GENERAL BINARY PARTIALLY BALANCED BLOCK DESIGNS

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Summary

Binary partially balanced incomplete block design theory has been generalized in the sense that the i'th treatment occurs either m_0 or m_1 times, $0 \le m_0 < m_1$, in the j'th block rather than zero or one time as in traditional design theory. The intrablock analysis for general binary partially balanced block designs with s associate classes is described along with solutions for effects, variances, and efficiencies. The existence of these designs is proved and a method of construction is given. Optimality criteria are developed for selecting an optimal design or designs from the constructed class of group divisible binary partially balanced block designs. Eigenvalues were evaluated for group divisible, triangular association scheme, and latin square association scheme general binary partially balanced block designs. An example is presented showing how to construct a class of general binary partially balanced block designs and to use the six optimality criteria developed in selecting an optimal design or designs.

1. Introduction

Statistical literature on binary partially balanced incomplete block designs has been confined almost entirely to the case wherein the occurrence, n_{ij}, of the i'th treatment in the j'th block is either zero or one. Cheng (1977), Shafiq (1978), and Shafiq and Federer (1979), have considered more general situations. The former considered, among other items, the case where n_{ij} was either m or m + 1, and the latter considered the case where n_{ij} was either m_0 or m_1, $0 \leq m_0 < m_1$, m_0 and m_1 being positive integers.

When n_{ij} is zero or one, the design is denoted as a basic binary partially balanced incomplete block design; the design parameters are the number of treatments v, the number of blocks b, the number of replicates r, of each treatment, the size of the block k, and the number of treatments $n_a < v$ $(a = 1, 2, \cdots, s)$ which have pairs of treatments occurring in exactly λ_a blocks where at least two of the λ_a are unequal. The parameters of the general binary partially balanced block design are defined as functions of the parameters of the basic design, m_0, and m_1. Complete, incomplete, and orthogonal general binary partially balanced block designs (GBPBBD) are defined. Conditions on the coefficient matrix $\underline{C}^*$ of a GBPBBD with s association classes are given for a design to be s-partially variance balanced. A result from Bose and Mesner (1959) is generalized and used extensively in obtaining the intrablock analysis for a GBPBBD. Intrablock solutions for treatment effects and the various variances of a difference between two effects are given. Relative efficiencies of two designs for special cases are also presented.

The eigenvalues of $\underline{n}^*\underline{n}^{*\prime}$, where $\underline{n}^*$ is the incidence matrix of a GBPBBD, and of the coefficient matrix $\underline{C}^*$ were obtained. The results are applied to find the eigenvalues of three classes of designs; group divisible GBPBBD, GBPBBD having a triangular association scheme, and GBPBBD having a latin square association scheme [see, e.g., Bose, Clatworthy, and Shrikhande (1954)].

The existence of a basic binary partially balanced incomplete block design (BBPBIBD) implies the existence of a class of GBPBBD's for given v and b. In Theorems 5.1, 5.2, and 5.3, criteria are developed for A-, E-, D-optimality of a design in the class of two associate group divisible GBPBBD's. An example is included to illustrate the consequences and uses of Theorems 5.1 to 5.3. It should be noted that the results can be extended to more than two associate classes, but the accompanying algebra becomes laborious.

2. Parameters of general binary partially balanced block design (GBPBBD) and some definitions

Definition 2.1. Given a basic binary partially balanced incomplete block design (BBPBIBD) with design parameters (v, b, r, k, λ_1, λ_2, $\cdots$, λ_s; n_{ij} = 0 or 1) and an association scheme with the parameters (n_u, p^i_{ju}; $i,j,u = 1, 2, \cdots, s$), a general binary partially balanced block design (GBPBBD) with parameters (v, b, r^*, k^*, λ^*_1, λ^*_2, $\cdots$, λ^*_s; $n^*_{ij} = m_0$ or m_1) and the same scheme as given above, is defined to be an arrangement of v treatments in b blocks each of size k^* (k^* not necessarily less than v) such that its incidence matrix is defined by

$$\underline{n}^* = \underline{n}(m_1 - m_0) + \underline{J}m_0 \qquad (2.1)$$

where $\underline{n}$ is a $v \times b$ incidence matrix of a BBPBIBD, $\underline{J}$ is a $v \times b$ matrix with unit entries everywhere, and $0 \leq m_0 < m_1$, m_0 and m_1 being any two positive integers.

