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A UNIFIED INTERPRETATION OF  
SEVERAL COMBINATORIAL DUALITIES

by

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### *Abstract*

Several combinatorial structures exhibit a duality relation that yields interesting theorems, and, sometimes useful explanations or interpretations of results that do not concern duality explicitly. We present a common characterization of the duality relations associated with matroids, Sperner families, oriented matroids, and weakly oriented matroids. The same conditions characterize the orthogonality relation on certain families of vector spaces. This leads to a notion of abstract duality. An example of a combinatorial structure having no abstract duality comes from antimatroids (convex geometries).

In this paper we will examine some combinatorial structures (Sperner families, matroids, oriented matroids, and weakly oriented matroids) and some algebraic structures (vector spaces coordinatized over such fields as  $\mathbb{R}$ ,  $\mathbb{Q}$ , or  $GF(p^n)$ ,  $n$  odd) in which there are interesting duality or orthogonality relations. Although there are known resemblances among the duality relations in these different settings, a much stronger connection can be made. Specifically, we give a brief set of conditions that characterize each of the duality relations within its domain.

Section 1 outlines some of the main results. Section 2 contains brief introductions to the structures of interest, and introduces some notation. Section 3 offers a general notation into which all of the structures fit and defines the notion of *abstract duality*. The main result of the paper is that each of the structures noted above has a unique abstract duality. Section 4 concerns some fundamental properties of abstract dualities that are used in Section 5 to prove the uniqueness results. This study was initially prompted by the question of whether *antimatroids* (also known as convex geometries or anti-exchange closures) [6, 7, 8, 15] admit a relation reminiscent of the duality relation in matroids. In Section 6 we show how antimatroids fit in the notation of Section 3 and demonstrate that there is no abstract duality on these structures.

The research announcement [1] presents a brief discussion of this work, which was first presented in the Ph.D. dissertation of the second author [5]. Additional details, beyond those in Section 2, on the combinatorial structures studied here can be found in [2, 3, 4, 5, 9, 10, 11, 13, 19, 21]. Note that our attention is limited to finite structures, i.e., matroids, etc. with only finitely many elements.

# 1 INTRODUCTION

Each of the structures under consideration can be put in the following form. Let  $\mathcal{F}$  be a family in which each  $F \in \mathcal{F}$  is associated with a finite set  $E(F)$ . Assume further that there are operations  $/$  (contraction) and  $\backslash$  (deletion) that take each  $F \in \mathcal{F}$  and  $e \in E(F)$  to  $F/e \in \mathcal{F}$  and  $F \backslash e \in \mathcal{F}$ , respectively, having  $E(F/e) = E(F \backslash e) = E(F) - \{e\}$ . We are concerned with relations  $D : \mathcal{F} \rightarrow \mathcal{F}$  having such properties as:

$$(1.1) \quad E(D(F)) = E(F) \quad (\forall F \in \mathcal{F})$$

$$(1.2) \quad D(D(F)) = F \quad (\forall F \in \mathcal{F})$$

$$(1.3) \quad D(F/e) = D(F) \backslash e \text{ and } D(F \backslash e) = D(F)/e \quad (\forall F \in \mathcal{F}, e \in E(F)).$$

It is not difficult to construct trivial examples of this type. Given  $\mathcal{F}$  and, say, the contraction operation, one could take deletion to be the same as contraction and  $D$  to be the identity. However, there are several interesting and well-known structures in which there are established contraction and deletion operations. We show that under these operations, (1.1) - (1.3) characterize the established duality relation.

We will be examining some structures that are combinatorial and some that are algebraic. The first example, which concerns vector spaces coordinatized over  $GF(2)$ , although explicitly algebraic, has the flavor of some of the combinatorial examples as well.

For the moment, let  $K$  denote the binary field  $GF(2)$ . For a given finite set  $E$  let  $K^E$  denote the vector space of all maps from  $E$  to  $K$ . We denote by  $\mathcal{F}_K$  the family of all subspaces of vector spaces of the form  $K^E$ , where  $E$  ranges over all finite sets. For any  $F \in \mathcal{F}_K$ , the common domain of all maps in  $F$  is denoted by  $E(F)$ . The operations of contraction ( $/$ ) and deletion ( $\backslash$ ) of a coordinate  $e^* \in E(F)$  correspond to projection of  $F$  onto,

and intersection of  $F$  with, the hyperplane  $x(e^*) = 0$ . Specifically,  $F/e^*$  has  $E(F/e^*) = E(F) - \{e^*\}$  and  $F/e^* = \{x : E(F/e^*) \rightarrow K \mid \exists x' \in F \text{ s.t. } x'(e) = x(e), \forall e \in E(F/e^*)\}$ , while  $F \setminus e^*$  has  $E(F \setminus e^*) = E(F) - \{e^*\}$  and  $F \setminus e^* = \{x : E(F \setminus e^*) \rightarrow K \mid \exists x' \in F \text{ s.t. } x'(e^*) = 0 \text{ and } x'(e) = x(e), \forall e \in E(F \setminus e^*)\}$ . The orthogonality relation  $D_K$  having  $D_K(F) = \{y : E(F) \rightarrow K \mid y \cdot x = 0, \forall x \in F\}$  for every  $F \in \mathcal{F}_K$ , satisfies (1.1) - (1.3). Later we will examine families  $\mathcal{F}_K$ , as above, for arbitrary fields  $K$ , so we denote by  $\mathcal{F}_{GF(2)}$  and  $D_{GF(2)}$  what had been denoted by  $\mathcal{F}_K$  and  $D_K$  in the case of  $K = GF(2)$ .

(1.4) **Theorem.** For  $\mathcal{F} = \mathcal{F}_{GF(2)}$ , the orthogonality relation  $D_{GF(2)}$  is the unique  $D : \mathcal{F} \rightarrow \mathcal{F}$  satisfying (1.1) - (1.3).

There is a natural bijection between  $\mathcal{F}_{GF(2)}$  and  $\mathcal{F}_\delta$ , the family of finite binary matroids. Under this bijection,  $/$  and  $\setminus$  act like ordinary matroid contraction and deletion, and orthogonality acts like matroid duality. This gives another interpretation of (1.4). Let  $D_\delta$  denote the matroid duality relation restricted to  $\mathcal{F}_\delta$ .

(1.5) **Corollary.** For  $\mathcal{F} = \mathcal{F}_\delta$ , the relation  $D_\delta$  is the unique  $D : \mathcal{F} \rightarrow \mathcal{F}$  satisfying (1.1) - (1.3).

In the next example we expand the viewpoint from  $\mathcal{F}_\delta$  to  $\mathcal{F}_M$ , the family of all matroids  $F$  on a finite set of elements  $E(F)$ . We take  $/$  and  $\setminus$ , respectively, to be the usual matroid contraction and deletion operations (see Section 2).

(1.6) **Theorem.** For  $\mathcal{F} = \mathcal{F}_M$ , the matroid duality relation  $D_M$  is the unique  $D : \mathcal{F} \rightarrow \mathcal{F}$  satisfying (1.1) - (1.3).

Another example comes from *Sperner families* (also called *clutters*). Let  $\mathcal{F}_S$  be the family of all Sperner families  $F$  on a finite set  $E(F)$ , take  $/$  and  $\backslash$ , respectively, to be the usual contraction and deletion operations in this setting, and let  $D_S$  be the blocking duality relation on  $\mathcal{F}_S$  (see Section 2).

(1.7) **Theorem.** For  $\mathcal{F} = \mathcal{F}_S$ ,  $D_S$  is the unique  $D : \mathcal{F} \rightarrow \mathcal{F}$  satisfying (1.1) - (1.3).

G. Kalai pointed out to us that Theorem 1.6 is a strengthening of a result of J. P. S. Kung [16]. Kung proved the version of Theorem 1.6 in which one imposes the additional restriction that  $D$  preserves isomorphisms,

$$(1.8) \quad F_1 = \psi(F_2) \Rightarrow D(F_1) = \psi(D(F_2))$$

$$(\forall F_1, F_2 \in \mathcal{F} \text{ and isomorphisms } \psi \text{ from } F_1 \text{ to } F_2).$$

An isomorphism  $\psi$ , as in (1.8), is a bijection from  $E(F_2)$  to  $E(F_1)$  that takes  $F_2$  to  $F_1$ . It is evident that (1.8) holds, not only for  $D_M$  on  $\mathcal{F}_M$  but also for  $D_S$  on  $\mathcal{F}_S$ , and for  $D_\delta$  on  $\mathcal{F}_\delta$ , or, equivalently, for  $D_{GF(2)}$  on  $\mathcal{F}_{GF(2)}$ . Moreover, (1.1) - (1.3) together with (1.8) characterize the standard duality relations  $D_O$  on  $\mathcal{F}_O$ , the family of all oriented matroids  $F$  on a finite set  $E(F)$ , and  $D_W$  on  $\mathcal{F}_W$ , the family of all weakly oriented matroids  $F$  on a finite set  $E(F)$ . Here we again take  $/$  and  $\backslash$  to be the usual contraction and deletion operations in these settings.

(1.9) **Theorem.** (a) For  $\mathcal{F} = \mathcal{F}_O$ ,  $D_O$  is the unique  $D : \mathcal{F} \rightarrow \mathcal{F}$  satisfying (1.1) - (1.3) and (1.8).

(b) For  $\mathcal{F} = \mathcal{F}_W$ ,  $D_W$  is the unique  $D : \mathcal{F} \rightarrow \mathcal{F}$  satisfying (1.1) - (1.3) and (1.8)

The inclusion of (1.8) with (1.1) - (1.3) also enables us to extend Theorem 1.4 to vector spaces coordinatized over fields other than  $GF(2)$ . First note that for any field  $K$ , all of

$\mathcal{F}_K$ ,  $/$ ,  $\backslash$ , and  $D_K$  remain well defined. The condition (1.8) on preserving isomorphisms takes the following form here. Let  $F_1, F_2 \in \mathcal{F}_K$  and let  $\psi$  be a bijection from  $E(F_2)$  to  $E(F_1)$  such that  $F_2 = \{x \circ \psi \mid x \in F_1\}$ . Then we require that  $D(F_2) = \{y \circ \psi \mid y \in D(F_1)\}$ .

(1.10) *Theorem.* For any field  $K$  having no nontrivial involutory automorphisms, (e.g.,  $K = \mathbb{R}, \mathbb{Q}$ , or  $GF(p^n)$ , for  $p$  prime and  $n$  odd) conditions (1.1), (1.2), (1.3), and (1.8) characterize the orthogonality relation  $D_K$  on  $\mathcal{F}_K$ .

The results outlined in this section indicate that properties (1.1), (1.2), (1.3), and (1.8) characterize the duality relations in each of several different examples, when we take the operations  $/$  and  $\backslash$  in (1.3) to be the standard contraction and deletion operations in the relevant example. In Section 3 we will also show that there is a common description of the contraction and deletion operations across these examples. This leads to the definition of an abstract duality relation. First we will give some background on the combinatorial examples.

## 2 BACKGROUND AND NOTATION

Although each of the combinatorial structures of interest can be described in many different, but equivalent, ways, the properties that presently concern us do not depend on the form of the description. For instance, matroids can be defined in terms of circuits, independent sets, bases, hyperplanes, rank, closure, etc.. It is sometimes convenient to think of the contraction and deletion operations and the duality relation in the notation of one particular description. However, the properties in Theorem 1.6 do not depend, even in their form, on whether we think of matroids  $F$  in terms of circuits  $\mathcal{C}(F)$ , or independent sets

$\mathcal{I}(F)$ , or any other characterizing feature. For example, the duality relation  $D_M$  is a map from  $\mathcal{F}_M$  to  $\mathcal{F}_M$  whose form, as a map, is independent of whether we choose to characterize it in terms of circuits, or independent sets, etc..

In the introductory remarks about matroids and the other combinatorial structures we will first give characterizations in terms of circuits. It will then be helpful to give equivalent characterizations in terms of what we call “spans” of circuits. The reason for this is that it will unify the results, in that the contraction operation, as well as the deletion operation, in the different examples can all be given the same description in terms of their spans.

### ***Matroids***

Let  $E$  be a finite set and let  $\mathcal{C}$  be a set of nonempty subsets of  $E$  satisfying

$$(2.1) \quad C_1, C_2 \in \mathcal{C}, \quad C_1 \subseteq C_2 \Rightarrow C_1 = C_2;$$

$$(2.2) \quad C_1, C_2 \in \mathcal{C}, \quad C_1 \neq C_2, \quad e \in C_1 \cap C_2 \Rightarrow \exists C_3 \in \mathcal{C} \text{ s.t. } C_3 \subseteq (C_1 \cup C_2) - \{e\}.$$

Then  $\mathcal{C}$  is the set of *circuits* of a matroid  $F$  on  $E$ . Given a matroid  $F$ , we denote by  $E(F)$  the set of elements on which it is defined, and by  $\mathcal{C}(F)$  the set of circuits of  $F$ . Given  $E(F)$  and  $\mathcal{C}(F)$ , it is easy to determine  $\mathcal{I}(F)$ , the independent sets of  $F$ ,  $\mathcal{B}(F)$ , the bases of  $F$ ,  $\rho_F$ , the rank function of  $F$ , etc.. It is also easy to determine the set of unions of circuits of  $F$ , which we denote by  $\mathcal{S}(F)$  and call the *span* of  $F$ . Like  $\mathcal{I}(F)$ ,  $\mathcal{B}(F)$ ,  $\rho_F$ , etc.,  $\mathcal{S}(F)$  together with  $E(F)$  determines the matroid  $F$ . Later it will be convenient to work with  $\mathcal{V}(F)$ , the set of  $(0,1)$  – incidence vectors of  $x \in \mathcal{S}(F)$ .



