MAXIMA OF CONTINUOUS TIME STATIONARY STABLE PROCESSES

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ABSTRACT. We study the suprema over compact time intervals of stationary locally bounded α-stable processes. The behaviour of these suprema as the length of the time interval increases turns out to depend significantly on the ergodic-theoretical properties of a flow generating the stationary process.

1. INTRODUCTION

Let $\mathbf{X} = \big(X(t), t \in \mathbb{R}\big)$ be a stationary, locally bounded, separable stochastic process. Then

$$M(t) = \sup_{0 \le s \le t} |X(s)|, \quad t \ge 0 \tag{1.1}$$

is a well defined finite valued stochastic process. Its distributional behavior as the length of the interval t increases is a subject of study of continuous time extreme value theory. Results are available for special classes of processes $\mathbf{X}$, such as certain Gaussian processes and diffusion processes; the two major references here are Leadbetter et al. (1983) and Berman (1992). A number of more recent general results is due to P. Albin (e.g. Albin (1990, 1992)).

In this paper we consider the class of real, stationary, locally bounded, separable and measurable symmetric α-stable (SαS) processes, $0 < \alpha < 2$. (Extensions to complex-valued processes will be mentioned in the sequel.) As we go along we will often drop the adjectives "separable and measurable" from the description of a process, but these assumptions remain in place. Such processes have an integral representation of the form

$$X(t) = \int_E a_t(x) \left(\frac{dm \circ \phi_t}{dm}(x) \right)^{1/\alpha} f \circ \phi_t(x)\, M(dx),\ t \in \mathbb{R} \tag{1.2}$$

where m is a σ-finite measure on E, M is a SαS random measure on E with control measure m, $f \in L^\alpha(m)$. Furthermore, (ϕ_t) is a measurable family of maps from E onto E such that $\phi_{t+s}(x) = \phi_t(\phi_s(x))$ for all $t, s \in \mathbb{R}$ and $x \in E$, $\phi_0(x) = x$ for all $x \in E$, and $m \circ \phi_t^{-1} \sim m$ for all $t \in \mathbb{R}$. The assumptions mean that the family (ϕ_t) forms a measurable nonsingular flow on E. Finally, (a_t) is a measurable family of $\{-1, 1\}$-valued functions on E such that for every $s, t \in \mathbb{R}$

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we have $a_{t+s}(x) = a_s(x)a_t(\phi_s(x))$ m-almost everywhere on E. This means that the family (a_t) forms a cocycle for the flow (ϕ_t).

This representation is due to Rosiński (1995). A general source of information on stable processes, stable random measures, integrals and integral representation of stable processes is Samorodnitsky and Taqqu (1994).

The basic ergodic theory (see e.g. Krengel (1985) or Aaronson (1997)) leads to a natural decomposition of stationary symmetric α-stable processes as follows. There exists a (unique up to a set of measure zero) measurable set $C \subset E$ invariant under each of the maps ϕ_t and such that each map ϕ_t is conservative on C and dissipative on $D = C^c$. A flow is called dissipative if $C = \emptyset$ and conservative if $C = E$ (all the set equalities are up to a set of measure zero). In these two cases we say that the stationary SαS process (1.2) is generated by a dissipative or conservative flow respectively, and, in general, a stationary SαS process given in the form (1.2) can be decomposed into a sum of two independent stationary SαS processes

$$X(t) = \int_C a_t(x) \left(\frac{dm \circ \phi_t}{dm}(x)\right)^{1/\alpha} f \circ \phi_t(x)\ M(dx), + \int_D a_t(x) \left(\frac{dm \circ \phi_t}{dm}(x)\right)^{1/\alpha} f \circ \phi_t(x)\ M(dx)$$

$$:= X_C(t) + X_D(t),\ t \in I\!R. \tag{1.3}$$

It has been proven by Rosiński (1995), Theorem 4.3, that this decomposition is unique in law.

It has also been shown by Rosiński (1995), Theorem 4.4, that any stationary SαS process generated by a dissipative flow has a mixed moving average representation

$$X(t) = \int_W \int_{I\!R} f(v, x - t)\ M(dv, dx)\,, \quad t \in I\!R, \tag{1.4}$$

with M a symmetric α-stable random measure on a product measurable space $(W \times I\!R, \mathcal{W} \times \mathcal{B})$ with control measure $m = \nu \times \text{Leb}$, where ν is a σ-finite measure on $(W, \mathcal{W})$, and $f \in L^\alpha(m, \mathcal{W} \times \mathcal{B})$.

It turns out that the maximal process $(M(t), t \geq 0)$ in (1.1) grows at a different rate for stationary stable processes generated by conservative flows and those having a nonzero component generated by a dissipative flow in the decomposition (1.3). This is proven in the next section. A parallel result in the discrete time case is in Samorodnitsky (2002).