The parameters of a GBPBBD are:

$$r^* = rm_1 + (b - r)m_0 , \qquad (2.2)$$

$$k^* = km_1 + (v - k)m_0 , \qquad (2.3)$$

$$\sum_{i=1}^{s} n_i\lambda^*_i = r^*(k^* - m_1 - m_0) + bm_1m_0 , \qquad (2.4)$$

$$\sum_{j=1}^{b} n^*_{gj} n^*_{hj} = \begin{cases} \lambda^*_0 & \text{if } g = h , & (2.5) \\ \lambda^*_i & \text{if } (g,h) \text{ are i'th associates,} & (2.6) \end{cases}$$

$$\lambda^*_0 = r^*(m_1 + m_0) - bm_1 m_0 , \tag{2.7}$$

$$\lambda^*_i = \lambda_i (m_1 - m_0)^2 + 2r(m_1 - m_0)m_0 + bm_0^2 , \tag{2.8}$$

and

$$vr^* = bk^* = N^* . \tag{2.9}$$

<u>Definition 2.2.</u> A GBPBBD is said to be incomplete if $m_0 = 0$, otherwise, it is complete.

<u>Definition 2.3.</u> A complete GBPBBD is said to be orthogonal if $n^*_{ij} = r^*_i k^*_j / N^*$, where N^* is the total number of observations, r^*_i is the number of replications of the i'th treatment, k^*_j is the number of entries in the j'th block and n^*_{ij} is the (i,j)'th entry of $\underline{n}^*$.

<u>Definition 2.4.</u> A GBPBBD with s association classes is said to be s-partially variance balanced if the coefficient matrix $\underline{C}^*$ can be expressed as

$$\underline{C}^* = c^*_0 \underline{I} + c^*_1 \underline{J} + c^*_2 B_2 + \cdots + c^*_s B_s , \tag{2.10}$$

where λ^*_0 and λ^*_i are as defined above,

$$c^*_0 = r^* - \lambda^*_0/k^* + \lambda^*_1/k^* ,$$

$$c^*_1 = -\lambda^*_1/k^* ,$$

$$c^*_i = -(\lambda^*_i - \lambda^*_1)/k^* , \quad i = 2, 3, \cdots, s,$$

and

$$\underline{J} = \underline{B}_0 + \underline{B}_1 + \cdots + \underline{B}_s ,$$

where

$$\underline{B}_0 = \underline{I} \quad \text{and} \quad \underline{B}_i = \left(b^h_{gi}\right)$$

and

$$b_{gi}^{h} = \begin{cases} 1 & \text{if } (g,h) \text{ are i'th associates} \\ 0 & \text{otherwise} \end{cases}$$

3. Intrablock analysis of general binary partially balanced block design (GBPBBD) with s association classes

Using the usual linear model for a GBPBBD under the assumptions of homoscedasticity and uncorrelated errors, the best linear unbiased estimate of the treatment effects may be obtained from the reduced normal equations as:

$$\underline{C}^{*}\underline{\tau} = \underline{Q} \; , \tag{3.1}$$

where

$$\underline{C}^{*} = r^{*}\underline{I} - \underline{n}^{*}\underline{n}^{*\prime}/k^{*} \; , \tag{3.2}$$

$$\underline{Q} = \underline{T} - \underline{n}^{*}\underline{B}/k^{*}, \; \underline{T} \text{ and } \underline{B} \text{ are vectors of treatment and block totals, respectively,} \tag{3.3}$$

$$\underline{n}^{*}\underline{n}^{*\prime} = \lambda_{0}^{*}\underline{B}_{0} + \lambda_{1}^{*}\underline{B}_{1} + \cdots + \lambda_{s}^{*}\underline{B}_{s} \; . \tag{3.4}$$

Equation (3.4) is an extension of the equation $\underline{n}\underline{n}' = r\underline{B}_0 + \lambda_1\underline{B}_1 + \lambda_2\underline{B}_2 + \cdots + \lambda_s\underline{B}_s$ from Bose and Mesner (1959), and it reduces to their result if $m_0 = 0$ and $m_1 = 1$. To derive (3.4) we write

$$\underline{n}^{*}\underline{n}^{*\prime} = [\underline{n}(m_1 - m_0) + \underline{J}m_0][\underline{n}'(m_1 - m_0) + \underline{J}'m_0]$$

$$= \lambda_{0}^{*}\underline{I} + \lambda_{1}^{*}\underline{B}_{1} + \cdots + \lambda_{s}^{*}\underline{B}_{s} \; .$$