The matroid duality relation  $D_M$  associates with each matroid  $F$  on a finite set  $E(F)$  a matroid  $D_M(F)$  with  $E(D_M(F)) = E(F)$  and  $\mathcal{C}(D_M(F))$  the collection of setwise minimal subsets of  $E(F)$  among the nonempty sets in

$$(2.3) \quad \{B \subseteq E(F) : |B \cap C| \neq 1, \forall C \in \mathcal{C}(F)\}.$$

It is an easy exercise to show that  $\mathcal{P}(D_M(F))$  is the entire set (2.3).

The *contraction* (/) and *deletion* (\) operations on matroids can be defined as follows. For  $F$  a matroid and  $e \in E(F)$ ,  $F/e$  and  $F \setminus e$  are matroids having  $E(F/e) = E(F \setminus e) = E(F) - \{e\}$ ,  $\mathcal{C}(F/e)$  is the collection of setwise minimal subsets of  $E$  among the nonempty members of  $\{C - \{e\} \mid C \in \mathcal{C}(F)\}$ , and  $\mathcal{C}(F \setminus e)$  is  $\{C \in \mathcal{C}(F) \mid e \notin C\}$ . The equivalent definition of these operations in terms of the span is even simpler:  $\mathcal{P}(F/e) = \{X - \{e\} \mid X \in \mathcal{P}(F)\}$  and  $\mathcal{P}(F \setminus e) = \{X \in \mathcal{P}(F) \mid e \notin X\}$ , which can be verified easily.

Two matroids  $F_1$  and  $F_2$  are *isomorphic* if there is a bijection  $\psi : E(F_2) \rightarrow E(F_1)$  such that  $\mathcal{C}(F_1) = \psi(\mathcal{C}(F_2)) = \{\bigcup_{e \in C} \{\psi(e)\} \mid C \in \mathcal{C}(F_2)\}$  or, equivalently,  $\mathcal{P}(F_1) = \psi(\mathcal{P}(F_2))$ . It is well known that the matroid duality relation  $D_M$  on  $\mathcal{F}_M$ , the family of all finite matroids, has properties (1.1), (1.2), (1.3), and (1.8) (see [19, 21]).

### ***Sperner families***

Removing (2.2), the second of the two circuit axioms for matroids, gives a standard characterization of *Sperner families* or *clutters*. Let  $\mathcal{C}$  be a set of subsets of a finite set  $E$  satisfying

$$(2.4) \quad C_1, C_2 \in \mathcal{C}, \quad C_1 \subseteq C_2 \Rightarrow C_1 = C_2.$$

Then we will say that  $\mathcal{C}$  is the set of *circuits* of a Sperner family on  $E$ . Given  $F \in \mathcal{F}_S$ , the family of all finite Sperner families, we denote by  $E(F)$  and  $\mathcal{C}(F)$ , respectively, the set of elements on which  $F$  is defined and the set of circuits of  $F$ . It is in terms of what we have called circuits that Sperner families are usually described, but, just as for matroids, one could give a different, but equivalent, description in terms of, say, independent sets - those subsets of  $E(F)$  containing no circuit. It will be useful for us to describe each  $F \in \mathcal{F}_S$  in terms of its *span*,  $\mathcal{P}(F) = \{X \subseteq E(F) \mid \exists Y \subseteq X, Y \in \mathcal{C}(F)\}$ . Later we will work with  $\mathcal{V}(F)$ , the set of (0,1)-incidence vectors of the subsets  $X \subseteq E(F)$  in  $\mathcal{P}(F)$ . The *blocking duality* relation  $D_S$  on  $\mathcal{F}_S$  takes each  $F \in \mathcal{F}_S$  to  $D_S(F) \in \mathcal{F}_S$  with  $E(D_S(F)) = E(F)$  and  $\mathcal{C}(D_S(F))$  the collection of setwise minimal subsets in

$$(2.5) \quad \{Y \subseteq E(F) \mid Y \cap X \neq \emptyset, \quad \forall X \in \mathcal{C}(F)\}.$$

It is easy to see that  $\mathcal{P}(D_S(F))$  is the entire set (2.5).

The contraction and deletion operations on  $\mathcal{F}_S$  can be defined as follows. For  $F \in \mathcal{F}_S$  and  $e \in E(F)$ ,  $F/e \in \mathcal{F}_S$  and  $F \setminus e \in \mathcal{F}_S$  have  $E(F/e) = E(F \setminus e) = E(F) - \{e\}$ ,  $\mathcal{C}(F/e)$  is the collection of setwise minimal subsets of  $E$  among  $\{C - \{e\} \mid C \in \mathcal{C}(F)\}$ , and  $\mathcal{C}(F \setminus e)$  is  $\{C \in \mathcal{C}(F) \mid e \notin C\}$ . The equivalent definition in terms of the span is, again, somewhat simpler:  $\mathcal{P}(F/e) = \{X - \{e\} \mid X \in \mathcal{P}(F)\}$  and  $\mathcal{P}(F \setminus e) = \{X \in \mathcal{P}(F) \mid e \notin X\}$ . This is easy to verify. Note that when described in terms of the span, the contraction and deletion operations on  $\mathcal{F}_S$  take the same form as on  $\mathcal{F}_M$ . Indeed, if we describe the contraction and deletion operations in terms of  $\mathcal{V}(F)$  rather than  $\mathcal{P}(F)$ , their form in  $\mathcal{F}_M$  and  $\mathcal{F}_S$  would be identical with their form in the earlier algebraic example  $\mathcal{F}_{GF(2)}$ . This will also be true for the examples that appear later in this section.

The definition of *isomorphisms* between  $F_1, F_2 \in \mathcal{F}_S$  is identical to the definition for  $\mathcal{F}_M$ . It is clear that the blocking duality  $D_S$  on  $\mathcal{F}_S$  preserves isomorphisms, i.e., it satisfies (1.8), and preserves the ground set (1.1). It is also well-known that  $D_S$  satisfies (1.2) and (1.3).

### ***Oriented Matroids***

While matroids can be regarded to be set systems that abstract linear dependence over a field, *oriented matroids* can be regarded to be *signed-set systems* that abstract signed linear dependence over an ordered field. A signed subset  $X$  of  $E$  is a pair  $X = (X^+, X^-)$  of disjoint subsets of  $E$ . We can also think of  $X$  in terms of its *signed incidence vector*, the map from  $E$  to  $\{-, 0, +\}$  that takes  $e \in X^+$  to  $+$ ,  $e \in X^-$  to  $-$ , and  $e \in E - (X^+ \cup X^-)$  to  $0$ . Given  $X$  a signed subset of  $E$ , let  $\underline{X}$  denote the *underlying set*  $\underline{X} = X^+ \cup X^-$ , let  $-X$  denote the signed subset of  $E$  having  $(-X)^+ = X^-$  and  $(-X)^- = X^+$ , and, for  $e \in E$ , let  $X - \{e\}$  denote the signed subset of  $E$  having  $(X - \{e\})^+ = X^+ - \{e\}$  and  $(X - \{e\})^- = X^- - \{e\}$ . Denote the set of all signed subsets of  $E$  by  $\{-, 0, +\}^E$ .

Like matroids, oriented matroids can be axiomatized in many different but equivalent ways (see [3, 10]). One axiomatization, in terms of signed circuits, follows. Let  $\mathcal{C}$  be a set of signed subsets of a finite set  $E$  such that the following properties hold:

$$(2.6) \quad C \in \mathcal{C} \Rightarrow \underline{C} \neq \emptyset \text{ and } -C \in \mathcal{C};$$

$$(2.7) \quad C_1, C_2 \in \mathcal{C}, \underline{C}_1 \subseteq \underline{C}_2 \Rightarrow C_1 = \pm C_2;$$

$$(2.8) \quad C_1, C_2 \in \mathcal{C}, C_1 \neq -C_2, e \in C_1^+ \cap C_2^- \Rightarrow \exists C_3 \in \mathcal{C} \\ \text{s.t. } C_3^+ \subseteq (C_1^+ \cup C_2^+) - \{e\}, C_3^- \subseteq (C_1^- \cup C_2^-) - \{e\}.$$

Then  $\mathcal{C}$  is the set of (signed) *circuits* of an *oriented matroid* on  $E$ . We denote by  $\mathcal{F}_O$  the family of all finite oriented matroids. Given an oriented matroid  $F \in \mathcal{F}_O$ , we will denote by  $E(F)$  and  $\mathcal{C}(F)$ , respectively, the set of elements on which  $F$  is defined, and the set of (signed) circuits of  $F$ . Again in this setting, it will be helpful to associate with each  $F \in \mathcal{F}_O$  a *span*  $\mathcal{P}(F)$  that determines  $F$  uniquely. Let  $\mathcal{P}(F)$  be the set of all *conformal unions* of circuits of  $F$  (see [3]), i.e.,  $\mathcal{P}(F)$  is the set of signed subsets  $X$  of  $E(F)$  that arise as  $X^+ = \bigcup \{Z^+ \mid Z \in \mathcal{U}\}$ ,  $X^- = \bigcup \{Z^- \mid Z \in \mathcal{U}\}$ , where  $\mathcal{U} \subseteq \mathcal{C}(F)$  is such that  $Z_1, Z_2 \in \mathcal{U}$  implies  $(Z_1^+ \cap Z_2^-) \cup (Z_1^- \cap Z_2^+) = \emptyset$ . Later we will work with  $\mathcal{V}(F)$ , the set of signed incidence vectors of all signed sets  $X \in \mathcal{P}(F)$ .

The oriented matroid duality relation  $D_O$  associates with each  $F \in \mathcal{F}_O$  an oriented matroid  $D_O(F) \in \mathcal{F}_O$  with  $E(D_O(F)) = E(F)$  and  $\mathcal{C}(D_O(F))$  the collection of all setwise minimal signed subsets of  $E(F)$  among the nonempty members of

$$(2.9) \quad \{Y \in \{-, 0, +\}^{E(F)} \mid (X^+ \cap Y^+) \cup (X^- \cap Y^-) = \emptyset \iff \\ (X^- \cap Y^+) \cup (X^+ \cap Y^-) = \emptyset, \forall X \in \mathcal{C}(F)\}.$$

That is,  $\mathcal{C}(D_O(F))$  consists of those  $Y$  in (2.9) such that  $\underline{Y} \neq \emptyset$  and there is no  $Y'$  in (2.9) with  $\emptyset \neq \underline{Y'} \subset \underline{Y}$ . It follows from [3] that the span  $\mathcal{P}(D_O(F))$  is the entire set (2.9).

Contraction and deletion in  $\mathcal{F}_O$  can be defined as follows. Let  $F \in \mathcal{F}_O$  and let  $e \in E(F)$ . Then  $F/e \in \mathcal{F}_O$  and  $F \setminus e \in \mathcal{F}_O$  have  $E(F/e) = E(F \setminus e) = E(F) - \{e\}$ ,  $\mathcal{C}(F/e)$  is the collection of setwise minimal signed subsets of  $E$  among the nonempty  $\{X - \{e\} \mid X \in \mathcal{C}(F)\}$ , and  $\mathcal{C}(F \setminus e) = \{X \in \mathcal{C}(F) \mid e \notin \underline{X}\}$ . In terms of the span  $\mathcal{P}(F)$ , we have  $\mathcal{P}(F/e) = \{X - \{e\} \mid X \in \mathcal{P}(F)\}$  and  $\mathcal{P}(F \setminus e) = \{X \in \mathcal{P}(F) \mid e \notin \underline{X}\}$  (see [3]).

Two oriented matroids  $F_1, F_2 \in \mathcal{F}_O$  are *isomorphic* if there is a bijection  $\psi : E(F_2) \rightarrow E(F_1)$  such that  $\mathcal{C}(F_1) = \psi(\mathcal{C}(F_2)) = \{(\psi(X^+), \psi(X^-)) \mid X \in \mathcal{C}(F_2)\}$  or, equivalently,  $\mathcal{P}(F_1) = \psi(\mathcal{P}(F_2))$ . It is easy to show that  $D_O$  preserves isomorphisms (1.8) and preserves the ground set (1.1). That it also satisfies (1.2) and (1.3) follows from [3].

### **Weakly Oriented Matroids**

Weakly oriented matroids [2, 13] constitute a natural intermediate between matroids and oriented matroids. Among several equivalent definitions is one in terms of signed circuits. Given  $X \in \{-, 0, +\}^E$  and  $e \in E$ , let  $\text{sgn}(e, X)$  be  $+1$  if  $e \in X^+$ ,  $-1$  if  $e \in X^-$ , and  $0$  if  $e \notin X$ . Let  $\mathcal{C}$  be a set of signed subsets of a finite set  $E$  such that

$\mathcal{C}$  is the set of circuits of a matroid on  $E$ ;

$C \in \mathcal{C} \Rightarrow -C \in \mathcal{C}$ ;

$\forall C_1, C_2 \in \mathcal{C}, C_1 \neq -C_2, e \in C_1^+ \cap C_2^-$

(i)  $f \in (C_1^+ - C_2^-) \cup (C_1^- - C_2^+) \Rightarrow \exists C_3 \in \mathcal{C} \text{ s.t. } f \in \underline{C}_3 \subseteq (\underline{C}_1 \cup \underline{C}_2) - \{e\}$ ;

(ii)  $\exists e_1 \in \underline{C}_1 - \underline{C}_2, e_2 \in \underline{C}_2 - \underline{C}_1, C_3 \in \mathcal{C} \text{ s.t.}$

$\underline{C}_3 \subseteq (\underline{C}_1 \cup \underline{C}_2) - \{e\} \text{ and } \text{sgn}(e_1, C_3)\text{sgn}(e_2, C_3) = \text{sgn}(e_1, C_1)\text{sgn}(e_2, C_2)$ .