2. Rate of growth of the maximal process

Consider a locally bounded stationary SαS process given in the form

$$X(t) = \int_E f_t(x)\, M(dx),\ t \in I\!R \tag{2.1}$$

where M is a SαS random measure on E with a σ-finite control measure m, and the functions $f_t \in L^\alpha(m)$ for $t \in I\!R$ may or may not be of the form given in (1.2). In this section we study the rate of growth of the maximal process (1.1) depending on the kind of the flow the process $\mathbf{X}$ is generated by. We will use the following procedure to avoid the usual measurability problems. Since the process $\mathbf{X}$ is measurable and stationary, it is continuous in probability (see, e.g. Aaronson (1997), page 48). Therefore, we can take its separable version such that the maximal process (1.1) is defined via

$$M(t) = \sup_{s\in[0,t]\cap Q_{\text{bin}}} |X(s)|, \quad t \geq 0,$$

where Q_{bin} is the set of the binary rational rational numbers in $[0,\infty)$. We will treat, without further discussion, all the suprema of functions encountered in the sequel in the same way: for example, when we write for some set A $\sup_{t\in A} |f_t(x)|$ we mean the measurable function $\sup_{t\in A\cap Q_{\text{bin}}} |f_t(x)|$.

Define for $T \geq 0$

$$b(T) = \left(\int_E \sup_{0\leq t\leq T} |f_t(x)|^\alpha \, m(dx) \right)^{1/\alpha} . \tag{2.2}$$

Since the process $\mathbf{X}$ is locally bounded, it turns out that $b(T) < \infty$ for all $T \geq 0$; see e.g. Theorem 10.2.3 in Samorodnitsky and Taqqu (1994). We remark, further, that $b(T)$ does not depend on a particular integral representation of the process; this can be easily deduced from, say, Corollary 4.4.6 *ibid.*

We start with proving that, as in the discrete time case, the rate of growth of the function $b(T)$ depends significantly on the flow generating the stable process.

Theorem 2.1. *(i) If the flow (ϕ_t) is conservative, then*

$$T^{-1/\alpha} b(T) \to 0 \quad \textit{as } T \to \infty. \tag{2.3}$$

(ii) If the flow is dissipative, then

$$\lim_{T\to\infty} T^{-1/\alpha} b(T) = \left(\int_W g(v)^\alpha \, \nu(dv) \right)^{1/\alpha} \in (0,\infty) \tag{2.4}$$

where one can use any mixed moving average representation (1.4) of the process and

$$g(v) = \sup_{x\in\mathbb{R}} |f(v,x)| \quad \textit{for } v \in W. \tag{2.5}$$

Proof. (i) Suppose that the flow (ϕ_t) is conservative. Let $\epsilon > 0$. Since $b(T) < \infty$ for all $T \geq 0$, monotone convergence theorem implies that there is a $k = 1, 2, \ldots$ such that

$$\int_E \max_{j=0,1,\ldots,k} |f_{j/k}(x)|^\alpha \, m(dx) \geq \int_E \sup_{0\leq t\leq 1} |f_t(x)|^\alpha \, m(dx) - \epsilon. \tag{2.6}$$

The discrete time stationary SαS process

$$Y_n = X(n/k) = \int_E f_{n/k}(x)\, M(dx),\ n = 0, 1, 2, \dots$$

is then generated by a conservative discrete-time flow (see e.g. Theorem 3.4 in Krengel (1985)). Defining

$$\tilde{b}_n = \left(\int_E \max_{j=0,1,\dots,n} |f_{j/k}(x)|^\alpha\, m(dx) \right)^{1/\alpha},\ n = 0, 1, \dots,$$

we conclude that

$$\lim_{n\to\infty} n^{-1/\alpha} \tilde{b}_n = 0; \tag{2.7}$$

see Theorem 3.1 in Samorodnitsky (2002).

Note that by stationarity, for every $N = 1, 2, \dots$

$$\int_E \sup_{0\le t\le N} |f_t(x)|^\alpha\, m(dx)$$

$$\le \int_E \max_{j=0,1,\dots,Nk} |f_{j/k}(x)|^\alpha\, m(dx) + \sum_{i=1}^N \int_E \left(\sup_{i-1\le t\le i} |f_t(x)|^\alpha - \max_{j=(i-1)k,\dots,ik} |f_{j/k}(x)|^\alpha \right) m(dx)$$

$$= \tilde{b}^\alpha_{NK} + N \int_E \left(\sup_{0\le t\le 1} |f_t(x)|^\alpha - \max_{j=0,\dots,k} |f_{j/k}(x)|^\alpha \right) m(dx)$$

$$\le \tilde{b}^\alpha_{Nk} + N\epsilon$$

by (2.6). Therefore, for every $T \ge 0$

$$b(T)^\alpha \le b([T]+1)^\alpha \le \tilde{b}^\alpha_{([T]+1)k} + (T+1)\epsilon,$$

and using (2.7) we conclude that

$$\limsup_{T\to\infty} \frac{1}{T} b(T)^\alpha \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (2.3) follows.