Thus, equation (3.2) may be rewritten as:

$$\underline{C}^{*} = \frac{(r^{*}k^{*} - \lambda_{0}^{*} + \lambda_{1}^{*})}{k^{*}}\,\underline{I} - \frac{\lambda_{1}^{*}}{k^{*}}\,\underline{J} - \frac{1}{k^{*}}\sum_{i=2}^{s}(\lambda_{i}^{*} - \lambda_{1}^{*})\underline{B}_{i} \; . \tag{3.5}$$

Now, normal equation (3.1) may be written as:

$$\left[\frac{(r^{*}k^{*} - \lambda_{0}^{*})}{k^{*}}\,\underline{B}_{0} - \frac{\lambda_{1}^{*}}{k^{*}}\,\underline{B}_{1} - \frac{\lambda_{2}^{*}}{k^{*}}\,\underline{B}_{2} - \cdots - \frac{\lambda_{s}^{*}}{k^{*}}\,\underline{B}_{s}\right]\underline{\tau} = \underline{Q} \; . \tag{3.6}$$

After some lengthy algebraic manipulations, a solution of (3.6) is obtained as:

$$\hat{\tau}_g = \frac{k^*}{r^*k^* - \lambda_0^*} Q_g - \frac{1}{r^*k^* - \lambda_0^*} \sum_{i=1}^{s} \sum_{u=1}^{s} d^{*iu} \lambda_i^* S_u(Q_g) , \qquad (3.7)$$

where $\sum_{g=1}^{v} \hat{\tau}_g = 0$, $S_u(\hat{\tau}_g)$ is the g'th element of the vector $\underline{B}_u \hat{\underline{\tau}}$, $S_u(Q_g)$ is the g'th element of the vector $\underline{B}_u \underline{Q}$, d^{*iu} are elements of the matrix $\underline{D}$ in the matrix equation $\underline{B}_u \hat{\underline{\tau}} = \underline{D} \underline{B}_u \underline{Q}$, and the other symbols are as defined previously; d^{*iu} is the (i,u)'th entry of $(d^*_{iu})^{-1}$, the inverse matrix (d^*_{iu}), where

$$d^*_{ui} = (n_u \lambda_u^* - p^i_{u1}\lambda_1^* - p^i_{u2}\lambda_2^* - \cdots - p^i_{us}\lambda_s^*)/k^*, \text{ if } u \neq i$$

$$d^*_{uu} = (r^*k^* - \lambda_0^* + n_u\lambda_u^* - p^u_{u1}\lambda_1^* - p^u_{u2}\lambda_2^* - \cdots - p^u_{us}\lambda_s^*)/k^* .$$

When treatments g and h are u'th associates,

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = \frac{2\sigma^2}{r^*} \left[\frac{k^* - \sum_{i=1}^{s} d^{*iu}\lambda_i^*}{k^* - \lambda_0^*/r^*} \right] . \qquad (3.8)$$

The estimate of error variance may be obtained from the analysis of variance for a GBPBBD for s association classes as given in Table 3.1.

Table 3.1. Analysis of Variance for a GBPBBD

Source of Variation	d.f.	S.S.	M.S.
Blocks (unadjusted)	b-1	$\hat{\underline{\beta}}'\underline{B}$	
Treatments (adjusted)	v-1	$\hat{\underline{\tau}}'\underline{Q} = \sum_{g=1}^{v} \hat{\tau}_g Q_g$	$\frac{\text{S.S. Treatment}}{v-1}$
Residual	N^*-b-v+1	$\underline{Y}'\underline{Y} - \frac{G^2}{N^*} - \hat{\underline{\beta}}'\underline{B} - \hat{\underline{\tau}}'\underline{Q}$	$\hat{\sigma}^2 = \frac{\text{S.S. Residual}}{N^*-b-v+1}$
Total	N^*-1	$\underline{Y}'\underline{Y} - G^2/N^*$	

For two associate class designs the equations (3.7) and (3.8) may be written as:

$$\hat{\tau}_g = \frac{k^*}{r^*k^* - \lambda_0^*} Q_g + \frac{1}{r^*k^* - \lambda_0^*} \Big[(d^{*11}\lambda_1^* + d^{*21}\lambda_2^*) S_1(Q_g) + (d^{*12}\lambda_1^* + d^{*22}\lambda_2^*) S_2(Q_g) \Big] \tag{3.9}$$

and

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = \left(\frac{k^* - d^{*1u}\lambda_1^* - d^{*2u}\lambda_2^*}{k^* - \lambda_0^*/r^*} \right) \left(\frac{2\sigma^2}{r^*} \right), \tag{3.10}$$

