the classical balanced incomplete block design (CBIB). Any such design will be designated by BIB(v,b,r,k,λ). Literature on this family of designs is vast. One can construct many families of nonproper (i.e., not proper) binary (n_{ij}=0,1) pairwise balanced incomplete block designs via CBIB designs either by deletion of some treatments from the known CBIB designs or simply by putting together several different CBIB designs. Note that all these generated designs will be equireplicated. This fact has led several authors (see the references) to consider the construction of non-equireplicate pairwise balanced incomplete block designs.

Denote by R and K the diagonal matrices $\mathrm{dig}(r_1,r_2,\cdots,r_v)$ and $\mathrm{dig}(k_1,k_2,\cdots,k_b)$ respectively. The matrix $R - NK^{-1}N'$ is known as the coefficient matrix of the design and will be denoted by C. We shall shortly see the importance of the C matrix in block designs. Since each row (or column) of C adds to zero, the rank of C is at most v-1.

Lemma 1.1. A block design is connected if and only if the rank of its coefficient matrix is v-1.

Under the usual homoscedastic linear model and the standard intrablock analysis of the responses it is known that

Lemma 1.2. If the rank of $C = v - \alpha$, *a set of* α-1 *independent treatment contrasts is not estimable.*

A block design is said to be *variance balanced* if every normalized estimable linear function of the treatment effects can be estimated with the same variance. Vartak (1963) proved that

Theorem 1.1. A block design is variance balanced if and only if the nonzero characteristic roots of its coefficient matrix are equal.

In particular, if the design is connected the following is true.

Corollary 1.1. The following are equivalent.

(*i*) *A connected block design is variance balanced if and only if its coefficient matrix has* v-1 *equal characteristic roots other than zero* [*Rao* (*1958*)].

(*ii*) *A connected block design is variance balanced if and only if every treatment effect is estimated with the same variance and every two treatment effects with the same covariance* [*Atiqullah* (*1961*)].

(*iii*) *A connected block design is variance balanced if and only if its coefficient matrix is of the form* $c_1 I + c_2 J$ *where* I *is the identity matrix of order* v, J *a matrix with unit entries everywhere,* c_1 *and* c_2 *are scalars. It can be shown that* c_1 *is the nonzero characteristic root of* C *and* $c_2 = -c_1/v$ *where* v *is the number of treatments* [*Rao* (*1958*)].

Corollary 1.2. <u>If a design is pairwise (variance) balanced this does not imply that it is variance (pairwise) balanced.</u>

Example 1. Let $\Sigma_1 = \{1,2,\cdots,7\}$. Then, $B_1 = \{1,2,4\}$, $B_2 = \{2,3,5\}$, $B_3 = \{4,5\}$, $B_4 = \{3,4,6\}$, $B_5 = \{5,6\}$, $B_6 = \{6,2\}$, and $B_7 = \{1,3\}$ is pairwise balanced. However, it can be shown that this design is not variance balanced.

Example 2. Let $\Sigma_2 = \{1,2,3,4,5\}$. Then, the following design is variance balanced but not pairwise balanced: $B_1 = \{1,2,3,4\}$, $B_2 = \{1,2,3,4\}$, $B_3 = \{1,5\}$, $B_4 = \{2,5\}$, $B_5 = \{3,5\}$, and $B_6 = \{4,5\}$. Besides the classical balanced incomplete block designs, there are other families of incomplete block designs which are balanced in both senses, as illustrated in Example 3.

Example 3. Let $\Sigma_3 = \{1,2,3,4\}$. Then, $B_1 = \{1,2,3\}$, $B_2 = \{1,2,4\}$, $B_3 = \{1,3,4\}$, $B_4 = \{2,3,4\}$, $B_5 = \{1,2\}$, $B_6 = \{1,3\}$, $B_7 = \{1,4\}$, $B_8 = \{2,3\}$, $B_9 = \{2,4\}$ and $B_{10} = \{3,4\}$.

Rao (1958), among other results, has proved that

Theorem 1.2. <u>If a binary variance balanced design is proper, then it must be equireplicate.</u>

However, if we relax the binary and/or the proper condition the conclusion of the above theorem is no longer true in general. Rao (1958) gave an example of an equireplicate binary (n_{ij}=0,1) variance balanced design with unequal block sizes.

From the results given in John (1964), the following two theorems may be stated.

Theroem 1.3. *If* $v \equiv 0$ *or* 1 (*mod* 3) *there exists a family of trinary* (n_{ij}=0,1,2) *proper variance balanced incomplete block designs for* v+1 *treatments with two different replications*.

Theorem 1.4. *If there exists a* BIB(v,b,r,k,λ) *such that* $\lambda = k/2$, $r = k(v-1)/2(k-1)$, *then there exists a family of binary* (n_{ij}=0,1) *variance balanced incomplete block designs for* v+1 *treatments with two different replications and block sizes*.