(ii) Fix any mixed moving average representation (1.4) of the process. Notice, first of all, that

$$\int_W g(v)^\alpha\, \nu(dv) \le \sum_{i=-\infty}^\infty \int_W \sup_{x\in[i,i+1)} |f(v,x)|^\alpha\, \nu(dv)$$

$$\le \sum_{i=-\infty}^\infty \int_W \left(\int_{i+1}^{i+2} \sup_{0\le t\le 2} |f(v, x-t)|^\alpha\, dx \right) \nu(dv)$$

$$= \int_W \int_{-\infty}^\infty \sup_{0\le t\le 2} |f(v,x-t)|^\alpha\, dx\ \nu(dv) = b(2)^\alpha < \infty.$$

The proof of (2.4) is similar to the corresponding statement in the discrete time case. Start with the case where f has a compact support, that is

$$f(v,x)=0 \quad \text{for all } (v,x) \text{ with } |x|>A, \text{ some } A>0. \tag{2.8}$$

Then for all $T>2A$

$$\begin{aligned} b(T)^\alpha &= \int_{-A-T}^{A}\int_W \sup_{0\le t\le T}|f(v,x+t)|^\alpha\,\nu(dv)\,dx = \int_{-A-T}^{A-T}\int_W \sup_{0\le t\le T}|f(v,x+t)|^\alpha\,\nu(dv)\,dx \\ &+ \int_{A-T}^{-A}\int_W \sup_{0\le t\le T}|f(v,x+t)|^\alpha\,\nu(dv)\,dx + \int_{-A}^{A}\int_W \sup_{0\le t\le T}|f(v,x+t)|^\alpha\,\nu(dv)\,dx \\ &:= R_T^{(1)}+G_T+R_T^{(2)}\,. \end{aligned} \tag{2.9}$$

Since

$$G_T=(T-2A)\int_W g(v)^\alpha\,\nu(dv)$$

and

$$R_T^{(i)}\le 2A\int_W g(v)^\alpha\,\nu(dv) \quad \text{for } i=1,2,$$

the statement (2.4) follows in the case of a compactly supported f.

For a general f, define for $m>0$

$$f_m(v,x)=f(v,x)\mathbf{1}(|x|\le m) \;\text{ and }\; g_m(v)=\sup_{x\in I\!R}|f_m(v,x)|=\sup_{|x|\le m}|f(v,x)|,$$

$v\in W$ and $x\in I\!R$. Then every f_m is compactly supported, and for every $v\in W$

$$g_m(v)\uparrow g(v) \quad \text{as } m\to\infty.$$

Therefore,

$$\int_W g_m(v)^\alpha\,\nu(dv)\uparrow\int_W g(v)^\alpha\,\nu(dv) \quad \text{as } m\to\infty. \tag{2.10}$$

Furthermore, for every $v\in W$ and $x\in I\!R$,

$$\sup_{0\le t\le 1}\left(|f(v,t-x)|^\alpha-|f_m(v,t-x)|^\alpha\right)\to 0 \quad \text{as } m\to\infty$$

(in fact, the expression is, eventually, equal to 0,) and so by the dominated convergence theorem,

$$\int_W\int_{-\infty}^{\infty}\sup_{0\le t\le 1}\left(|f(v,t-x)|^\alpha-|f_m(v,t-x)|^\alpha\right)\,dx\,\nu(dv)\to 0 \quad \text{as } m\to\infty. \tag{2.11}$$

We have for all $T>0$ and $m>0$

$$\begin{aligned} &\left|\frac{1}{T}b(T)^\alpha-\int_W g(v)^\alpha\,\nu(dv)\right| \\ &\le\frac{1}{T}\left|\int_W\int_{-\infty}^{\infty}\sup_{0\le t\le T}|f(v,t-x)|^\alpha\,dx\,\nu(dv)-\int_W\int_{-\infty}^{\infty}\sup_{0\le t\le T}|f_m(v,t-x)|^\alpha\,dx\,\nu(dv)\right| \end{aligned}$$

$$+\left| \frac{1}{T} \int_W \int_{-\infty}^{\infty} \sup_{0\le t\le T} |f_m(v,t-x)|^\alpha \, dx \, \nu(dv) - \int_W g_m(v)^\alpha \, \nu(dv) \right|$$
$$+\left| \int_W g(v)^\alpha \, \nu(dv) - \int_W g_m(v)^\alpha \, \nu(dv) \right| := A_{m,T} + B_{m,T} + C_m.$$