where (g, h) are the u'th associates $(u = 1, 2)$. The values of (d^{*iu}) and the value of determinant of D^{-1} may be obtained as:

$$d^{*11} = \left(r^*k^* - \lambda_0^* + \lambda_2^* + p_{12}^2(\lambda_2^* - \lambda_1^*) \right) / k^*\det$$

$$d^{*12} = p_{12}^2(\lambda_2^* - \lambda_1^*)/k^*\det$$

$$d^{*21} = p_{12}^1(\lambda_1^* - \lambda_2^*)/k^*\det$$

$$d^{*22} = \left(r^*k^* - \lambda_0^* + \lambda_1^* + p_{12}^1(\lambda_1^* - \lambda_2^*) \right) / k^*\det$$

where

$$k^{*2}\det = (r^*k^* - \lambda_0^* + \lambda_1^*)(r^*k^* - \lambda_0^* + \lambda_2^*) + (\lambda_1^* - \lambda_2^*) \left\{ p_{12}^1(r^*k^* - \lambda_0^* + \lambda_2^*) - p_{12}^2(r^*k^* - \lambda_0^* + \lambda_1^*) \right\} .$$

Then, we may rewrite equations (3.7) and (3.8) as:

$$\hat{\tau}_g = \frac{r^*k^* - \lambda_0^* + \lambda_1^* + (p_{12}^1 - p_{12}^2)(\lambda_1^* - \lambda_2^*)}{k^*\det} Q_g + \frac{\lambda_1^* - \lambda_2^*}{k^*\det} S_1(Q_g), \tag{3.11}$$

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = \frac{2\sigma^2 k^*}{k^{*2}\det} \left[(r^*k^* - \lambda_0^* + \lambda_2^*) + (p_{12}^1 - p_{12}^2)(\lambda_1^* - \lambda_2^*) \right],$$

if (g,h) are first associates, (3.12)

and

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = \frac{2\sigma^2 k^*}{k^{*2}\det} \left[(r^*k^* - \lambda_0^* + \lambda_1^*) + (p_{12}^1 - p_{12}^2)(\lambda_1^* - \lambda_2^*) \right],$$

if (g,h) are second associates. (3.13)

The eigenvalues of $\underline{C}^*$, denoted by $\theta_i(\underline{C}^*)$ may be written as:

$$\left.\begin{aligned}
\theta_0(\underline{C}^*) &= r^* - r^*k^*/k^* = 0 , \\
\theta_1(\underline{C}^*) &= \frac{r^*k^* - \lambda_0^* + \tfrac{1}{2}\{(\lambda_1^* - \lambda_2^*)(p_{12}^1 - p_{12}^2 - \sqrt{\Delta}) + \lambda_1^* + \lambda_2^*\}}{k^*} , \\
\text{and} \quad \theta_2(\underline{C}^*) &= \frac{r^*k^* - \lambda_0^* + \tfrac{1}{2}\{(\lambda_1^* - \lambda_2^*)(p_{12}^1 - p_{12}^2 + \sqrt{\Delta}) + \lambda_1^* + \lambda_2^*\}}{k^*} ,
\end{aligned}\right\} \quad (3.14)$$

where $\Delta = (p_{12}^2 - p_{12}^1)^2 + p_{12}^2 + p_{12}^1 + 1$.

The multiplicities of the roots of $\underline{C}^*$ are:

$$\left.\begin{aligned}
\alpha_0 &= v - \alpha_1 - \alpha_2 = 1 , \\
\alpha_1 &= \frac{n_1 + n_2}{2} - \frac{(p_{12}^2 - p_{12}^1)(n_1 + n_2) + n_1 - n_2}{2\sqrt{\Delta}} , \\
\text{and} \quad \alpha_2 &= \frac{n_1 + n_2}{2} + \frac{(p_{12}^2 - p_{12}^1)(n_1 + n_2) + n_1 - n_2}{2\sqrt{\Delta}} .
\end{aligned}\right\} \quad (3.15)$$

For special classes of two associate class designs, the θ_i and α_i, $i = 0, 1, 2$, take on the following values:

(i) For a GBPBBD having a group divisible association scheme, $\theta_0(\underline{C}^*) = 0$, $\theta_1(\underline{C}^*) = v\lambda_2^*/k^*$, and $\theta_2(\underline{C}^*) = (r^*k^* - \lambda_0^* + \lambda_1^*)/k^* = [n\lambda_1^*(v - n)\lambda_2^*]/k^*$, with multiplicities $\alpha_0(\underline{C}^*) = 1$, $\alpha_1(\underline{C}^*) = m - 1$, and $\alpha_2(\underline{C}^*) = m(n - 1)$.