Then  $\mathcal{C}$  is the set of (signed) circuits of a weakly oriented matroid on  $E$ . The symbol  $\mathcal{F}_W$  denotes the family of all weakly oriented matroids, and for  $F \in \mathcal{F}_W$ ,  $E(F)$  and  $\mathcal{C}(F)$ , respectively, denote the ground set on which  $F$  is defined, and the set of circuits of  $F$ .

Some additional notation will ease the definition of the span of  $F \in \mathcal{F}_W$ . Given  $X \in \{-, 0, +\}^E$  and  $e, f \in E$  let  $\text{sgn}(e, f, X) = \text{sgn}(e, X)\text{sgn}(f, X)$ . A signed subset  $X$  of  $E$  is a *consistent union* of a set  $\mathcal{U} \subseteq \{-, 0, +\}^E$  if

$$\begin{aligned}
\underline{X} &= \cup \{ \underline{Z} \mid Z \in \mathcal{U} \}; \\
\{ \cup Z^+ \mid Z \in \mathcal{U} \} - \{ \cup Z^- \mid Z \in \mathcal{U} \} &\subseteq X^+ \subseteq \{ \cup Z^+ \mid Z \in \mathcal{U} \}, \\
\{ \cup Z^- \mid Z \in \mathcal{U} \} - \{ \cup Z^+ \mid Z \in \mathcal{U} \} &\subseteq X^- \subseteq \{ \cup Z^- \mid Z \in \mathcal{U} \}; \\
\text{sgn}(x, y, X) \neq 0 &\Rightarrow \exists Z_1, Z_2 \in \mathcal{U} \text{ with } x \in \underline{Z}_1, y \in \underline{Z}_2 \\
&\text{s.t. either } \text{sgn}(x, y, Z_1) \neq -\text{sgn}(x, y, X) \text{ or } \text{sgn}(x, y, Z_2) \neq -\text{sgn}(x, y, X).
\end{aligned}$$

The *span*  $\mathcal{P}(F)$  of  $F \in \mathcal{F}_W$  is the set of all consistent unions of circuits of  $F$ ;  $\mathcal{P}(F)$  can also be defined in terms of series classes in deletion minors of  $F$ . In the next section we will work with  $\mathcal{V}(F)$ , the set of signed incidence vectors of  $X \in \mathcal{P}(F)$ .

The weakly oriented matroid duality relation  $D_W$  (see [2, 13]) takes each  $F \in \mathcal{F}_W$  to  $D_W(F) \in \mathcal{F}_W$  such that  $E(D_W(F)) = E(F)$  and  $\mathcal{C}(D_W(F))$  is the collection of setwise minimal signed subsets among the nonempty members of

$$\begin{aligned}
(2.10) \quad & \{ Y \in \{ -, 0, + \}^{E(F)} \mid (X^+ \cap Y^+) \cup (X^- \cap Y^-) = \emptyset \iff \\
& (X^- \cap Y^+) \cup (X^+ \cap Y^-) = \emptyset, \forall X \in \mathcal{C}(F) \text{ s.t. } |\underline{X} \cap \underline{Y}| \leq 2 \}.
\end{aligned}$$

The characterizations of contraction and deletion for  $\mathcal{F}_W$  in terms of circuits are exactly the same as for  $\mathcal{F}_O$ . That the characterization in terms of spans are the same -

$$\mathcal{P}(F/e) = \{ X - \{e\} \mid X \in \mathcal{P}(F) \} \text{ and } \mathcal{P}(F \setminus e) = \{ X \in \mathcal{P}(F) \mid e \notin \underline{X} \}$$

is proved in [5], as is the result that  $\mathcal{P}(D_W(F))$  is the entire set (2.10). Isomorphism of weakly oriented matroids is defined as for oriented matroids. Again it is easy to see that  $D_W$  on  $\mathcal{F}_W$  satisfies (1.8) and (1.1), and, it is proved in [2,13] that  $D_W$  also satisfies (1.2) and (1.3).

### 3 ABSTRACT DUALITIES

In Section 2 we presented for each of the families  $\mathcal{F} = \mathcal{F}_M, \mathcal{F}_S, \mathcal{F}_O, \mathcal{F}_W$ , a description of the  $F \in \mathcal{F}$  in terms of the ground set  $E(F)$  and span  $\mathcal{P}(F)$ . Recall that the duality relations on these families and the contraction operations take a somewhat simpler form when described in terms of spans, rather than, say, circuits. More importantly, note that the descriptions of the contraction operations in terms of spans are essentially identical, while the descriptions in terms of circuits differ noticeably. This can be carried further by a simple change in notation, which gives the combinatorial families a closer resemblance to the algebraic families  $\mathcal{F}_K$ . In place of the description of  $F \in \mathcal{F}$  in terms of  $E(F)$  and  $\mathcal{P}(F)$ , consider the same information presented in the form of  $\mathcal{V}(F)$ , the set of incidence vectors (for  $\mathcal{F}_M$  and  $\mathcal{F}_S$ ) or signed incidence vectors (for  $\mathcal{F}_O$  and  $\mathcal{F}_W$ ) of all  $X \in \mathcal{P}(F)$ . The contraction operations for  $\mathcal{F}_M, \mathcal{F}_S, \mathcal{F}_O$ , and  $\mathcal{F}_W$  have a single characterization in terms of  $\mathcal{V}(F)$  - contraction of an element  $e^*$  in the domain of the maps  $\mathcal{V}(F)$  is just projection onto  $x(e^*) = 0$ , the same as for  $\mathcal{F}_K$ ; deletion is intersection with  $x(e^*) = 0$ , again, as for  $\mathcal{F}_K$ . The significance of a common description of the contraction and deletion operations across all of the examples is twofold. First it facilitates a common approach to proving the five characterizations embodied in Theorems 1.6, 1.7, 1.9a, 1.9b, and 1.10. Second, it gives a cleaner unification of the five dualities under examination, since condition (1.3) can be regarded to be a universal condition applied in the same way to each of the five, rather than a set of five distinct conditions of similar form. So, for example, instead of working with the family  $\mathcal{F}_M$  of all matroids on finitely many elements, we will work with the family  $\mathcal{G}_M = \{ \mathcal{V}(F) \mid F \in \mathcal{F}_M \}$ . The operations of contraction and deletion, and the matroid duality relation  $D_M$ , behave in exactly the same way on  $\mathcal{F}_M$  and  $\mathcal{G}_M$ , under the natural bijection between  $\mathcal{F}_M$  and  $\mathcal{G}_M$ . Therefore, to prove Theorem 1.6, it is sufficient to prove

the analogous result on  $\mathcal{G}_M$ . Similarly, we will deal with  $\mathcal{F}_S, \mathcal{F}_O$ , and  $\mathcal{F}_W$  implicitly, through the consideration of the families  $\mathcal{G}_S, \mathcal{G}_O$ , and  $\mathcal{G}_W$ , in which the role of each member  $F$  of the relevant family  $\mathcal{F}_i$  is played by  $G \in \mathcal{G}_i$  with  $G = \mathcal{V}(F)$ ; we denote by  $E(G)$  the common domain of the maps in  $G$ .

We will now formalize this notation. Given  $T$  a nonempty set with a distinguished element  $t^*$ , let  $\mathcal{G}$  be a family of sets  $G$  of maps from  $E(G)$ , a finite set, to  $T$ . Suppose further that  $\mathcal{G}$  is closed under the operations of *contraction* ( $/$ ) and *deletion* ( $\backslash$ ), which are defined as follows. For any  $G \in \mathcal{G}$  and  $e^* \in E(G)$ ,

$$\begin{aligned}
 E(G/e^*) &= E(G \backslash e^*) = E - \{e^*\}, \\
 (3.0) \quad G/e^* &= \{x : E(G/e^*) \rightarrow T \mid \exists x' \in G \text{ with } x'(e) = x(e), \forall e \in E(G/e^*)\}, \\
 G \backslash e^* &= \{x : E(G \backslash e^*) \rightarrow T \mid \exists x' \in G \text{ with } x'(e^*) = t^*, \\
 &\quad \text{and } x'(e) = x(e), \forall e \in E(G \backslash e^*)\}.
 \end{aligned}$$

A function  $D : \mathcal{G} \rightarrow \mathcal{G}$  is a *weak abstract duality* on  $\mathcal{G}$  if it preserves the ground set -

$$(3.1) \quad E(D(G)) = E(G) \quad (\forall G \in \mathcal{G}),$$

it is an involution -

$$(3.2) \quad D(D(G)) = G \quad (\forall G \in \mathcal{G}),$$

and it interchanges contraction and deletion -

$$(3.3a) \quad D(G/e) = D(G) \backslash e \quad (\forall G \in \mathcal{G}, e \in E(G)),$$

$$(3.3b) \quad D(G \backslash e) = D(G)/e \quad (\forall G \in \mathcal{G}, e \in E(G)).$$



An isomorphism from  $G_1 \in \mathcal{G}$  to  $G_2 \in \mathcal{G}$  is a bijection  $\phi : E(G_2) \rightarrow E(G_1)$  such that  $G_2 = \{x \circ \phi \mid x \in G_1\}$ . A weak abstract duality on  $\mathcal{G}$  is an *abstract duality* if it preserves isomorphisms -

$$(3.4) \quad D(G_2) = \{y \circ \phi \mid y \in D(G_1)\}$$

$$(\forall G_1, G_2 \in \mathcal{G}, \text{ and } \phi \text{ an isomorphism from } G_1 \text{ to } G_2).$$

<u>Example</u>	<u><math>\mathcal{G}</math></u>	<u><math>T</math></u>	<u><math>t^*</math></u>
<i>Matroids</i>	$\mathcal{G}_M = \{ \mathcal{V}(F) \mid F \in \mathcal{F}_M \}$	$\{0,1\}$	0
<i>Sperner families</i>	$\mathcal{G}_S = \{ \mathcal{V}(F) \mid F \in \mathcal{F}_S \}$	$\{0,1\}$	0
<i>Oriented Matroids</i>	$\mathcal{G}_O = \{ \mathcal{V}(F) \mid F \in \mathcal{F}_O \}$	$\{ -, 0, + \}$	0
<i>Weakly Oriented Matroids</i>	$\mathcal{G}_W = \{ \mathcal{V}(F) \mid F \in \mathcal{F}_W \}$	$\{ -, 0, + \}$	0
<i>Vector spaces coordinatized over a field <math>K</math></i>	$\mathcal{G}_K = \mathcal{F}_K$	$K$	0

Table 1. The Unifying Framework

Table 1 indicates how the earlier examples fit into this notation. It is important to note that: (1) there is a natural bijection between  $\mathcal{F}_i$  and  $\mathcal{G}_i$  ( $i = M, S, O, W, K$ ); (2) for each of these  $\mathcal{F}_i$  and the associated  $D_i$  the natural bijection from  $\mathcal{F}_i$  to  $\mathcal{G}_i$  permits us to regard  $D_i$  to (also) be a function from  $\mathcal{G}_i$  to  $\mathcal{G}_i$ ; and (3) contraction and deletion in  $\mathcal{G}_i$ , as defined by (3.0) correspond, under the natural bijection from  $\mathcal{F}_i$  to  $\mathcal{G}_i$ , exactly to the usual contraction and deletion in  $\mathcal{F}_i$ , as described in Section 2.

It should be evident that each of  $\mathcal{G}_M, \mathcal{G}_S, \mathcal{G}_O, \mathcal{G}_W$ , and  $\mathcal{G}_K$  (for arbitrary field  $K$ ) has at least one abstract duality, namely  $D_M, D_S, D_O, D_W$ , and  $D_K$ , respectively. In the next two sections we will prove

- (3.5) **Theorem.** (a)  $D_S$  is the unique weak abstract duality on  $\mathcal{G}_S$ ;  
 (b)  $D_M$  is the unique weak abstract duality on  $\mathcal{G}_M$ ;  
 (c)  $D_O$  is the unique abstract duality on  $\mathcal{G}_O$ ;  
 (d)  $D_W$  is the unique abstract duality on  $\mathcal{G}_W$ ;  
 (e) for every field  $K$  having no nontrivial involutory automorphism  $D_K$  is the unique abstract duality on  $\mathcal{G}_K$ .

Theorem 3.5 implies Theorems 1.6, 1.7, 1.9a, 1.9b, and 1.10.

#### 4 RECONSTRUCTIBILITY

In this section we present some general results about weak abstract dualities and abstract dualities. These results facilitate a common approach to the proofs of the five parts of Theorem 3.5, based on a reconstructibility property that we will describe now.