Since the claim (2.4) has been proved for compactly supported f, we see that for every $m > 0$, $B_{m,T} \to 0$ as $T \to \infty$. Further, $C_m \to 0$ as $m \to \infty$ by (2.10). Finally,

$$A_{m,T} \le \frac{1}{T} \int_W \int_{-\infty}^{\infty} \sup_{0\le t\le T} \left(|f(v,t-x)|^\alpha - |f_m(v,t-x)|^\alpha \right) \, dx \, \nu(dv)$$
$$\le \frac{1}{T} \int_W \int_{-\infty}^{\infty} \sum_{j=1}^{\lceil T \rceil} \sup_{j-1\le t\le j} \left(|f(v,t-x)|^\alpha - |f_m(v,t-x)|^\alpha \right) \, dx \, \nu(dv)$$
$$= \frac{\lceil T \rceil}{T} \int_W \int_{-\infty}^{\infty} \sup_{0\le t\le 1} \left(|f(v,t-x)|^\alpha - |f_m(v,t-x)|^\alpha \right) \, dx \, \nu(dv).$$

We now obtain (2.4) by letting first $T \to \infty$ and then $m \to \infty$. □

Theorem 2.1 is the main ingredient in both the statement and the proof of the main result of this paper, which we state next. This is a continuous time extension of the result on the behavior of the maxima of discrete time stationary SαS processes in Theorem 4.1 in Samorodnitsky (2002).

Theorem 2.2. *Let $\mathbf{X} = \big(X(t), t \in \mathbb{R}\big)$ be a stationary, locally bounded SαS process, $0 < \alpha < 2$.*

(i) Suppose that the dissipative component $\mathbf{X}_D$ in the decomposition (1.3) of the process $\mathbf{X}$ is not zero. Then

$$T^{-1/\alpha} M(T) \Rightarrow C_\alpha^{1/\alpha} K_X Z_\alpha \tag{2.12}$$

weakly as $T \to \infty$, where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1}, \tag{2.13}$$

and Z_α is a standard Frechét random variable with distribution function

$$P\big(Z_\alpha \le z\big) = e^{-z^{-\alpha}}, \ z > 0.$$

Finally,

$$K_X = \left(\int_W g(v)^\alpha \, \nu(dv) \right)^{1/\alpha},$$

in the notation of Theorem 2.1, where one can use an arbitrary representation of $\mathbf{X}_D$ as a mixed moving average (1.4).

(ii) Suppose that $\mathbf{X}$ is generated by a conservative flow. Then

$$T^{-1/\alpha} M(T) \to 0 \quad \text{in probability} \tag{2.14}$$

as $T \to \infty$.

Furthermore, for every positive function $c(T) = o(b(T))$ as $T \to \infty$ we have

$$\big(c(T)^{-1} M(T)\big) \quad \text{is not tight,} \tag{2.15}$$

while if for some $\theta > 0$ and $c > 0$

$$b(T) \geq cT^{\theta} \quad \text{for all } T \text{ large enough,} \tag{2.16}$$

then

$$\big(b(T)^{-1} M(T)\big) \quad \text{is tight.} \tag{2.17}$$

Finally, for $T > 0$ define a probability measure ν_T on $(E, \mathcal{E})$ by

$$\frac{d\eta_T}{dm}(x) = b(T)^{-\alpha} \sup_{0 \leq t \leq T} |f_t(x)|^{\alpha}\,,\ x \in E\,. \tag{2.18}$$

Let $U_j^{(T)}$, $j = 1, 2$ be independent E-valued random variables with common law η_T. Suppose that (2.16) holds and that, in addition, for any $\epsilon > 0$

$$P\left(\text{for some } 0 \leq t \leq T,\ \frac{\left|f_t\left(U_j^{(T)}\right)\right|}{\sup_{0 \leq s \leq T}\left|f_s\left(U_j^{(T)}\right)\right|} > \epsilon,\ j = 1, 2\right) \to 0 \tag{2.19}$$

as $T \to \infty$. Then

$$b(T)^{-1} M(T) \Rightarrow C_{\alpha}^{1/\alpha}\, Z_{\alpha} \tag{2.20}$$

weakly as $T \to \infty$. A sufficient condition for (2.19) is

$$\lim_{T \to \infty} \frac{b(T)}{T^{1/2\alpha}} = \infty. \tag{2.21}$$

Proof. Similarly to the argument in the proof of the discrete time result in Samorodnitsky (2002), we will use a series representation (in law) of the process $\{X(t),\, 0 \leq t \leq T\}$ in the form

$$X(t) = b(T)\, C_{\alpha}^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j\, \Gamma_j^{-1/\alpha}\, \frac{f_t\left(U_j^{(T)}\right)}{\sup_{0 \leq s \leq T}\left|f_s\left(U_j^{(T)}\right)\right|}\,,\ 0 \leq t \leq T\,, \tag{2.22}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are iid Rademacher random variables (symmetric ± 1-valued random variables), $\Gamma_1, \Gamma_2, \ldots$ is a sequence of the arrival times of a unit rate Poisson process on $(0, \infty)$, and $\big(U_j^{(T)}\big)$ are iid E-valued random variables with common law given by (2.18). All three sequences are independent. See Samorodnitsky and Taqqu (1994) for details.