(ii) For a GBPBBD having a triangular association scheme, $\theta_0(\underline{C}^*) = 0$, $\theta_1(\underline{C}^*) = [r^*k^* - \lambda_0^* - (n - 4)\lambda_1^* + (n - 3)\lambda_2^*]/k^*$, and $\theta_2(\underline{C}^*) = [r^*k^* - \lambda_0^* + \lambda_1^* + (\lambda_1^* - \lambda_2^*)]/k^*$, with multiplicities $\alpha_0(\underline{C}^*) = 1$, $\alpha_1(\underline{C}^*) = n - 1$, and $\alpha_2(\underline{C}^*) = n(n - 3)/2$.

(iii) For a GBPBBD having a latin square type association scheme, $\theta_0(\underline{C}^*) = 1$, $\theta_1(\underline{C}^*) = [r^*k^* - \lambda_0^* - (s - i)\lambda_1^* + (s - i + 1)\lambda_2^*]/k^*$, and $\theta_2(\underline{C}^*) = [r^*k^* - \lambda_0^* + i\lambda_1^* - (i - 1)\lambda_2^*]/k^*$, with multiplicities $\alpha_0(\underline{C}^*) = 1$, $\alpha_1(\underline{C}^*) = i(s - 1)$, and $\alpha_2(\underline{C}^*) = (s - i + 1)(s - 1)$.

4. Relative efficiency of a GBPBBD

A method of comparing two designs is to compute the relative efficiency of one design over the other. Thus, given (v, b, r^*, k^*), the efficiency of a GBPBBD relative to a general binary balanced block design (GBBBD) [if such designs exist] may be obtained as follows. Let (g,h) be the first associates, then for a GBPBBD, the

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = \frac{2\sigma^2}{r^*} \cdot \frac{r^*k^*}{v\lambda^*}\left[\frac{v\lambda^*\{r^*k^* - \lambda_0^* + \lambda_2^*) + (p_{12}^1 - p_{12}^2)(\lambda_1^* - \lambda_2^*)\}}{k^{*2}\det}\right]$$

and for a GBBBD, the $\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = (2\sigma^2)/r^*)(r^*k^*/v\lambda^*)$ [see Shafiq and Federer (1979)]. The efficiency of a GBPBBD relative to a GBBBD, where (g,h) are first associates, is:

$$\begin{aligned} E_1^*(\mathrm{GBPBBD/GBBBD}) &= \frac{k^{*2}\det}{v\lambda^*\{r^*k^* - \lambda_0^* + \lambda_2^* + (p_{12}^1 - p_{12}^2)(\lambda_1^* - \lambda_2^*)\}} \\ &= \frac{r^*k^* - \lambda_0^* + \lambda_1^*}{v\lambda^*} - \frac{p_{12}^1(\lambda_1^* - \lambda_2^*)^2}{v\lambda^*[r^*k^* - \lambda_0^* + \lambda_2^* + (p_{12}^1 - p_{12}^2)(\lambda_1^* - \lambda_2^*)]} . \end{aligned} \tag{4.1}$$

Similarly, $E_2^*(\mathrm{GBPBBD/GBBBD})$ denotes the efficiency of GBPBBD relative to GBBBD, where (g,h) are second associates, thus:

$$E_2^*(\mathrm{GBPBBD/GBBBD}) = \frac{r^*k^* - \lambda_0^* + \lambda_2^*}{v\lambda^*} - \frac{p_{12}^2(\lambda_1^* - \lambda_2^*)^2}{v\lambda^*[r^*k^* - \lambda_0^* + \lambda_1^* + (p_{12}^1 - p_{12}^1)(\lambda_1^* - \lambda_2^*)]} . \tag{4.2}$$

For group divisible designs having two association classes, we may write $\hat{\tau}_g$, variances of the difference of $\hat{\tau}_g - \hat{\tau}_h$, and efficiencies as:

$$\hat{\tau}_g = \frac{k^*}{v\lambda_2^*}\left[\left(\frac{\lambda_1^* + (v-1)\lambda_2^*}{n\lambda_1^* + (v-n)\lambda_2^*}\right) Q_g + \frac{\lambda_1^* - \lambda_2^*}{n\lambda_1^* + (v-n)\lambda_2^*} S_1(Q_g)\right], \tag{4.3}$$