2. THE RESULTS

Our object in this paper is to utilize the ideas of Das (1958), Federer (1961), John (1964), and Rao (1958) to construct a new family of binary variance balanced incomplete block designs with two different replications and two different block sizes. The method is as follows: Let $\Sigma = \{1,2,\cdots,v\}$, and let D_1 be a BIB(v,b,r,k,λ); let D_2 be a BIB(v,b_1,r_1,k_1,λ_1) on Σ. It is easy to see that $D_1 \cup D_2$ is pairwise balanced, but in general is not variance balanced. One can seek a condition(s) on the parameters of D_1 and D_2 such that $D_1 \cup D_2$ is also variance balanced. But then, the design will be equireplicate. Now augment every block of D_1 with a new treatment designated as "0". Call the resulting design D_1^*. Only under certain conditions is $D_1^* \cup D_2$ a binary (n_{ij}=0,1) variance balanced incomplete block design with two different replications and block sizes on $\Sigma \cup \{0\}$. A common trick in combinatorial theory is to take a constant, say α, times D_1^* and a constant, say β, times D_2 and try to find if there are solutions for α and β such that the incidence matrix of $\alpha D_1^* \cup \beta D_2$ is of the form $C_1 I + C_2 J$. After some algebra we see that N will have this structure if we let $\alpha = \lambda_2(k_1+1)/d$, $\beta = k_2(r_1-\lambda_2)/d$, where d is the greatest common divisor of $\lambda_2(k_1+1)$ and $k_2(r_1-\lambda_1)$. Note that for this design $C_1 = r_1\lambda_2(v+1)/d$ and $C_2 = -r_1\lambda_2/d$. Thus, we have proved:

Theorem 2.1. The existence of a BIB(v,b,r,k,λ) *and a* BIB$(v,b_1,r_1,k_1,\lambda_1)$ *implies the existence of a binary* $(n_{ij}=0,1)$ *variance balanced incomplete block design for* v+1 *treatments with two different replications and two different block sizes.*

Concluding remark. In reading the literature it is imperative that the reader understand the kind of balance the authors are describing, as there are several kinds of balance. Two forms of balance are used in the above. A paper discussing the various forms of balance used in the literature is being prepared for publication.

REFERENCES

ADHIKARY, B. (1965). On the properties and construction of balanced block designs with variable replications. *Calcutta Statist. Assoc. Bull.* *14*, 36-64.

ADHIKARY, B. (1965). A difference theorem for the construction of balanced block designs with variable replications. *Calcutta Statist. Assoc. Bull.* *14*, 167-170.

AGRAWAL, H. (1963). On balanced block designs with two different numbers of replications. *J. Indian Statist. Assoc.* *1*, 145-151.

AGRAWAL, H. & RAGHAVACHARI, R. (1964). On balanced block designs with three different numbers of replications. *Calcutta Statist. Assoc. Bull.* 13, 80-86.

ATIQULLAH, M. (1961). On a property of balanced designs. *Biometrika* *48*, 215-218.

CALINSKI, T. (1971). On some desirable patterns in block designs (with discussion). *Biometrics* *27*, 275-292.

DAS, M. N. (1958). On reinforced incomplete block designs. *J. Indian Soc. of Agri. Statist.* *10*, 73-76.

DAS, M. N. & RAO, S.V.S.P. (1968). On balanced n-ary designs. J. Indian Statist. Assoc. 6, 137-146.

FEDERER, W. T. (1961). Augmented designs with one-way elimination of heterogeneity. Biometrics 17, 447-472.

JOHN, P.W.M. (1964). Balanced designs with unequal numbers of replicates. Ann. Math. Statist. 35, 897-899.

MURTY, J. S. & DAS, M. N. (1967). Balanced n-ary block designs and their uses. J. Indian Statist. Assoc. 5, 73-82.

RAGHAVARAO, D. (1962). On balanced unequal block designs. Biometrika 49, 561-562.

RAO, M. B. (1966). A note on equi-replicate balanced designs with b=v. Calcutta Statist. Assoc. Bull. 15, 43-44.

RAO, S.V.S.P. & DAS, M. N. (1969). Incomplete weighing designs through balanced ternary designs. J. Indian Soc. of Agri. Statist. 21, 67-72.

RAO, V. R. (1958). A note on balanced designs. Ann. Math. Statist. 29, 290-294.

THOMPSON, W. A. (1956). A note on balanced incomplete block designs. Ann. Math. Statist. 27, 842-846.

TOCHER, K. D. (1952). The design and analysis of block experiments (with discussion). J. R. Statist. Soc. B, 14, 45-91.

VARTAK, M. N. (1963). Disconnected balanced designs. J. Indian Statist. Assoc. 1, 104-107.