We start with proving (2.15). We use the above series representation. For $T > 0$ and $m = 1, 2, \ldots$ let

$$K_m(T) = \min\left(k = 0, 1, 2, \ldots : \left|f_{k/2^m}\left(U_1^{(n)}\right)\right| = \max_{i = 0, 1, \ldots;\, i/2^m \leq T}\left|f_{i/2^m}\left(U_1^{(n)}\right)\right|\right).$$

Let $\mathcal{G}$ be the σ-field generated by ε_1, $\left(\Gamma_j,\, j \geq 1\right)$ and $\left(U_j^{(n)},\, j \geq 1\right)$, and note that $K_m(T)$ is measurable $\mathcal{G}$. Notice that for any $x > 0$,

$$P\left(c(T)^{-1}M(T) > x\right) = P\left(\sup_{t\in[0,T]\cap Q_{\rm bin}} |X(t)| > c(T)x\right)$$

$$= \lim_{m\to\infty} P\left(\max_{i=0,1,\dots;\, i/2^m\leq T} |X(i/2^m)| > c(T)x\right).$$

We have by the symmetry, for any $m = 1, 2\dots$

$$P\left(\max_{i=0,1,\dots;\, i/2^m\leq T} |X(i/2^m)| > c(T)x\right) \geq P\left(|X(K_m(T)/2^m)| > c(T)x\right)$$

$$= E\left(P\left(b(T)\, C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j\, \Gamma_j^{-1/\alpha} \frac{f_{K_m(T)/2^m}\left(U_j^{(T)}\right)}{\sup_{0\leq s\leq T}\left|f_s\left(U_j^{(T)}\right)\right|} > c(T)x \middle| \mathcal{G}\right)\right)$$

$$\geq \frac{1}{2} E\left(P\left(\Gamma_1^{-1/\alpha} \frac{f_{K_m(T)/2^m}\left(U_1^{(T)}\right)}{\sup_{0\leq s\leq T}\left|f_s\left(U_1^{(T)}\right)\right|} > C_\alpha^{1/\alpha} \frac{c(T)}{b(T)} x \middle| \mathcal{G}\right)\right)$$

$$= \frac{1}{2} P\left(\Gamma_1^{-1/\alpha} \frac{f_{K_m(T)/2^m}\left(U_1^{(T)}\right)}{\sup_{s\in[0,T]\cap Q_{\rm bin}}\left|f_s\left(U_1^{(T)}\right)\right|} > C_\alpha^{1/\alpha} \frac{c(T)}{b(T)} x\right)$$

$$= \frac{1}{2} P\left(\Gamma_1^{-1/\alpha} \frac{\max_{i=0,1,\dots;\, i/2^m\leq T}\left|f_{i/2^m}\left(U_1^{(n)}\right)\right|}{\sup_{s\in[0,T]\cap Q_{\rm bin}}\left|f_s\left(U_1^{(T)}\right)\right|} > C_\alpha^{1/\alpha} \frac{c(T)}{b(T)} x\right).$$

Therefore,

$$P\left(c(T)^{-1}M(T) > x\right) \geq \frac{1}{2} P\left(\Gamma_1^{-1/\alpha} > C_\alpha^{1/\alpha} \frac{c(T)}{b(T)} x\right)$$

for all $x > 0$ and $T > 0$. Since the right hand side above converges to $1/2$ as $T \to \infty$ for all $x > 0$, the lack of tightness follows.

Suppose now that (2.16) holds. We will prove that there is $\epsilon > 0$ so small that for any $\lambda > 0$

$$\lim_{T\to\infty} P\left(M(T) > b(T)\lambda,\, \Gamma_1^{-1/\alpha} \leq \epsilon\lambda\right) = 0. \tag{2.23}$$

To this end observe that for any $T > 0$ and $\epsilon > 0$

$$P\left(M(T) > b(T)\lambda,\, \Gamma_1^{-1/\alpha} \leq \epsilon\lambda\right) \tag{2.24}$$

$$\leq \sum_{i=1}^{\lceil T\rceil} P\left(\sup_{i-1\leq t\leq i} |X(t)| > b(T)\lambda,\, \Gamma_j^{-1/\alpha} \frac{\sup_{i-1\leq t\leq i}\left|f_t\left(U_j^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left|f_s\left(U_j^{(T)}\right)\right|} \leq \epsilon\lambda \;\text{ for all } j = 1, 2, \dots\right).$$