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = 2\sigma^2 \frac{k^*}{n\lambda_1^* + (v-n)\lambda_2^*} \text{ if } (g,h) \text{ are first associates }, \tag{4.4}$$

$$\mathrm{Var}(\hat{\tau}_g - \hat{\tau}_h) = 2\sigma^2 \frac{k^*}{v\lambda_2^*}\left(\frac{\lambda_1^* + (v-1)\lambda_2^*}{n\lambda_1^* + (v-n)\lambda_2^*}\right) \text{ if } (g,h) \text{ are second associates}, \tag{4.5}$$

$$E_1^*(\mathrm{GDGBPBBD/GBBBD}) = \frac{n\lambda_1^* + (v-n)\lambda_2^*}{v\lambda^*} \text{ if } (g,h) \text{ are first associates}, \tag{4.6}$$

and

$$E_2^*(\mathrm{GDGBPBBD/GBBBD}) = \frac{\lambda_2^*}{\lambda^*}\left(\frac{n\lambda_1^* + (v-n)\lambda_2^*}{\lambda_1^* + (v-1)\lambda_2^*}\right) \text{ if } (g,h) \text{ are second associates}. \tag{4.7}$$

The average variance for a group divisible GBPBBD is obtained as:

$$\bar{V} = \frac{2\sigma^2 k^*}{(v-1)\lambda_2^*}\left[\frac{v\lambda_2^* + (\lambda_1^* - \lambda_2^*)}{v\lambda_2^* + n(\lambda_1^* - \lambda_2^*)} - \frac{1}{v}\right]. \tag{4.8}$$

The average efficiency is:

$$E_{\cdot}^*(\mathrm{GDGBPBBD/GBBBD}) = (v-1)\lambda_2^*/v\lambda^*\left[\frac{v\lambda_2^* + (\lambda_1^* - \lambda_2^*)}{v\lambda_2^* + n(\lambda_1^* - \lambda_2^*)} - \frac{1}{v}\right]. \tag{4.9}$$

5. Existence and optimality of GBPBBD

The existence of a basic binary partially balanced incomplete block design (BBPBIBD) with parameters $(v, b, r, k, \lambda_1, \lambda_2, \cdots, \lambda_s;\ n_{ij} = 0 \text{ or } 1)$ and the parameters of the association scheme $(n_1, n_2, \cdots, n_s, p_{jk}^i\ (i,j,k = 1, 2, \cdots, s))$ implies the existence of a general binary partially balanced block design (GBPBBD) with parameters $(v, b, r^*, k^*, \lambda_1^*, \lambda_2^*, \cdots, \lambda_s^*;\ n_{ij}^* = m_0 \text{ or } m_1)$ and with the same parameters of the association scheme. In the class of all equi-replicated and equi-sized block GBPBBDs, the question arises as to which one(s) of these partially balanced block designs has(have) the smallest average variance. We answer this question for group divisible general binary partially balanced block designs with two association classes in the next three theorems.

Theorem 5.1. In the class of all equi-replicated equi-sized block group divisible general binary partially balanced block designs (GD GBPBBD) with two association classes with parameters $(m_d n_d, b_d, r_d^*, k_d^*, \lambda_{1d}^*, \lambda_{2d}^*; n_{ij}^* = m_{0d}$ or $m_{1d})$ which are derived from group divisible basic binary partially balanced incomplete block designs (GD BBPBIBD) having two association classes with parameters $(m_d n_d, b_d, r_d, k_d, \lambda_{1d}, \lambda_{2d}; n_{ij} = 0$ or $1)$ such that $\lambda_{1d} = \lambda_{2d} + 1$, the design(s) which maximizes the value of

$$\left\{\mathrm{tr}(\underline{C}_d^*) - \frac{(m_d - 1)(v - m_d)\{n_d b_d(m_{1d} - m_{0d})^2/vr_d^*\}^2}{\mathrm{tr}(\underline{C}^*) + (2m_d - 1 - v)n_d b_d(m_{1d} - m_{0d})^2/vr_d^*}\right\}$$

is(are) A-optimal, where $\mathrm{tr}(\underline{C}_d^*) = [(v - 1)r_d^* - r_d(b_d - r_d)(m_{1d} - m_{0d})^2/r_d^*]$.