For each of the five examples in Table 1 of Section 3, it is easy to show that for all  $G \in \mathcal{G}$  with  $E(G)$  sufficiently large,  $G$  is determined uniquely by its set of *simple minors*:

$$(4.1) \quad \{G/e \mid e \in E(G)\} \cup \{G \setminus e \mid e \in E(G)\}.$$

Such a  $G$  is called *reconstructible*. It is not difficult to see why one might expect reconstructibility. For  $G \in \mathcal{G}$  having  $|E(G)| \geq 2$ , let  $\mathcal{M}$  be the set (4.1) of simple minors of  $G$ . First note that  $E(G)$  is just the union of  $E(G')$  over  $G' \in \mathcal{M}$ . Also note that: (a) for

any  $e \in E(G)$  there are exactly two  $G' \in \mathcal{M}$  having  $E(G') = E(G) - \{e\}$ , namely  $G/e$  and  $G \setminus e$ ; (b) it is easy to distinguish between these two because  $G \setminus e \subseteq G/e$ . Next note that if we extend each  $x \in G \setminus e$  to a map  $x' : E(G) \rightarrow T$  having  $x'(e) = t^*$ , then  $x' \in G$ . Typically  $G$  can be generated unambiguously from these  $x'$ , except in the degenerate cases, where the structure of  $G$  can be determined from the contraction minors. For example, if  $G \in \mathcal{G}_{GF(2)}$ , then the only ambiguity arises when for every choice of  $e \in E(G)$ ,  $G \setminus e$  contains only the zero vector  $(0, \dots, 0)$ . In this circumstance,  $G$  could be either  $\{(0, \dots, 0)\}$  or  $\{(0, \dots, 0), (1, \dots, 1)\}$ , but the contraction minors  $G/e$  immediately reveal which of the two possibilities holds. Note that if  $|E(G)| = 1$ , then  $G$  cannot be reconstructed, since we cannot even recover  $E(G)$ .

Let  $r(\mathcal{G})$  be the least integer  $r$  such that every  $G \in \mathcal{G}$  having  $|E(G)| \geq r$  is reconstructible;  $r(\mathcal{G}) = \infty$  if there exists no such integer  $r$ . It is easy to verify that  $r(\mathcal{G}_M) = r(\mathcal{G}_S) = 2$ ;  $r(\mathcal{G}_O) = r(\mathcal{G}_W) = 3$ ;  $r(\mathcal{G}_{GF(2)}) = 2$ ; and  $r(\mathcal{G}_K) = 3$  for other  $K$ . We will say that  $G \in \mathcal{G}$  is *small* if  $|E(G)| < r(\mathcal{G})$ .

(4.2) **Theorem.** *If  $D_1$  and  $D_2$  are distinct weak abstract dualities on  $\mathcal{G}$ , then  $D_1(G) \neq D_2(G)$  for a small  $G \in \mathcal{G}$ .*

The proof of Theorem (4.2) uses the following lemma. For  $q \in \mathbb{Z}_+$ , denote by  $\mathcal{G}^q$  the subfamily  $\{G \in \mathcal{G} : q \geq |E(G)|\}$  of  $\mathcal{G}$ .

(4.3) **Lemma.** *Suppose  $D_1$  and  $D_2$  are weak abstract dualities on  $\mathcal{G}$  and for some positive integer  $q$ : (i)  $D_1(G) = D_2(G)$ ,  $\forall G \in \mathcal{G}^q$ ; (ii) every  $\hat{G} \in \mathcal{G} \setminus \mathcal{G}^q$  is reconstructible. Then  $D_1 = D_2$ .*

*Proof.* Suppose that  $\hat{G} \in \mathcal{G}$  has  $|E(\hat{G})| = q + 1$ . Then  $D_1(\hat{G})$  is determined uniquely by

$$(4.4) \quad \mathcal{M} = \{D_1(\hat{G}) \setminus e \mid e \in E(\hat{G})\} \cup \{D_1(\hat{G})/e \mid e \in E(\hat{G})\}.$$

By (3.3) and (4.4)

$$(4.5) \quad \mathcal{M} = \{D_1(\hat{G}/e) \mid e \in E(\hat{G})\} \cup \{D_1(\hat{G} \setminus e) \mid e \in E(\hat{G})\}.$$

Now each simple minor of  $\hat{G}$  in (4.5) has  $q$  elements, so by (i)

$$(4.6) \quad \mathcal{M} = \{D_2(\hat{G}/e) \mid e \in E(\hat{G})\} \cup \{D_2(\hat{G} \setminus e) \mid e \in E(\hat{G})\}.$$

Again using (3.3) with (4.6) we get

$$(4.7) \quad \mathcal{M} = \{D_2(\hat{G} \setminus e) \mid e \in E(\hat{G})\} \cup \{D_2(\hat{G})/e \mid e \in E(\hat{G})\}.$$

Since  $E(D_1(\hat{G}))$  and  $E(D_2(\hat{G}))$  both have  $q + 1$  elements, by (ii)  $D_1(\hat{G})$  and  $D_2(\hat{G})$  are reconstructible from their sets of simple minors, which, by (4.4) and (4.7), are identical. Therefore,  $D_1(\hat{G}) = D_2(\hat{G})$ , for all  $\hat{G} \in \mathcal{G}^{q+1}$ , i.e., (i) holds with  $q$  replaced by  $q + 1$ . Since  $\mathcal{G} - \mathcal{G}^{q+1} \subseteq \mathcal{G} - \mathcal{G}^q$ , (ii) also holds with  $q$  replaced by  $q + 1$ . By induction,  $D_1 = D_2$ . ■

*Proof of Theorem 4.2.* If  $r(\mathcal{G}) = \infty$ , then there is nothing to prove, since all  $G \in \mathcal{G}$  are small. So assume that  $r(\mathcal{G})$  is finite. Let  $D_1$  and  $D_2$  be weak abstract dualities on  $\mathcal{G}$ , and, suppose that the conclusion of (4.2) fails. Then  $D_1(G) = D_2(G)$  for all  $G \in \mathcal{G}^q$ , with  $q = r(\mathcal{G}) - 1$ . Conditions (i) and (ii) of Lemma 4.3 both hold for this choice of  $q$ . Hence,  $D_1 = D_2$ , contradicting the hypothesis that  $D_1$  and  $D_2$  are distinct. ■

(4.8) **Corollary.** *If all (weak) abstract dualities  $D_1$  and  $D_2$  on  $\mathcal{G}$  agree on all small  $G \in \mathcal{G}$ , then  $\mathcal{G}$  has at most one (weak) abstract duality.*

Establishing Theorem 3.5 now reduces to establishing the hypothesis of (4.8) in each of the five families. In the combinatorial cases,  $\mathcal{G}_M, \mathcal{G}_S, \mathcal{G}_O, \mathcal{G}_W$ , this is facilitated by the modest size of  $r(\mathcal{G})$  and by the following lemma. For any finite set  $E$  let  $\mathcal{G}(E)$  be the subfamily  $\{G \in \mathcal{G} \mid E(G) = E\}$ .

(4.9) **Lemma.** Let  $D$  be a weak abstract duality on  $\mathcal{G}$ , let  $E$  be a finite set, and let  $e^* \in E$  and  $G^* \in \mathcal{G}(E - \{e^*\})$  be fixed. Then the restriction of  $D$  to  $\mathcal{G}_1 = \{G \in \mathcal{G}(E) \mid G \setminus e^* = G^*\}$  is a bijection from  $\mathcal{G}_1$  to  $\mathcal{G}_2 = \{G \in \mathcal{G}(E) \mid G/e^* = D(G^*)\}$ .

*Proof:* Suppose  $G \in \mathcal{G}_1$ . Then  $G \setminus e^* = G^*$ , so  $D(G \setminus e^*) = D(G^*)$ , and by (3.3)  $D(G \setminus e^*) = D(G)/e^*$ . Therefore,  $D(G) \in \mathcal{G}_2$ , so  $D$  restricted to  $\mathcal{G}_1$  has its range in  $\mathcal{G}_2$ . Also, it must be one-to-one, since  $D$  is one-to-one on all of  $\mathcal{G}$ , by (3.2). Now suppose that  $G \in \mathcal{G}_2$ , which implies that  $G/e^* = D(G^*)$ . Then  $D(G/e^*) = D(D(G^*))$ , which is  $G^*$  by (3.2). Furthermore, by (3.3),  $D(G/e^*) = D(G) \setminus e^*$ , so  $D(G) \setminus e^* = G^*$ , which implies that  $D(G) \in \mathcal{G}_1$ . By (3.2)  $G$  is the image under  $D$  of  $D(G)$ , so  $D$  restricted to  $\mathcal{G}_1$  is onto  $\mathcal{G}_2$ . ■

In the next section we will complete the proofs of the five parts of Theorem 3.5, by establishing the hypothesis of Corollary 4.8 separately for each of the five cases.

## 5 UNIQUENESS PROOFS

The details of the proofs of each of the parts of Theorem 3.5 are presented in this section. First we introduce some additional notation. For each map  $x \in G \in \mathcal{G}$ , a family as above, and each  $e \in E(G)$  let  $\hat{x}_e$  denote the *restriction* of  $x$  to the set  $E(G) - \{e\}$ , that is,  $\hat{x}_e : E - \{e\} \rightarrow T$  has  $\hat{x}_e(e') = x(e')$ , for all  $e' \in E(G) - \{e\}$ . For an element  $e \notin E(G)$  let  $\mathcal{G}_e(G)$  denote the *extension* to  $E(G) \cup \{e\}$  of the maps in  $G$ ,

$$\mathcal{G}_e(G) = \{x : E(G) \cup \{e\} \rightarrow T \mid x(e) = t^*, \text{ and } \exists y \in G \ni x(e') = y(e'), \forall e' \in E(G)\}.$$

The symbol  $\underline{x}$  denotes the *support* of a map  $x \in G$ , that is the set  $\{e \in E(G) \mid x(e) \neq t^*\}$ , where  $t^*$  is the distinguished element of the target set  $T$  of the maps in  $G$ . For any  $t \in T$  let  $t^E$  denote the map from  $E$  to  $T$  having  $t^E(e) = t$ , for all elements  $e \in E$ . In particular,  $0^E$  denotes the zero map (vector) on  $E$ . The unique map from the empty set to any  $T$  is denoted

by  $0^\emptyset$ . Some additional notation will be helpful in the cases where  $T = \{-, 0, +\}$ . Given a map  $x : E \rightarrow \{-, 0, +\}$ , we write  $x^+$  for the set  $\{e \in E(G) \mid x(e) = +\}$  and  $x^-$  for the set  $\{e \in E(G) \mid x(e) = -\}$ . We write  $-x$  to denote the map from  $E$  to  $\{-, 0, +\}$  having  $(-x)^+ = x^-$  and  $(-x)^- = x^+$ .

### *Matroids and Sperner Families*

The following lemmas establish the hypothesis of Corollary 4.8 for the families  $\mathcal{G}_M$  and  $\mathcal{G}_S$  of matroids and Sperner families, respectively.

(5.1) **Lemma.** (a) Every  $G \in \mathcal{G}_M$  with  $|E(G)| \geq 2$  is reconstructible.

(b) All weak abstract dualities on  $\mathcal{G}_M$  agree on the subfamily  $\mathcal{G}_M^1$ .

*Proof:* (a) Choose  $G \in \mathcal{G}_M$  having  $|E(G)| \geq 2$ . Let  $G' = \{x : E(G) \rightarrow \{0,1\} \mid \underline{x} = \bigcup_{e \in E(G)} \underline{x}_e, \text{ where } x_e \in \mathcal{G}_e(G \setminus e)\}$ . If  $G' \neq \{0^{E(G)}\}$ , then  $G = G'$ . Suppose  $G' = \{0^{E(G)}\}$ . If  $G/e = \{0^{E(G)-\{e\}}\}$  for any  $e \in E(G)$ , then  $G = \{0^{E(G)}\}$ ; otherwise  $G = \{0^{E(G)}, 1^{E(G)}\}$ .

(b) Let  $D : \mathcal{G}_M \rightarrow \mathcal{G}_M$  be a weak abstract duality. There is only one matroid with  $E(G) = \emptyset$  and, by (3.1), it must be its own image under  $D$ . For  $E = \{e'\}$ , there are two elements of  $\mathcal{G}_M(E)$ :  $G = \{(0), (1)\}$  and  $G' = \{(0)\}$ . Here each map is denoted by the image of the element  $e'$ . For  $E = \{e, e'\}$ , there are five elements of  $\mathcal{G}_M(E')$ :

$$\begin{array}{ll} G_1 = \{(0,0)\} & G_4 = \{(0,0), (1,0), (0,1), (1,1)\}. \\ G_2 = \{(0,0), (1,0)\} & G_5 = \{(0,0), (1,1)\} \\ G_3 = \{(0,0), (0,1)\} & \end{array}$$

Here each map is denoted by the ordered pair  $(x(e), x(e'))$ . Let  $\mathcal{Y} = \{G_i \mid G_i/e = G\}$ ,  $\mathcal{Y}' = \{G_i \mid G_i/e = G'\}$ ,  $\mathcal{Z} = \{G_i \mid G_i \setminus e = G\}$ , and  $\mathcal{Z}' = \{G_i \mid G_i \setminus e = G'\}$ . Note that  $|\mathcal{Y}| = 3$ ,  $|\mathcal{Y}'| = 2$ ,  $|\mathcal{Z}| = 2$ , and  $|\mathcal{Z}'| = 3$ . Lemma 4.9 implies that  $D : \mathcal{Y} \rightarrow \mathcal{Z}'$ ,

$D : \mathcal{Y}' \rightarrow \mathcal{Z}$ ,  $D : \mathcal{Z} \rightarrow \mathcal{Y}'$ , and  $D : \mathcal{Z}' \rightarrow \mathcal{Y}$ , so  $D(G) = G'$ . This is independent of the choice of  $e'$ , so  $D$  is uniquely determined on  $\mathcal{G}_M^1$ . ■

(5.2) **Lemma.** (a) Every  $G \in \mathcal{G}_S$  with  $|E(G)| \geq 2$  is reconstructible.