Note that for a fixed i the points

$$\left\{ b(T)\epsilon_j \Gamma_j^{-1/\alpha} \left(\sup_{0\le s\le T} \left| f_s\left(U_j^{(T)}\right)\right| \right)^{-1} \left(f_t\left(U_j^{(T)}\right),\, i-1\le t\le i\right),\, j=1,2,\dots \right\} \tag{2.25}$$

form a (symmetric) Poisson random measure on $I\!R^{[0,1]}$ equipped with the cylindrical σ-field, whose mean measure is given by

$$\eta_i(A) = \frac{1}{2}\int_E \int_0^\infty \Big[\mathbf{1}\left(y^{-1/\alpha}\left(f_t(s),,\, i-1\le t\le i\right) \in A\right)$$
$$+\mathbf{1}\left(y^{-1/\alpha}\left(f_t(s),,\, i-1\le t\le i\right) \in -A\right)\Big]\, dy\, m(ds),$$

A a cylindrical set. We claim that η_i is the same for all i. To this end, it is enough to check that $\eta_i(A)$ is independent of i for sets A of the form

$$A = \left\{ g\in I\!R^{[0,1]} :\, (g(t_1),\dots,g(t_d))\in B\right\}$$

for $d=1,2,\dots$, $0\le t_1<\dots<t_d\le 1$ and $B\in\mathcal{B}^d$. For a fixed d and $t_1,\dots,t_d$, we obtain this way a σ-finite measure on $I\!R^d$; let us denote it by γ_i. Observe that for any $\delta>0$

$$\gamma_i\left(\left([-\delta,\delta]^d\right)^c\right) = \int_E\int_0^\infty \mathbf{1}\left(y^{-1/\alpha}|f_{t_j}(s)|>\delta \ \text{ for some } j=1,\dots,d\right) dy\, m(ds)$$
$$\le \sum_{j=1}^d \int_E\int_0^\infty \mathbf{1}\left(y^{-1/\alpha}|f_{t_j}(s)|>\delta\right) dy\, m(ds) = d\delta^{-\alpha}\int_E f_0(s)^\alpha\, m(ds).$$

Therefore, each γ_i is a symmetric Lévy measure on $I\!R^d$, and so to show that γ_i is the same for all i, it is enough to check that $\gamma_i(B)$ is independent of i for sets B of the form

$$B = \left\{ (z_1,\dots,z_d)\in I\!R^d :\, \left|\theta_1 z_1+\dots+\theta_d z_d\right| > \lambda\right\}$$

for $\theta_1,\dots,\theta_d\in I\!R$ and $\lambda>0$.

However, for each B of this form

$$\gamma_i(B) = \int_E\int_0^\infty \mathbf{1}\left(y^{-1/\alpha}\left|\sum_{j=1}^d \theta_j f_{t_j+i-1}(s)\right| > \lambda\right) dy\, m(ds)$$
$$= \lambda^{-\alpha}\int_E \left|\sum_{j=1}^d \theta_j f_{t_j+i-1}(s)\right|^\alpha m(ds) = \lambda^{-\alpha}\int_E \left|\sum_{j=1}^d \theta_j f_{t_j}(s)\right|^\alpha m(ds)$$

by stationarity of the process, providing us with the required independence of i.

The immediate conclusion is that the terms in the right hand side of (2.24) are independent of i, and so

$$P\left(M(T)>b(T)\lambda,\, \Gamma_1^{-1/\alpha}\le\epsilon\lambda\right) \tag{2.26}$$

$$\leq \lceil T\rceil P\left(\sup_{0\leq t\leq 1}|X(t)|>b(T)\lambda,\,\Gamma_j^{-1/\alpha}\frac{\sup_{0\leq t\leq 1}\left|f_t\left(U_j^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left|f_s\left(U_j^{(T)}\right)\right|}\leq \epsilon\lambda\ \text{ for all } j=1,2,\ldots\right).$$

Furthermore, the above argument also shows the the Poisson random measure with the points (2.25) with $i=1$ can be also represented by

$$\left\{b(1)\epsilon_j\Gamma_j^{-1/\alpha}\left(\sup_{0\leq s\leq 1}|f_s(V_j)|\right)^{-1}\left(f_t(V_j)\,,\,0\leq t\leq 1\right),\,j=1,2,\ldots\right\},$$

where (V_j) are iid E-valued random variables with the same law as $\left(U_j^{(1)}\right)$, independent of the sequences (ε_j) and (Γ_j). Therefore,

$$P\left(M(T)>b(T)\lambda,\,\Gamma_1^{-1/\alpha}\leq \epsilon\lambda\right) \tag{2.27}$$

$$\leq \lceil T\rceil P\left(b(1)C_\alpha^{1/\alpha}\sup_{0\leq t\leq 1}\left|\sum_{j=1}^\infty \varepsilon_j\,\Gamma_j^{-1/\alpha}\frac{f_t(V_j)}{\sup_{0\leq s\leq 1}|f_s(V_j)|}\right|>b(T)\lambda,\ b(1)\Gamma_1^{-1/\alpha}\leq \epsilon\, b(T)\lambda\right).$$