Theorem 5.2. In the class of all equi-replicated and equi-sized block group divisible general binary partially balanced block designs (GD GBPBBD) having two association classes with parameters $(m_d n_d, b_d, r_d^*, k_d^*, \lambda_{1d}^*, \lambda_{2d}^*; n_{ij}^* = m_{0d}$ or $m_{1d})$ which are derived from group divisible basic binary partially balanced incomplete block designs (GD BBPBIBD) having two association classes with parameters $(m_d n_d, b_d, r_d, k_d, \lambda_{1d}, \lambda_{2d}; n_{ij} = 0$ or $1)$ such that $\lambda_{12} = \lambda_{2d} + 1$, and design(s) having the minimal value of $\{r_d(b_d - r_d) + b_d(n_d - 1)\}(m_{1d} - m_{0d})^2$ is(are) E-optimal.

Theorem 5.3. In the class of all equi-replicated and equi-sized blocks group divisible general binary partially balanced block designs (GD GBPBBD) having two association classes with parameters $(m_d n_d, b_d, r_d^*, k_d^*, \lambda_{1d}^*, \lambda_{2d}^*; n_{ij}^* = m_{0d}$ or $m_{1d})$ which are derived from group divisible basic binary partially balanced incomplete block designs (GD BBPBIBD) having two association classes with parameters $(m_d n_d,$

b_d, r_d, k_d, λ_{1d}, λ_{2d}; $n_{ij} = 0$ or 1) such that $\lambda_1 = \lambda_2 + 1$, the design(s) having the minimal values of

$$\{r_d(b_d - r_d) + b_d(n_d - 1)\}(m_{1d} - m_{0d})^2$$

and

$$\{r_d(b_d - r_d) - b_d(1 - \frac{1}{m_d})\}(m_{1d} - m_{0d})^2 ,$$

is(are) D-optimal.

The proofs of the theorems have been omitted for lack of space. However, they are straightforward, and if any difficulty ensues, the reader is invited to write one of the authors. Instead, an example illustrating the results of the theorems is presented next.

6. An example

Some consequences of the theorems considered in the last section are illustrated by the following example. Before proceeding to the example, some items should be noted. The real world situation is important in applications, not some statistician's assumptions. A frequent assumption of statisticians is that block size must be relatively small. Although this assumption may be true in many situations, it is not universally true. In sugar cane and pineapple plantations in Hawaii, sugar beet fields in Colorado, wheat fields in Kansas and Oklahoma, in a single growth chamber, etc. blocking is often of no avail in reducing variation in an experiment. Minimum, not maximum, blocking should be used to control heterogeneity in the experimental material. In some situations, quite large numbers of experimental units can be included in a block without increasing the estimated residual variance. The example given by Shafiq and Federer (1979) illustrates the efficiency of a GBBBD relative to traditional designs. Also, in some experiments, the experimental technique and procedure induce heterogeneity between blocks, whereas none may be present if uniform techniques and procedures were used. Finally, the statistician should provide designs for all situations, not merely a subset of them.

Example 6.1. Suppose an experimenter wants to test 12 treatments in, at most, 12 blocks of homogeneous material. We know that a balanced design would require at least 22 blocks [see Raghavarao (1971)]. Possible candidates for performing this experiment would be group divisible (GD) partially balanced incomplete block designs. Suppose all the homogeneous material must be used and $r^* = 64$ is fixed. There are four GD basic binary partially balanced incomplete block designs having $\lambda_1 = \lambda_2 + 1$ [Bose, Clatworthy and Shrikhande (1954)]. These designs would be used to construct GD GBPBBD for $v = 12$ and $r^* = 64$. The plans for these GD BBPBIBD's are given in Table 6.1. The parameters of GD GBPBBD constructed from GD BBPBIBD-1 to GD BBPBIBD-4 in Table 6.1 and various optimality measures are presented in Table 6.2.

Twenty-four designs listed in Table 6.2 form a complete class of GD GBPBBD with $v = 12$ and $r^* = 64$ derived from GD BBPBIBD having $\lambda_1 = \lambda_2 + 1$. Six different optimality measures described in Table 6.2 result in six subclasses. Optimal designs in each subclass marked * are given in Table 6.3.

We note a few interesting results from Table 6.3. Design 24 is the only member of its class which achieves $m_1 - m_0 = 1$, but optimality criterion IV_d^* would select Design 11 as the optimal one; and this design estimates all the elementary contrasts with the minimum average variance of $2\sigma^2(.015627)$, and therefore is optimal. The average variance of all elementary contrasts is $2\sigma^2(.015632)$ for Design 18 and $2\sigma^2(.015636)$ for Design 24. Designs 11, 18 and 24 have the same minimal value 44 of V_d^*; hence, they are equivalent in the sense of E-optimality.