(b) All weak abstract dualities on  $\mathcal{G}_S$  agree on the subfamily  $\mathcal{G}_S^1$ .

*Proof:* (a) Choose  $G \in \mathcal{G}_S$  with  $|E(G)| \geq 2$ . Let  $G' = \{x : E(G) \rightarrow \{0,1\} \mid \exists e \in E(G) \text{ and } y \in \mathcal{E}_e(G \setminus e) \text{ with } y \subseteq x\}$ . If  $G' \neq \emptyset$ , then  $G = G'$ . If  $G' = \emptyset$ , and  $G/e = \emptyset$  for any  $e \in E(G)$ , then  $G = \emptyset$ . Otherwise  $G = \{1^{E(G)}\}$ .

(b) There are two elements of  $\mathcal{G}_S^0$ ,  $G_{0,1} = \emptyset$  which contains no maps, and  $G_{0,2} = \{0^\emptyset\}$ , which contains only the empty map. For  $E = \{e\}$ , there are three elements of  $\mathcal{G}_S(E)$ :  $G_{1,1} = \emptyset$ ,  $G_{1,2} = \{(0),(1)\}$ , and  $G_{1,3} = \{(1)\}$  respectively. The minors are given by

$$\begin{array}{lll} G_{1,1}/e = \emptyset & G_{1,2}/e = \{0^\emptyset\} & G_{1,3}/e = \{0^\emptyset\} \\ G_{1,1} \setminus e = \emptyset & G_{1,2} \setminus e = \{0^\emptyset\} & G_{1,3} \setminus e = \emptyset. \end{array}$$

Let  $D : \mathcal{G}_S \rightarrow \mathcal{G}_S$  be a weak abstract duality. Since Lemma 4.9 implies that  $D$  takes  $\{G_{1,i} \mid G_{1,i}/e = \emptyset\}$  to  $\{G_{1,i} \mid G_{1,i} \setminus e = \{0^\emptyset\}\}$ , we must have  $D(G_{0,1}) = G_{0,2}$ . With property 3.3 and the minors listed above, this implies that  $D(G_{1,1}) = G_{1,2}$ . By (3.2),  $D(G_{1,2}) = G_{1,1}$ , and, therefore,  $D(G_{1,3}) = G_{1,3}$ . This is independent of the choice of  $e$ . ■

### ***Oriented Matroids and Weakly Oriented Matroids***

To show that  $\mathcal{G}_O$ , the family of oriented matroids, has a unique abstract duality, we demonstrate reconstructibility and then consider the subfamily  $\mathcal{G}_O(E)$  for small sets  $E$ . The proof for  $\mathcal{G}_W$ , the family of weakly oriented matroids, is similar.

(5.3) **Lemma:** (a) Every  $G \in \mathcal{G}_O$  with  $|E(G)| \geq 3$  is reconstructible.

(b) All abstract dualities on  $\mathcal{G}_O$  agree on the subfamily  $\mathcal{G}_O^2$ .

*Proof:* (a) Let  $G'$  be the set of maps from  $E(G)$  to  $\{-, 0, +\}$  whose signed support is the conformal union of the signed supports of maps in  $\cup \{\mathcal{G}_e(G \setminus e) \mid e \in E(G)\}$ . If  $G'$  contains only the zero map, then  $G = \{x : E(G) \rightarrow \{-, 0, +\} \mid \hat{x}_e \in G/e, \forall e \in E(G)\}$ . If  $G' \neq \{0^E\}$ ,  $G = G'$ . Note that this construction holds only for  $|E(G)| \geq 3$ . For  $|E| = 2$ , there are distinct elements of  $\mathcal{G}_O(E)$  having the same set of minors.

(b) There is only one oriented matroid on  $\emptyset$  and it must be its own image under any abstract duality  $D : \mathcal{G}_O \rightarrow \mathcal{G}_O$ . For each singleton  $\{e\}$  there are two elements of  $\mathcal{G}_O(\{e\})$ :  $G(e) = \{0^{\{e\}}\}$  and  $G'(e) = \{0^{\{e\}}, +^{\{e\}}, -^{\{e\}}\}$ . For  $E = \{e_1, e_2\}$ , there are six elements of  $\mathcal{G}_O(E)$ :

$$\begin{array}{ll} G_1 = \{(0,0)\}; & G_4 = \{(0,0), \pm(+,0), \pm(+,+), \pm(0,+), \pm(+,-)\} \\ G_2 = \{(0,0), \pm(+,0)\}; & G_5 = \{(0,0), \pm(+,+)\}; \\ G_3 = \{(0,0), \pm(0,+)\}; & G_6 = \{(0,0), \pm(+,-)\}. \end{array}$$

Here each map is expressed as the image of the ordered pair  $(e_1, e_2)$ . Note that  $|\{G_i \mid G_i/e_1 = G'(e_1)\}| = 4$ ,  $|\{G_i \mid G_i/e_1 = G(e_1)\}| = 2$ ,  $|\{G_i \mid G_i \setminus e_1 = G'(e_1)\}| = 2$  and  $|\{G_i \mid G_i \setminus e_1 = G(e_1)\}| = 4$ . By Lemma 4.9,  $D(G(e_1)) = G'(e_1)$  and  $D(G'(e_1)) = G(e_1)$ . Similarly,  $D(G(e_2)) = G'(e_2)$  and  $D(G'(e_2)) = G(e_2)$ . This, together with (3.2), determines the value of  $D$  on the first four oriented matroids:

$$(5.4) \quad D(G_1) = G_4; \quad D(G_2) = G_3; \quad D(G_3) = G_2; \quad D(G_4) = G_1.$$

The remaining two oriented matroids,  $G_5$  and  $G_6$ , have the same set of minors, and either (i)  $D(G_5) = G_6$  and  $D(G_6) = G_5$ , or (ii)  $D(G_5) = G_5$  and  $D(G_6) = G_6$ . Let  $E' = \{e_1, e_2, e_3\}$ , and  $G = \{(0,0,0), (+,+,+), (-,-,-)\} \in \mathcal{G}_O(E')$ . For each  $i = 1,2,3$ ,  $G/e_i$  is isomorphic to  $G_5$  and  $G \setminus e_i$  is isomorphic to  $G_1$ . If (ii) holds, by (5.4) and property (3.4) there exists  $G' = D(G) \in \mathcal{G}_O$  such that



$$(5.5) \quad G'/e_i = D(G \setminus e_i) \text{ is isomorphic to } G_4, \quad (i = 1, 2, 3);$$

$$(5.6) \quad G' \setminus e_i = D(G/e_i) \text{ is isomorphic to } G_5, \quad (i = 1, 2, 3).$$

By (5.6),  $G' = \{ \pm (+, +, +), \pm (+, +, 0), \pm (+, 0, +), \pm (0, +, +), (0, 0, 0) \}$ , as in the proof of (a). This contradicts (5.5) and (ii) cannot hold. Therefore, (i) holds,  $D$  is determined on  $\mathcal{G}_O^2$ , and the lemma is proved.  $\blacksquare$

For oriented matroids the unique abstract duality corresponds to oriented matroid duality. The analogous result holds for the family  $\mathcal{G}_W$  of weakly oriented matroids.

(5.7) **Lemma:** (a) Every  $G \in \mathcal{G}_W$  with  $|E(G)| \geq 3$  is reconstructible.

(b) All abstract dualities on  $\mathcal{G}_W$  agree on the subfamily  $\mathcal{G}_W^2$ .

*Proof:* (a) Let  $G'$  be the set of maps from  $E(G)$  to  $\{-, 0, +\}$  whose signed support is a consistent union of the signed supports of maps in  $\cup \{\mathcal{E}_e(G \setminus e) \mid e \in E(G)\}$ . From here the proofs of (a) and (b) follow the proof of Lemma 5.3.

The proof of Lemma 5.3 used the isomorphism-preserving property (3.4) to determine the behavior of  $D$  on  $\mathcal{G}_O^2$ . The families  $\mathcal{G}_O$  and  $\mathcal{G}_W$  have many weak abstract dualities. We will show that each of these weak dualities arises from the usual (weakly) oriented matroid duality by “reversing signs” on some set of elements. We begin by formalizing the notion of reversing signs. If  $x$  is a map from a finite set  $E$  to  $\{-, 0, +\}$ , and  $S$  is a set, then  $\bar{S}x$  is the map from  $E$  to  $\{-, 0, +\}$  having  $(\bar{S}x)^+ = (x^+ - S) \cup (x^- \cap S)$  and  $(\bar{S}x)^- = (x^- - S) \cup (x^+ \cap S)$ . The map  $\bar{S}x$  is said to be obtained from  $x$  by *reversing signs* on  $S$ , and for any collection of maps  $G$  from  $E$  to  $\{-, 0, +\}$ , we write  $\bar{S}G$  for the collection  $\{\bar{S}x \mid x \in G\}$ . Note that we do not require that  $S \subseteq E$ .

To characterize the weak abstract dualities on  $\mathcal{G}_O$  we first note that reversing signs on a set  $S$  commutes with oriented matroid duality, weakly oriented matroid duality, and with contraction and deletion in both  $\mathcal{G}_O$  and  $\mathcal{G}_W$ .

(5.8) **Lemma.** (a) If  $G \in \mathcal{G}_O$  and  $S$  is a set, then  $D_O(\bar{S}G) = \bar{S}(D_O(G))$ .

(b) If  $G \in \mathcal{G}_W$  and  $S$  is a set, then  $D_W(\bar{S}G) = \bar{S}(D_W(G))$ .

(c) If  $G \in \mathcal{G}_O$  and  $e \in E(G)$  then  $\bar{S}(G \setminus e) = (\bar{S}G) \setminus e$  and  $\bar{S}(G/e) = (\bar{S}G)/e$ .

(d) If  $G \in \mathcal{G}_W$  and  $e \in E(G)$  then  $\bar{S}(G \setminus e) = (\bar{S}G) \setminus e$  and  $\bar{S}(G/e) = (\bar{S}G)/e$ .

Part (a) is proved by applying (2.9) and reversing signs on  $S$ . The proof of part (b) uses (2.10), while (c) and (d) are immediate results of the appropriate definitions.

For each set  $U$ , let  $\underline{\mathcal{G}}_O(U) = \{G \in \mathcal{G}_O \mid E(G) \subseteq U\}$ , be the family of oriented matroids with ground sets in  $U$  and let  $\underline{\mathcal{G}}_W(U) = \{G \in \mathcal{G}_W \mid E(G) \subseteq U\}$ . If  $S \subseteq U$ , then a weak abstract duality on  $\underline{\mathcal{G}}_O(U)$  (respectively,  $\underline{\mathcal{G}}_W(U)$ ) is obtained from  $D_O$  ( $D_W$ ) by reversing signs on  $S$ . Furthermore, every weak abstract duality on  $\underline{\mathcal{G}}_O(U)$  ( $\underline{\mathcal{G}}_W(U)$ ) is of this form.

(5.9) **Theorem.** A function  $D : \underline{\mathcal{G}}_O(U) \rightarrow \underline{\mathcal{G}}_O(U)$  is a weak abstract duality if and only if there exists  $S \subseteq U$  such that, for every  $G \in \underline{\mathcal{G}}_O(U)$ ,  $D(G) = \bar{S}D_O(G)$ .

*Proof:* Let  $D : \underline{\mathcal{G}}_O(U) \rightarrow \underline{\mathcal{G}}_O(U)$  and suppose that for some set  $S \subseteq U$ ,  $D(G) = \bar{S}D_O(G)$ , for every  $G \in \underline{\mathcal{G}}_O(U)$ . It is clear that the function  $D$  preserves ground sets, (3.1). For each  $G \in \underline{\mathcal{G}}_O(U)$  we have  $D(D(G)) = D(\bar{S}D_O(G)) = \bar{S}D_O(\bar{S}D_O(G)) = G$ , so  $D$  satisfies (3.2). For every element  $e \in E(G)$ , we have  $D(G/e) = \bar{S}(D_O(G/e)) = (\bar{S}D_O(G))/e = D(G)/e$  and  $D(G \setminus e) = \bar{S}(D_O(G \setminus e)) = (\bar{S}D_O(G)) \setminus e = D(G) \setminus e$ . Therefore,  $D$  satisfies (3.1), (3.2), and (3.3), and is a weak abstract duality for the family  $\underline{\mathcal{G}}_O(U)$ .

The proof of the converse is more difficult. Let  $D : \underline{\mathcal{G}}_O(U) \rightarrow \underline{\mathcal{G}}_O(U)$  be a weak abstract duality. First note that since any  $G \in \underline{\mathcal{G}}_O(U)$  having  $|E(G)| \geq 3$  is reconstructible, it is sufficient to determine the behavior of  $D$  on the subfamily  $\underline{\mathcal{G}}_O^2(U)$ . For each two element set  $E \subseteq U$ , there are six  $G \in \underline{\mathcal{G}}_O(U)$  having  $E(G) = E$ . Properties (3.1) - (3.3) determine  $D$  on four of these oriented matroids, as in the proof of Lemma 5.3. It is only on the remaining elements of  $\underline{\mathcal{G}}_O^2(U)$  that the function  $D$  can differ from  $D_O$ . For each two element set  $E \subseteq U$ , let  $G(E) = \{(0,0), \pm(+, +)\} \in \underline{\mathcal{G}}_O(U)$  and  $G'(E) = \{(0,0), \pm(+, -)\} \in \underline{\mathcal{G}}_O(U)$ . Either  $D(G(E)) = G'(E)$  or  $D(G(E)) = G(E)$ . The set  $S$  on which signs are reversed is determined from the behavior of  $D$  on these elements of  $\underline{\mathcal{G}}_O^2(U)$ .