Let $K=1,2,\ldots$ be such that

$$\frac{1}{\alpha\theta}-1<K<\frac{1}{\epsilon C_\alpha^{1/\alpha}}. \tag{2.28}$$

Note that this choice of K is possible as long as we choose $\epsilon>0$ appropriately small. We see by (2.27) and (2.28) that

(2.29)
$$P\left(M(T)>b(T)\lambda,\,\Gamma_1^{-1/\alpha}\leq \epsilon\lambda\right)\leq \lceil T\rceil P\left(\sup_{0\leq t\leq 1}\left|\sum_{j=K+1}^\infty \varepsilon_j\,\Gamma_j^{-1/\alpha}\frac{f_t(V_j)}{\sup_{0\leq s\leq 1}|f_s(V_j)|}\right|>\beta\, b(T)\right),$$

with $\beta=\left(1-K\epsilon C_\alpha^{1/\alpha}\right)b(1)^{-1}C_\alpha^{-1/\alpha}\lambda>0$.

Denote

$$g_t(x)=\frac{f_t(x)}{\sup_{0\leq s\leq 1}|f_s(x)|},\quad 0\leq t\leq 1,\ s\in E.$$

Clearly,

$$\sup_{0\leq t\leq 1}|g_t(x)|=1\ \text{ for all } x\in E, \tag{2.30}$$

and we have

$$P\left(\sup_{0\leq t\leq 1}\left|\sum_{j=K+1}^\infty \varepsilon_j\,\Gamma_j^{-1/\alpha}g_t(V_j)\right|>\beta\, b(T)\right) \tag{2.31}$$

$$=\int_0^\infty e^{-x}\frac{x^K}{K!}P\left(\sup_{0\leq t\leq 1}\left|\sum_{j=1}^\infty \varepsilon_j\,\left(x+\Gamma_j\right)^{-1/\alpha}g_t(V_j)\right|>\beta\, b(T)\right)dx.$$

Notice that by the contraction inequality for Rademacher series (see e.g. Proposition 1.2.1 in Kwapień and Woyczyński (1992)), for every $x > 0$ and $u > 0$ we have

$$P\left(\sup_{0\le t\le 1}\left|\sum_{j=1}^{\infty}\varepsilon_j\ (x+\Gamma_j)^{-1/\alpha}\ g_t(V_j)\right| > u\right) \le 2P\left(\sup_{0\le t\le 1}\left|\sum_{j=1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\ g_t(V_j)\right| > u\right).$$

Hence there is $r > 0$ independent of x such that

$$P\left(\sup_{0\le t\le 1}\left|\sum_{j=1}^{\infty}\varepsilon_j\ (x+\Gamma_j)^{-1/\alpha}\ g_t(V_j)\right| > \frac{r}{8}\right) \le \frac{1}{32}$$

for all $x > 0$. Since for every $x > 0$

$$\left\{\sum_{j=1}^{\infty}\varepsilon_j\ (x+\Gamma_j)^{-1/\alpha}\ g_t(V_j),\ 0\le t\le 1\right\}$$

is an infinitely divisible random vector in $\mathbb{R}^{[0,1]}$ whose Lévy measure is supported by the set

$$\left\{y\in\mathbb{R}^{[0,1]}:\ \sup_{0\le t\le 1}|y(t)|\le x^{-1/\alpha}\right\},$$

it follows by standard arguments (see e.g. proof of Lemma 2.2 in Rosiński and Samorodnitsky (1993)) that for all $x > 0$

$$E\exp\left\{\frac{\log 2}{r+2x^{-1/\alpha}}\sup_{0\le t\le 1}\left|\sum_{j=1}^{\infty}\varepsilon_j\ (x+\Gamma_j)^{-1/\alpha}\ g_t(V_j)\right|\right\}\le 4. \tag{2.32}$$

We use now an exponential Markov inequality in (2.31) to obtain the following bound:

$$\begin{aligned}P\left(\sup_{0\le t\le 1}\left|\sum_{j=K+1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\,g_t(V_j)\right| > \beta\, b(T)\right)\\ \le 4\int_0^\infty e^{-x}\frac{x^K}{K!}\exp\left\{-\frac{\beta\log 2}{r+2x^{-1/\alpha}}b(T)\right\}dx.\end{aligned} \tag{2.33}$$