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Table 6.1. Plans for group divisible basic binary partially balanced incomplete block designs (GD BBPBIBD)

BBPBIBD-1

$v = 12, \quad b = 3, \ r = 2, \ k = 8$

$\lambda_1 = 2, \ \lambda_2 = 1, \ m = 3, \ n = 4$

Blocks		
1	2	3
1	2	3
2	3	4
4	5	6
5	6	7
7	8	9
8	9	10
10	11	12
11	12	1

BBPBIBD-2

$v = 12, \quad b = 4, \ r = 3, \ k = 9$

$\lambda_1 = 3, \ \lambda_2 = 2, \ m = 4, \ n = 3$

Blocks			
1	2	3	4
1	2	3	4
2	3	4	5
3	4	5	6
5	6	7	8
6	7	8	9
7	8	9	10
9	10	11	12
10	11	12	1
11	12	1	2

BBPBIBD-3

$v = 12, \ b = 6, \ r = 5, \ k = 10$

$\lambda_1 = 5, \ \lambda_2 = 4, \ m = 6, \ n = 2$

Blocks					
1	2	3	4	5	6
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
7	8	9	10	11	12
8	9	10	11	12	1
9	10	11	12	1	2
10	11	12	1	2	3
11	12	1	2	3	4

BBPBIBD-4

$v = 12, \ b = 12, \ r = 4, \ k = 4$

$\lambda_1 = 2, \ \lambda_2 = 1, \ m = 6, \ n = 2$

Blocks											
1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12
2	3	4	5	6	7	8	9	10	11	12	1
5	6	7	8	9	10	11	12	1	2	3	4
7	8	9	10	11	12	1	2	3	4	5	6

Table 6.2. Plans for group divisible general binary partially balanced block designs derived from the designs in Table 6.1

GD BBPBIBD	d	Parameters of GD GBPBBD k^*_d	λ^*_{1d}	λ^*_{2d}	m_{0d}	m_{1d}	Optimality measures $m_{1d}-m_{0d}$	I^*_d	II^*_d	III^*_d	IV^*_d	V^*_d
-1	1	256	2048	1024	0	32	32	2*	2048	672.0	663.8	11264
	2	256	1926	1085	2	31	29	2*	1682	677.7	672.4	9251
	3	256	1816	1140	4	30	26	2*	1352	682.9	679.6	7436
	4	256	1718	1189	6	29	23	2*	1058	687.5	685.5	5819
	5	256	1632	1232	8	28	20	2*	800	691.5	690.4	4400
	6	256	1558	1269	10	27	17	2*	578	695.0	694.4	3179
	7	256	1496	1300	12	26	14	2*	392	698.0	697.6	2156
	8	256	1446	1325	14	25	11	2*	242	700.2	700.1	1331
	9	256	1408	1344	16	24	8	2*	128	702.0	702.0	704
	10	256	1382	1357	18	23	5	2*	50	703.2	703.2	275
	11	256	1368	1364	20	22	2	2*	8*	703.9*	703.9*	44*
-2	12	192	1324	924	1	21	20	3	1200	685.3	683.8	4400
	13	192	1216	960	4	20	16	3	768	692.0	691.4	2816
	14	192	1132	988	7	19	12	3	432	697.3	697.1	1584
	15	192	1072	1008	10	18	8	3	192	701.0	701.0	704
	16	192	1036	1020	13	17	4	3	48	703.3	703.2	176
-3	17	128	736	672	4	12	8	5	320	703.4	703.4	704
	18	128	686	682	9	11	2	5	20	703.7	703.7	44*
-4	19	64	512	256	0	16	16	32	8192	576.0	572.6	11264
	20	64	454	285	1	14	13	32	5408	619.5	618.1	7430
	21	64	408	308	2	12	10	32	3200	654.0	653.5	4400
	22	64	374	325	3	10	7	32	1568	679.5	679.4	2156
	23	64	352	336	4	8	4	32	512	696.0	696.0	704
	24	64	342	341	5	6	1*	32	32	703.5	703.5	44*

(1) $I^*_d = r_d(b_d - r_d)$

(2) $II^*_d = r_d(b_d - r_d)(m_{1d} - m_{0d})^2$

(3) $III^*_d = tr(\underline{C}^*_d)$

(4) IV^*_d = A-optimality criterion

(5) V^*_d = E-optimality criterion

Table 6.3. Optimal designs in each of six subclasses

Optimality measure	Optimal design number(s)
$m_{1d} - m_{0d}$	24
I_d^*	1 - 11
II_d^*	11
III_d^*	11
IV_d^*	11
V_d^*	11, 18, 24

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