For each element  $e \in U$ , let  $S(e) = \{e' \in U \mid D(G(\{e, e'\})) = G(\{e, e'\})\}$ . It is clear that  $f \in S(e)$  if and only if  $e \in S(f)$ . By proving the following series of claims, we show that  $\overline{S(e)}G = \overline{S(f)}G$ , for all  $G \in \underline{\mathcal{G}}_O(U)$ , and any choice of  $e, f \in U$ .

*Claim 5.9.1 :* If  $e, f$ , and  $g$  are distinct,  $f \in S(e)$  and  $g \in S(f)$ , then  $g \notin S(e)$ .

Suppose that the claim fails for the elements  $e, f$ , and  $g$ . Let  $\tilde{E} = \{e, f, g\}$  and consider

$$\tilde{G} = \{x : \tilde{E} \rightarrow \{-, 0, +\} \mid x^+ = \tilde{E} \text{ or } x^- = \tilde{E}\} \cup \{0^{\tilde{E}}\} \in \underline{\mathcal{G}}_O(U).$$

Note that  $\tilde{G}/e = G(\{f, g\})$ ,  $\tilde{G}/f = G(\{e, g\})$ , and  $\tilde{G}/g = G(\{e, f\})$ . Since  $f \in S(e)$ ,  $g \in S(f)$ , and  $g \notin S(e)$ , (3.3) implies that  $D(\tilde{G}) \setminus e = D(\tilde{G}/e) = G(\{f, g\})$ ,  $D(\tilde{G}) \setminus f = D(\tilde{G}/f) = G(\{e, g\})$ , and  $D(\tilde{G}) \setminus g = D(\tilde{G}/g) = G(\{e, f\})$ . This is impossible, since reconstructing  $D(\tilde{G})$  from its minors, as in the proof of Lemma 5.3, gives  $D(\tilde{G}) = \{x : \tilde{E} \rightarrow \{-, 0, +\} \mid x^+ = \emptyset \text{ or } x^- = \emptyset, \text{ and } |x| \neq 1\}$ , which is not in  $\underline{\mathcal{G}}_O$ .

*Claim 5.9.2:* If  $f \in U - S(e)$ , then  $S(e) = S(f)$ .

Suppose that  $f \in U - S(e)$ ,  $g \in S(e) - S(f)$ , and let  $\tilde{G}$  be as above. By property (3.3),  $D(\tilde{G}) \setminus e = D(\tilde{G}/e) = G'(\{f, g\})$ ,  $D(\tilde{G}) \setminus f = D(\tilde{G}/f) = G(\{e, g\})$ , and  $D(\tilde{G}) \setminus g = D(\tilde{G}/g) =$

$G'(\{e, f\})$ . Again, reconstructing  $D(\tilde{G})$  from its minors yields a collection of maps that is not in  $\mathcal{G}_O$ . Therefore  $S(e) \subseteq S(f)$ . Since  $e \in U - S(f)$ , the opposite containment also holds.

*Claim 5.9.3:* If  $S(e) \cap S(f) \neq \emptyset$  then  $S(e) = S(f)$ .

Let  $g \in S(e) \cap S(f)$ , so  $f \in S(g)$  and  $g \in S(e)$ . Applying Claim 1, with the roles of  $f$  and  $g$  interchanged,  $f \notin S(e)$ , and by Claim 2,  $S(e) = S(f)$ .

*Claim 5.9.4:* For any pair,  $e, f \in U$ , either  $S(e) = S(f)$ , or  $S(e) = U - S(f)$ .

If  $S(e) \neq S(f)$ , then by Claim 3,  $S(f) \subseteq U - S(e)$ . Suppose  $g \in U - (S(e) \cup S(f))$ . Then by Claim 2  $S(e) = S(g) = S(f)$ .

*Claim 5.9.5:* For every  $G \in \mathcal{G}_O(U)$ , and every  $e, f \in U$ ,  $\overline{S(e)}G = \overline{S(f)}G$ .

If  $S(e) = S(f)$  then the result is clear. Otherwise  $S(e) = U - S(f)$ . Since each  $G \in \mathcal{G}_O(U)$ , has  $G = \overline{U}G$ , we have  $\overline{S(e)}G = \overline{S(e)}(\overline{U}G) = \overline{U - S(e)}G = \overline{S(f)}G$ .

Choose an element  $e^0 \in U$  and let  $S = S(e^0)$ . We use induction on  $|E(G)|$  to show that  $D(G) = \overline{S}G$ , for all  $G \in \mathcal{G}_O(U)$ . If  $|E(G)| \leq 1$ , the result is obvious, since reversing signs has no effect. For  $|E(G)| = 2$ , the definition of  $S(e)$  implies that for either choice of  $e \in E(G)$ ,  $D(G) = \overline{S(e)}D_O(G)$ . By Claim 5, we have  $D(G) = \overline{S}D_O(G)$ . Suppose that the result holds whenever  $|E(G)| \leq k$ , for some  $k \geq 2$ , and let  $G \in \mathcal{G}_O(U)$  with  $|E(G)| = k + 1$ . By (3.3), for each  $e \in E(G)$ ,  $D(G)/e = D(G \setminus e)$ . By the induction hypothesis, property (3.3) applied to  $D_O$ , and Lemma 5.8,  $D(G)/e = D(G \setminus e) = \overline{S}D_O(G \setminus e) = \overline{S}(D_O(G)/e) = (\overline{S}(D_O(G)))/e$ . Similarly,  $D(G) \setminus e = (\overline{S}D_O(G)) \setminus e$ , for all  $e \in E(G)$ . Since  $D(G)$  and  $\overline{S}D_O(G)$  have the same simple minors and  $|E(D(G))| \geq 3$ , Lemma 5.3.a implies that  $D(G) = \overline{S}D_O(G)$ . This completes the induction and the proof of the theorem. ■

It is easy to extend the proof of Theorem 5.9 to the family of weakly oriented matroids with ground sets in  $U$ .

(5.10) **Theorem.** *A function  $D : \mathcal{G}_W(U) \rightarrow \mathcal{G}_W(U)$  is a weak abstract duality if and only if there exists  $S \subseteq U$  such that, for every  $G \in \mathcal{G}_W(U)$ ,  $D(G) = \bar{S}D_W(G)$ .*

### Vector Spaces

Uniqueness proofs for families  $\mathcal{G}_K$ ,  $K$  a field, are more difficult than for the combinatorial examples. However, as in the matroid case, it is easy to see that for  $|E(G)|$  large enough,  $G \in \mathcal{G}_K$  can be reconstructed from its simple minors.

(5.11) **Lemma.** *Every  $G \in \mathcal{G}_K$  with  $|E(G)| \geq 3$  is reconstructible.*

*Proof:* If  $\dim(G) \geq 2$ , then  $G = \sum_{e \in E(G)} \mathcal{G}_e(G \setminus e)$ , the vector sum of the sets  $\mathcal{G}_e(G \setminus e)$ . Otherwise,  $\dim(\sum_{e \in E(G)} \mathcal{G}_e(G \setminus e)) \leq 1$ , and  $G = \{x : E(G) \rightarrow K \mid \hat{x}_e \in G/e, \forall e \in E(G)\}$ . ■

The vector space orthogonality function  $D_K$  is an abstract duality for  $\mathcal{G}_K$ . By considering the behavior on  $\mathcal{G}_K^2$ , we show that each abstract duality on  $\mathcal{G}_K$  is the composition of vector space orthogonality with an automorphic involution on  $K$ . First we show that any such composition is indeed an abstract duality.

(5.12) **Theorem.** *Let  $d : K \rightarrow K$  be an involutory automorphism, and, for each finite set  $E$ , let  $\bar{d} : K^E \rightarrow K^E$  be given by  $(\bar{d}(x))(e) = d(x(e))$ , for all  $e \in E$ . For every  $G \in \mathcal{G}_K$ , let  $D_d(G) = \{\bar{d}(x) \mid x \in D_K(G)\}$ . Then  $D_d : \mathcal{G}_K \rightarrow \mathcal{G}_K$  is an abstract duality on  $\mathcal{G}_K$ .*

*Proof:* It is clear that  $D_d$  preserves the ground set (3.1) and that  $D_d$  preserves isomorphisms (3.4). Note that  $d(0) = 0$  for any field automorphism. Since  $d$  is additive, multiplicative,

and an involution, for each pair  $x, y \in E^K$ , the inner product  $x \bullet y = 0$  if and only if  $\bar{d}(x) \bullet \bar{d}(y) = 0$ . Therefore,  $D_K(\{\bar{d}(x) \mid x \in G\}) = \{\bar{d}(x) \mid x \in D_K(G)\}$ . This implies that  $D_d(D_d(G)) = \{\bar{d}(x) \mid x \in D_K(D_d(G))\} = D_K(\{\bar{d}(x) \mid x \in D_d(G)\}) = D_K(\{\bar{d}(\bar{d}(x)) \mid x \in D_K(G)\}) = \{x \mid x \in D_K(D_K(G))\} = G$ , for each  $G \in \mathcal{G}_K$ , so  $D_d$  satisfies (3.2). For each  $e \in E(G)$ ,  $\bar{d}(\hat{x}_e) = (\bar{d}x)_e$ , and  $x(e) = 0$  if and only if  $(\bar{d}(x))(e) = 0$ , so  $\bar{d}$  commutes with contraction and deletion. Since  $D_K(G \setminus e) = (D_K(G))/e$ , we have  $D_d(G \setminus e) = \{\bar{d}(x) \mid x \in (D_K(G))/e\} = \{\bar{d}(x) \mid x \in (D_K(G))\}/e = D_d(G)/e$ . Similarly,  $D_d(G/e) = D_d(G) \setminus e$ , so  $D_d$  satisfies (3.3). Therefore,  $D_d$  is an abstract duality on  $\mathcal{G}_K$ . ■

For now, assume that  $D : \mathcal{G}_K \rightarrow \mathcal{G}_K$  is an abstract duality. The following lemmas list conditions that  $D$  must satisfy. These conditions imply that  $D$  is determined by an involutory automorphism on the field  $K$ , as in Theorem 5.12. For each  $e \in E(G)$ ,  $u^e$  denotes the map in  $K^{E(G)}$  having  $u^e(e) = 1$  and  $u^e(e') = 0$  for all  $e' \in E(G) - \{e\}$ ;  $LS(S)$  denotes the linear span of the set of maps  $S$ .

(5.13) **Lemma.** For all finite sets  $E$ ,  $D(K^E) = \{0^E\}$  and  $D(\{0^E\}) = K^E$ .

*Proof:* For  $|E| = 0$  the result is clear, since there is only one vector space in  $\mathcal{G}_K(\emptyset)$ . For each one element set  $E$ , say  $E = \{e\}$ , there are two vector spaces in  $\mathcal{G}_K(E)$ :  $\{0^E\}$  and  $K^E$ . For the same choice of  $e$ , consider the set  $E' = \{e, f\}$ . Then  $\mathcal{G}_K(E') = \{\{0^{E'}\}, K^{E'}, LS(u^e)\} \cup \{LS(u^e + \alpha u^f) \mid \alpha \in K\}$ . The following minors are produced by contracting or deleting  $f$ :

$$\begin{aligned}
\{0^{E'}\}/f &= \{0^E\}; & \{0^{E'}\}\setminus f &= \{0^E\}; \\
K^{E'}/f &= K^E; & K^{E'}\setminus f &= K^E; \\
LS(u^f)/f &= \{0^E\}; & LS(u^f)\setminus f &= \{0^E\}; \\
LS(u^e)/f &= K^E; & LS(u^e)\setminus f &= K^E; \\
LS(u^e + \alpha u^f)/f &= K^E, \quad \forall \alpha \neq 0; & LS(u^e + \alpha u^f)\setminus f &= \{0^E\}, \quad \forall \alpha \neq 0.
\end{aligned}$$

If  $D(\{0^E\}) = \{0^E\}$  then, by Lemma 4.9,  $2 = |\{G \in \mathcal{G}_K(E') : G/f = \{0^E\}\}| = |\{G \in \mathcal{G}_K(E') : G\setminus f = \{0^E\}\}| = |K| + 1$ , which is impossible. Therefore,  $D(\{0^E\}) = K^E$  and  $D(K^E) = \{0^E\}$ . This is independent of the choice of  $e$  and so the result holds for all of  $\mathcal{G}_K^1$ . Now assume that the result holds for all  $E$  having  $|E| \leq q$ , for some  $q \geq 1$ . Let  $E'$  have  $|E'| = q + 1$ , and consider  $\{0^{E'}\} \in \mathcal{G}_K(E')$ . For each  $e \in E'$ ,  $D(\{0^{E'}\}) \setminus e = D(\{0^{E'}\}/e)$ . By the induction hypotheses,  $D(\{0^{E'}\}/e) = K^{E' - \{e\}} = (K^{E'}) \setminus e$ , so  $D(\{0^{E'}\}) = K^{E'}$ . By (3.2),  $D(K^{E'}) = \{0^{E'}\}$ , and the lemma is proved. ■

(5.14) **Lemma.** For any two element set  $E = \{e, f\}$ ,  $D(LS(u^e)) = LS(u^f)$  and  $D(LS(u^f)) = LS(u^e)$ .

*Proof.* Note that  $LS(u^f)/f = LS(u^f)\setminus f = \{0^{\{e\}}\}$ . Hence  $(D(LS(u^f)))\setminus f = (D(LS(u^f)))/f = K^{\{e\}}$ , by (3.3) and (5.13). The only  $G \in \mathcal{G}_K(E)$  satisfying  $G/f = G\setminus f = K^{\{e\}}$  are  $K^E$  and  $LS(u^e)$ . By (5.13)  $D(K^E) = \{0^E\}$ , so by (3.2)  $K^E \neq D(LS(u^f))$ . Therefore,  $D(LS(u^f)) = LS(u^e)$ , and by (3.2),  $D(LS(u^e)) = LS(u^f)$ . ■

Since any  $G \in \mathcal{G}_K$  with  $|E(G)| \geq 3$  is reconstructible, the abstract duality  $D$  is determined by its behavior on  $\mathcal{G}_K^2$ . Lemmas 5.13 and 5.14 indicate that  $D$  is fixed on all  $G \in \mathcal{G}_K^2$  other than those with  $|E(G)| = 2$ ,  $\dim(G) = 1$ , and containing no unit vector. Since  $D$  preserves isomorphisms, it suffices now to examine the behavior of  $D$  on those  $G$  as above

in  $\mathcal{G}_K(E)$ , where  $E$  is a fixed two element set, say  $\{e, e'\}$ . A map  $x \in K^E$  will be denoted by the ordered pair  $(x(e), x(e'))$ . The behavior of  $D$  on  $\{K^E, \{0^E\}, LS(0,1)\} \subseteq \mathcal{G}_K(E)$  has already been established. The action of  $D$  on the remaining one-dimensional subspaces in  $\mathcal{G}_K(E)$ ,  $\{LS(1, \alpha) \mid \alpha \in K\}$  determines a function  $f : K \rightarrow K$  as follows:

$$(5.15) \quad D(LS(1, \alpha)) = LS(f(\alpha), 1).$$

Since  $D(LS(1, 0)) = LS(0, 1)$ , we have  $f(0) = 0$ . By (3.2),  $f$  is a bijection and, since  $D(\{0^K\}) = K^E$ ,  $\dim(D(G)) < n$  whenever  $\dim(G) > 0$ . By property (3.3), for each vector space  $G \in \mathcal{G}_K$ , and each  $e \in E(G)$ ,

$$(5.16) \quad \mathcal{G}_e(D(G/e)) = \mathcal{G}_e(D(G) \setminus e) \subseteq D(G).$$

In the following proof, each map  $x \in K^{E'}$ , where  $E' = \{e_1, e_2, \dots, e_n\}$ , will be denoted by the ordered n-tuple  $(x(e_1), x(e_2), \dots, x(e_n))$ . If  $E'' = \{e_{i_1}, e_{i_2}, \dots, e_{i_s}\} \subseteq E'$ ,  $i_1 < i_2 < \dots < i_s$ , and  $x \in K^{E''}$ , then we will denote  $x$  by the ordered s-tuple  $(x(e_{i_1}), x(e_{i_2}), \dots, x(e_{i_s}))$ .

(5.17) **Lemma.** (a)  $f(1) = -1$ ;

(b) If  $\alpha \in K - \{0\}$  then  $f(1/\alpha) = 1/f(\alpha)$ .

(c) If  $\alpha, \beta \in K$  then  $f(\alpha\beta) = -f(\alpha)f(\beta)$ .

(d) If  $\alpha \in K$  then  $f(-\alpha) = -f(\alpha)$ .

(e) If  $\alpha \in K$  then  $f(f(\alpha)) = \alpha$ .

(f) Let  $\alpha, \beta \in K - \{0\}$ ,  $|E'| = 3$  and  $G = LS\{(1,0, \alpha), (0,1, \beta)\} \in \mathcal{G}_K(E')$ .

Then  $D(G) = LS(f(\alpha), f(\beta), 1)$ .

(g) The function  $f$  is additive: for any  $\alpha, \beta \in K$ ,  $f(\beta) = f(\alpha) + f(\beta - \alpha)$ .

*Proof:* (a) Let  $E' = \{e_1, e_2, e_3\}$ , let  $\alpha \in K - \{0\}$ , and let  $G = LS(1, 1, \alpha) \in \mathcal{G}_K(E')$ . Then  $G/e_1 = LS(1, \alpha) \in \mathcal{G}_K(\{e_2, e_3\})$ . Applying the definition of  $f$ , (with  $e = e_2, e' = e_3$ ) we have



$D(G) \setminus e_1 = D(G/e_1) = LS(f(\alpha), 1)$ . So  $LS(0, f(\alpha), 1) = \mathcal{E}_{e_1}(D(G) \setminus e_1) \subseteq D(G)$ , by (5.16).

Similarly,  $LS(f(\alpha), 0, 1) \subseteq D(G)$  and  $LS(f(1), 1, 0) \subseteq D(G)$ . Since  $\dim(D(G)) < 3$ , the vectors  $(0, f(\alpha), 1)$ ,  $(f(\alpha), 0, 1)$ , and  $(f(1), 1, 0)$  are linearly dependent. Taking the determinant of the  $3 \times 3$  matrix given by these vectors, we get  $0 = f(\alpha) + f(1)f(\alpha)$ , so  $f(1) = -1$ .

(b) Let  $\alpha \in K - \{0\}$ , and consider  $G = LS(1, \alpha^{-1}, 1) \in \mathcal{G}_K(\{e_1, e_2, e_3\})$ . Applying property (5.16) to  $G$ , for each  $e_i$ , shows that the vectors  $(0, f(\alpha), 1)$ ,  $(f(1), 0, 1)$ ,  $(f(\alpha^{-1}), 1, 0)$  are in  $D(G)$ . Since  $\dim(D(G)) < 3$ , these vectors are linearly dependent. Taking the determinant of the  $3 \times 3$  matrix given by these vectors, we get  $0 = f(1) + f(\alpha^{-1})f(\alpha)$ , so  $f(1/\alpha) = -f(1)/f(\alpha) = 1/f(\alpha)$ .

(c) Since  $f(0) = 0$ , the result is clear when  $\alpha\beta = 0$ . Let  $\alpha, \beta \in K - \{0\}$ , and consider  $G = LS(1, 1/\alpha, \beta) \in \mathcal{G}_K(\{e_1, e_2, e_3\})$ . Applying property (5.16) to  $G$ , for each  $e_i$ , shows that the vectors  $(0, f(\alpha\beta), 1)$ ,  $(f(\beta), 0, 1)$ ,  $(1/f(\alpha), 1, 0)$  are in  $D(G)$ . Since  $\dim(D(G)) < 3$ , these vectors are linearly dependent. Taking the determinant of the  $3 \times 3$  matrix given by these vectors, and applying (b) we get  $0 = f(\beta) + f(\alpha\beta)/f(\alpha)$ , so  $f(\alpha\beta) = -f(\beta)f(\alpha)$ .

(d) Note that  $-1 = f(1) = f(-1 \ -1) = -f(-1) f(-1)$ . Therefore  $f(-1) = \pm 1$ , and since  $f(1) = -1$  and  $f$  must be a bijection,  $f(-1) = 1$ . Moreover,  $f(-\alpha) = f(-1 \ \alpha) = -f(\alpha)$ , by (c).

(e) The result is clear for  $\alpha = 0$ . Let  $\alpha \in K - \{0\}$ , and let  $G = LS(\alpha^{-1}, 1) \in \mathcal{G}_K(E)$ . By property (3.2),  $LS(\alpha^{-1}, 1) = G = D(D(G)) = D(LS(f(\alpha), 1)) = D(LS(1, f(\alpha)^{-1})) = LS(f(f(\alpha)^{-1}), 1)$ . This implies that  $\alpha^{-1} = f(f(\alpha)^{-1})$ . Taking inverses and applying (b) we have  $\alpha = (f(f(\alpha)^{-1}))^{-1} = f((f(\alpha)^{-1})^{-1}) = f(f(\alpha))$ .

(f) Let  $E' = \{e_1, e_2, e_3\}$  and  $G = LS\{(1, 0, \alpha), (0, 1, \beta)\} \in \mathcal{G}_K(E')$ . The contraction minor  $G/e_3 = K^{\{e_1, e_2\}}$  so  $D(G/e_3) = D(G) \setminus e_3 = \{(0, 0)\}$ . This implies that  $D(G) = LS(x)$ , for some  $x \in K^{E'} - \{0^{E'}\}$ . Since the deletion minor  $G \setminus e_1 = LS(1, \beta)$ , we have

$D(G)/e_1 = LS(f(\beta), 1)$  and  $D(G) = LS(\lambda_1, f(\beta), 1)$ , for some  $\lambda_1 \in K$ . Similarly, we have  $G \setminus e_2 = LS(1, \alpha)$ , so  $D(G) = LS(f(\alpha), \lambda_2, 1)$ , some  $\lambda_2 \in K$ . Therefore,  $D(G) = LS(f(\alpha), f(\beta), 1)$ .

(g) If  $\beta = \alpha$ , then the result is clear. Otherwise, let  $E' = \{e_1, e_2, e_3, e_4\}$  and consider  $G = LS\{(1, 0, 1, \alpha), (0, 1, 1, \beta)\} \in \mathcal{G}_K(E')$ . Taking linear combinations, we can also express  $G$  as  $G = LS\{(\beta/(\beta - \alpha), -\alpha/(\beta - \alpha), 1, 0), (-1/(\beta - \alpha), 1/(\beta - \alpha), 0, 1)\}$ . Observe, therefore, that

$$G/e_1 = LS\{(-\alpha/(\beta - \alpha), 1, 0), (1/(\beta - \alpha), 0, 1)\};$$

$$G/e_2 = LS\{(\beta/(\beta - \alpha), 1, 0), (-1/(\beta - \alpha), 0, 1)\};$$

$$G/e_3 = LS\{(1, 0, \alpha), (0, 1, \beta)\};$$

$$G/e_4 = LS\{(1, 0, 1), (0, 1, 1)\};$$

Now,  $D(G) \setminus e_i = D(G/e_i)$ , for  $i = 1, 2, 3$ , so by applying (5.16), and (f), we have

$$\bigcup_{i=1}^4 \mathcal{G}_{e_i}(D(G) \setminus e_i) \subseteq D(G) \text{ where}$$

$$D(G) \setminus e_1 = LS(1, f(\alpha)/f(\beta - \alpha), 1/f(\beta - \alpha));$$

$$D(G) \setminus e_2 = LS(1, -f(\beta)/f(\beta - \alpha), -1/f(\beta - \alpha));$$

$$D(G) \setminus e_3 = LS(f(\alpha), f(\beta), 1);$$

$$D(G) \setminus e_4 = LS(-1, -1, 1).$$

By Lemma 5.13, and the fact that  $D$  is a bijection,  $\dim(D(G)) < 4$ , so the set of vectors  $\{(0, 1, f(\alpha)/f(\beta - \alpha), 1/f(\beta - \alpha)), (1, 0, -f(\beta)/f(\beta - \alpha), -1/f(\beta - \alpha)), (f(\alpha), f(\beta), 0, 1), (-1, -1, 1, 0)\}$  is linearly dependent. The determinant of the  $4 \times 4$  matrix given by these vectors has magnitude  $0 = (\frac{f(\alpha) - f(\beta)}{f(\beta - \alpha)} + 1)^2$ . Therefore,  $f(\beta) - f(\alpha) = f(\beta - \alpha)$ . ■

Lemma 5.17 implies that the function  $d = -f$  is an involutory automorphism on  $K$ . The abstract duality  $D$  on  $\mathcal{F}_K$  is the composition of vector space orthogonality with the natural extension  $\bar{d} : K^E \rightarrow K^E$  of the function  $d$  to finite sets.

(5.18) **Theorem:** If  $D : \mathcal{G}_K \rightarrow \mathcal{G}_K$  is an abstract duality, then there is an involutory automorphism  $d : K \rightarrow K$  such that  $D = D_d$ , that is, for every  $G \in \mathcal{G}_K$ ,  $D(G) = \{\bar{d}(y) \mid y \in D_K(G)\}$ .

*Proof:* Let  $f$  be the function determined by  $D$ , as in (5.15). As remarked above,  $d = -f$  is an involutory automorphism. It is clear from Lemmas 5.13, 5.14, and 5.17 that for each  $G \in \mathcal{G}_K^2$ ,  $D(G) = D_d(G)$ . Lemma 5.11 implies that  $D$  is determined by its behavior on  $\mathcal{G}_K^2$ , hence  $D = D_d$ . ■

For every field  $K$ , the identity function is trivially an involutory automorphism. The theorem implies that if the identity is the unique involutory automorphism on  $K$ , then  $\mathcal{G}_K$  has a unique abstract duality, vector space orthogonality. It is easy to show that for  $\mathbb{Q}$ , the field of rational numbers, there is no non-trivial automorphism  $d$ . The field  $\mathbb{R}$  of real numbers also has no non-trivial automorphism. It is particularly easy to show that  $\mathbb{R}$  has no non-trivial involutory automorphism, using the fact that every non-negative real number has a real square root. For any real number  $\gamma > 0$ ,  $d(\gamma) = (d(\sqrt{\gamma}))^2 > 0$ , so  $d(\alpha + \gamma) = d(\alpha) + d(\gamma) > d(\alpha)$ , and  $d$  is increasing. The only increasing involution on  $\mathbb{R}$  is the identity.

A field  $K$  has a non-trivial involutory automorphism if and only if it has a subfield  $K'$  such that, when  $K$  is regarded as a vector space over  $K'$ , the dimension of  $K$  is 2. Such a sub-field is said to have *index 2* in  $K$ . A finite field  $K$  has no non-trivial involutory automorphism if  $|K| = p^{2n+1}$ . If  $K$  has  $|K| = p^{2n}$ , then  $K$  has a unique non-trivial involutory automorphism given by  $d(\alpha) = \alpha^{p^n}$  for all  $\alpha \in K$  (see [20]); hence  $\mathcal{G}_K$  has exactly two abstract dualities. Infinite fields can also have non-trivial involutory automorphisms. For example, the function  $d : \mathbb{Q}\sqrt{2} \rightarrow \mathbb{Q}\sqrt{2}$  having  $d(\alpha + \sqrt{2}\beta) = \alpha - \sqrt{2}\beta$  is an involutory

automorphism. In the field  $C$  of complex numbers, the function that maps  $\alpha + \beta i$  to its complex conjugate,  $\alpha - \beta i$  is an involutory automorphism. These functions determine abstract dualities on  $\mathcal{G}_{\mathbb{Q}\sqrt{2}}$  and  $\mathcal{G}_C$ , respectively.

If a field  $K$  has characteristic two, every element is its own additive inverse, so “reversing signs” has no effect. For all other fields  $K$ , reversing signs on a set  $S$  corresponds to replacing the image of each element of  $S$  by its additive inverse, i.e., for a map  $x : E \rightarrow K$ , the map  $\bar{x} : E \rightarrow K$  has  $\bar{x}(e) = -x(e)$  if  $e \in E \cap S$  and  $\bar{x}(e) = x(e)$  if  $e \in E - S$ . If  $S \neq \emptyset$  and  $\alpha \neq -\alpha$  for some  $\alpha \in K$ , then the function consisting of vector space orthogonality ( $D_K$ ) followed by reversing signs on  $S$  is a weak abstract duality on  $\mathcal{G}_K$ . It is easy to verify that reversing signs on a set commutes with  $D_K$  and with contraction and deletion. Properties (3.1)-(3.3) follow immediately.

## 6 ANTIMATROIDS

Antimatroids generalize the notion of convexity in much the same way that matroids generalize the notion of linear dependence. They are equivalent to the abstract convexity structures studied by Edelman and Jamison-Waldner [7, 8, 12]. Both matroids and antimatroids are subclasses of a more general combinatorial structure, *greedoids*, introduced by Korte and Lovász (see [14, 15]). In this section we define antimatroids in terms of *circuits*, by conditions that are reminiscent of the (signed) circuit characterization of (oriented) matroids. Contraction and deletion operations, analogous to those for matroids, are defined for antimatroids. It is shown that antimatroids fit naturally into the unifying framework of Section 3, but that the resulting family  $\mathcal{G}$  has no abstract duality.

In the literature, several different, but equivalent, definitions and several different names (including shelling structures, APS greedoids, and upper interval greedoids) have been given for antimatroids. Like matroids, antimatroids can be characterized in several ways: in terms of feasible (independent) sets, rooted circuits, or a convex hull operator. The following characterization is appealing because of its similarity to the circuit set axiomatization of oriented matroids. Let  $\mathcal{C}$  be a collection of signed subsets of a finite set  $E$  such that the following properties hold.

$$(6.1) \quad |X^-| = 1, \quad \forall X \in \mathcal{C}$$

$$(6.2) \quad X_1, X_2 \in \mathcal{C}, X_1 \subseteq X_2 \Rightarrow X_1 = X_2;$$

$$(6.3) \quad X_1, X_2 \in \mathcal{C}, x \in X_1^- \cap X_2^+ \Rightarrow \exists X_3 \in \mathcal{C}$$

$$s.t. X_3^+ \subseteq (X_1^+ \cup X_2^+) - \{x\}, \quad X_3^- \subseteq (X_1^- \cup X_2^-) - \{x\}.$$

Then  $\mathcal{C}$  is the set of (*rooted*) *circuits* of an antimatroid on  $E$ . By (6.1), in (6.3) we have  $X_3^- = X_2^-$ . This definition differs slightly from the definition of shelling structures found in [15], which required that  $|\underline{C}| \geq 2$ , for all  $C \in \mathcal{C}$ . Here we allow an antimatroid to have “loops,” i.e., circuits with only one element. We denote by  $\mathcal{F}_A$  the family of all finite antimatroids. Given an antimatroid  $F \in \mathcal{F}_A$ , we will denote by  $E(F)$  and  $\mathcal{C}(F)$ , respectively, the set of elements on which  $F$  is defined, and the set of circuits of  $F$ . Note that the conditions above could have been expressed in terms of the collection rooted sets  $\{(\underline{C}, x) \mid C \in \mathcal{C}, \{x\} = C^-\}$ ; this characterization is presented in [5, 6].

We associate with each  $F \in \mathcal{F}_A$  a *span*  $\mathcal{P}(F)$  that determines  $F$  uniquely. Some additional terminology will ease the definition of  $\mathcal{P}(F)$ . We say that a signed set  $X$  is a *positive enlargement* of a signed set  $Y$  if  $Y^+ \subseteq X^+$  and  $Y^- = X^-$ . Then  $\mathcal{P}(F)$  is the collection of all positive enlargements of conformal unions of the circuits of  $F$ . We will later work with

$\mathcal{V}(F)$ , the set of all signed incidence vectors of  $X \in \mathcal{P}(F)$ . Note that  $\mathcal{C}(F)$  is the collection of setwise minimal elements  $X$  of  $\mathcal{P}(F)$  having  $|X^-| = 1$ .

Contraction and deletion operations can be defined for antimatroids. These operations act on the circuits and are analogous to oriented matroid contraction and deletion. Let  $F \in \mathcal{F}_A$  and let  $e \in E(F)$ . Then  $F/e$  and  $F \setminus e$  have  $E(F/e) = E(F \setminus e) = E(F) - \{e\}$ ,  $\mathcal{C}(F/e)$  is the collection of setwise minimal elements of  $\{X - \{e\} \mid X \in \mathcal{C}(F)\}$  having  $|X^-| = 1$ , and  $\mathcal{C}(F \setminus e) = \{X \in \mathcal{C}(F) \mid e \notin X\}$ . The family  $\mathcal{F}_A$  is closed under contraction and deletion (see [6]). In terms of the span  $\mathcal{P}(F)$ , we have  $\mathcal{P}(F/e) = \{X - \{e\} \mid X \in \mathcal{P}(F)\}$  and  $\mathcal{P}(F \setminus e) = \{X \in \mathcal{P}(F) \mid e \notin X\}$  (see [6]). Note that when described in terms of the span, the contraction and deletion operations of  $\mathcal{F}_A$  take the same form as for the earlier combinatorial examples. Indeed, if we take  $\mathcal{G}_A = \{\mathcal{V}(F) \mid F \in \mathcal{F}_A\}$  then the operations of contraction and deletion in  $\mathcal{G}_A$  are described by (3.0).

The definitions above of contraction and deletion resemble the circuit definitions of contraction and deletion in a matroid. However, they are not equivalent to the definitions given by Korte and Lovász [14], which are direct extensions of the independent set definitions of matroid contraction and deletion to general greedoids. Instead, our contraction operation is equivalent to the deletion operation of [14], while our deletion operation is equivalent to the *trace* operation of [15]. In other words, in  $\mathcal{G}_A$  different operations result from extending the independent set and circuit definitions of matroid deletion to the feasible sets and circuits, respectively, of antimatroids. This hints that there might not be a duality relation on  $\mathcal{G}_A$  which relates contraction and deletion (as defined by (3.0)) in the way that matroid duality relates contraction and deletion in  $\mathcal{G}_M$ . Indeed, although antimatroids having

$|E(G)| \geq 2$  can be reconstructed from their proper minors, there is no weak abstract duality for the family  $\mathcal{G}_A$ .

(6.4) **Theorem.** *There is no function  $D : \mathcal{G}_A \rightarrow \mathcal{G}_A$  satisfying (3.1), (3.2), and (3.3).*

*Proof:* Assume that a function  $D : \mathcal{G}_A \rightarrow \mathcal{G}_A$  satisfies properties (3.1), (3.2), and (3.3). For a fixed  $e$  there are two members of  $\mathcal{G}_A(\{e\})$ :  $G = \{(-), (0), (+)\}$  and  $G' = \{(0), (+)\}$ . Either  $D(G) = G$  or  $D(G) = G'$ . For  $e' \neq e$ , and  $E = \{e', e\}$ , there are six antimatroids:

$$G_1 = \{(0,0), (0,+), (+,0), (+,+)\};$$

$$G_2 = \{(-,0), (-,+), (0,0), (0,+), (+,+), (+,0)\};$$

$$G_3 = \{(0,-), (0,0), (0,+), (+,-), (+,0), (+,+)\};$$

$$G_4 = \{(-,-), (-,0), (-,+), (0,-), (0,0), (0,+), (+,-), (+,0), (+,+)\};$$

$$G_5 = \{(0,0), (0,+), (+,-), (+,0), (+,+)\};$$

$$G_6 = \{(-,+), (0,0), (0,+), (+,0), (+,+)\}.$$

where each ordered pair is the image of  $(e, e')$ . Let  $\mathcal{Y}' = \{G_i \mid G_i/e' = G'\} = \{G_1, G_2, G_6\}$  and let  $\mathcal{Y} = \{G_i \mid G_i/e' = G\} = \{G_3, G_4, G_5\}$ . Then  $|\mathcal{Y}'| = 3$  and  $|\mathcal{Y}| = 3$ . Similarly, let  $\mathcal{Z}' = \{G_i \mid G_i \setminus e' = G'\} = \{G_1, G_2, G_6, G_5\}$ , and let  $\mathcal{Z} = \{G_i \mid G_i \setminus e' = G\} = \{G_3, G_4\}$ . Then  $|\mathcal{Z}'| = 4$  and  $|\mathcal{Z}| = 2$ . By Lemma 4.9, the function  $D$  restricted to  $\mathcal{Z}'$  is a bijection from  $\mathcal{Z}'$  to either  $\mathcal{Y}$  or  $\mathcal{Y}'$ , both of which are impossible since  $|\mathcal{Y}| = |\mathcal{Y}'| < |\mathcal{Z}'|$ . ■

## 7 CONCLUSION

It has been appreciated previously that there are resemblances among the duality relations on the combinatorial structures examined here. For example, in each setting there is a variation on Minty's Coloring Property (see [17]) that characterizes the duality relation in that setting (see [9, 11, 17, 4, 2, 13]). Here we have shown that in fact there is a common

characterization of these duality relations, as well as the orthogonality relation on vector spaces coordinatized over fields having no subfield of index 2.

Some of the results here can be strengthened. For example under conditions (3.1) and (3.2), either half of condition (3.3) implies the other. We included both parts of (3.3) to emphasize symmetry. The combinatorial parts, (a) -(d), of Theorem 3.5 remain valid when (3.2) is relaxed to require only that  $D$  be one-to-one. Furthermore, the form of the proofs of the five parts of Theorem 3.5 yields similar uniqueness results for subfamilies of  $\mathcal{G}_M$ ,  $\mathcal{G}_S$ ,  $\mathcal{G}_O$ ,  $\mathcal{G}_W$ ,  $\mathcal{G}_K$ . Consider first  $\mathcal{G}_M$ .

Theorem 3.5a is proved by Lemma 5.1, which shows that: (a) all  $G \in \mathcal{G}_M - \mathcal{G}_M^1$  are reconstructible; and (b) all weak abstract dualities agree on  $\mathcal{G}_M^1$ . Suppose that

$$(7.1) \quad \tilde{\mathcal{G}}_M \subseteq \mathcal{G}_M \text{ is closed under contraction, deletion, and } D_M.$$

Reconstructibility of all  $G \in \tilde{\mathcal{G}}_M - \tilde{\mathcal{G}}_M^1$  is immediate. If, in addition to (7.1),

$$(7.2) \quad \mathcal{G}_M^2 \subseteq \tilde{\mathcal{G}}_M,$$

then it follows that all weak abstract dualities on  $\tilde{\mathcal{G}}_M$  agree on  $\tilde{\mathcal{G}}_M^1$ , since the proof of Lemma 5.1.b appealed only to  $G \in \mathcal{G}_M^2$ . Therefore  $D_M$  restricted to  $\tilde{\mathcal{G}}_M$  is the unique weak abstract duality on  $\tilde{\mathcal{G}}_M$ . Among the subfamilies  $\tilde{\mathcal{G}}_M$  of  $\mathcal{G}_M$  that satisfy (7.1) and (7.2) are those that arise from planar graphic matroids, matroids representable over a particular field, matroids representable over all fields in some specified set, and all unions of the subfamilies noted above.



One can extend the other parts of Theorem 3.5 to subfamilies  $\tilde{\mathcal{G}}_i \subseteq \mathcal{G}_i$  in a similar fashion.

The condition analogous to (7.2) is that

$$\mathcal{G}_i^q \subseteq \tilde{\mathcal{G}}_i$$

where  $q$  is the maximum cardinality of any  $G \in \mathcal{G}_i$  used in our proof that all abstract dualities on  $\mathcal{G}_i$  agree on  $\mathcal{G}_i^{r(\mathcal{G}_i)-1}$ . For each of the combinatorial examples  $q = r(\mathcal{G}_i)$ , for general fields  $K$ ,  $q = 4 = r(\mathcal{G}_K) + 1$ .

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