Using (2.16) we see that

$$\int_1^\infty e^{-x}\frac{x^K}{K!}\exp\left\{-\frac{\beta\log 2}{r+2x^{-1/\alpha}}b(T)\right\}dx \le e^{-C_1T^\theta}\int_1^\infty e^{-x}\frac{x^K}{K!}\,dx = C_2e^{-C_1T^\theta}$$

for some positive constants $C_i = C_i(\epsilon, K)$, $i = 1, 2$. Furthermore,

$$\int_0^1 e^{-x}\frac{x^K}{K!}\exp\left\{-\frac{\beta\log 2}{r+2x^{-1/\alpha}}b(T)\right\}dx \le \int_0^1 e^{-x}\frac{x^K}{K!}\exp\left\{-C_3x^{1/\alpha}T^\theta\right\}dx \le C_4T^{-\alpha\theta(K+1)}$$

for some other positive constants $C_i = C_i(\epsilon, K)$, $i = 3, 4$.

Putting everything together we see that

$$P\left(M(T) > b(T)\lambda,\,\Gamma_1^{-1/\alpha}\le\epsilon\lambda\right)\le\lceil T\rceil\left(C_2e^{-C_1T^\theta}+C_4T^{-\alpha\theta(K+1)}\right)\to 0$$

as $T\to\infty$ by (2.28). This proves (2.23) for ϵ small enough.

We note at this point that the same argument proves that for any $\epsilon_1, \epsilon_2 > 0$ such that ϵ_2/ϵ_1 is small enough,

$$\lim_{T\to\infty} P\Biggl(M(T) > b(T)\lambda,\, \Gamma_1^{-1/\alpha} \leq (1-\epsilon_1)\lambda, \text{ and for each } 0 \leq t \leq T, \tag{2.34}$$

$$\Gamma_j^{-1/\alpha} \frac{\left| f_t\left(U_j^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left| f_s\left(U_j^{(T)}\right)\right|} > \epsilon_2\lambda \text{ for at most one } j = 1,2,\ldots \Biggr) = 0$$

and

$$\lim_{T\to\infty} P\Biggl(M(T) \leq b(T)\lambda,\, \Gamma_1^{-1/\alpha} > (1+\epsilon_1)\lambda, \text{ and for each } 0 \leq t \leq T, \tag{2.35}$$

$$\Gamma_j^{-1/\alpha} \frac{\left| f_t\left(U_j^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left| f_s\left(U_j^{(T)}\right)\right|} > \epsilon_2\lambda \text{ for at most one } j = 1,2,\ldots \Biggr) = 0.$$

Indeed, the probability in the left hand side of, say, (2.34) can be bounded from above by

$$\lceil T\rceil P\left(\sup_{0\leq t\leq 1} \left| \sum_{j=K+1}^{\infty} \varepsilon_j\, \Gamma_j^{-1/\alpha}\, \frac{f_t(V_j)}{\sup_{0\leq s\leq 1}|f_s(V_j)|}\right| > \beta'\, b(T)\right)$$

in the notation of (2.29), where β' is a positive function of $K, \lambda, \epsilon_1, \epsilon_2$, and K can be chosen arbitrarily large by making the ratio ϵ_2/ϵ_1 is small enough. Therefore, the argument leading to (2.23) applies. One approaches (2.35) in the similar way.

Fix now ϵ for which (2.23) holds, and notice that

$$P\left(b(T)^{-1}M(T) > \lambda\right) \leq P\left(M(T) > b(T)\lambda,\, \Gamma_1^{-1/\alpha} \leq \epsilon\lambda\right) + P\left(\Gamma_1^{-1/\alpha} > \epsilon\lambda\right).$$

Given $\delta > 0$, choose λ so large that the second term above is less than $\delta/2$ and then choose T_0 such that the first term above is less than $\delta/2$ for all $T > T_0$. By the local boundedness of the process $\mathbf{X}$, for every $0 \leq T \leq T_0$

$$P\left(b(T)^{-1}M(T) > \lambda\right) \leq P\left(M(T_0) > b(0)\lambda\right) \leq \delta$$

if λ is large enough. This establishes (2.17).

Furthermore, the claim (2.14) follows from the just proven statement (2.17) by using part (i) of Theorem 2.1 and adding to $\mathbf{X}$, if necessary, an independent term to ensure that (2.16) holds. See a similar argument in Samorodnitsky (2002).

The next step is to check that the condition (2.21) implies (2.19). To this end, denote by $r(T)$ the probability in (2.19), and note that

$$r(T) \leq \lceil T \rceil \left(P\left(\frac{\sup_{0\leq t\leq 1}\left|f_t\left(U_1^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left|f_s\left(U_1^{(T)}\right)\right|} > \epsilon \right)\right)^2$$

$$= \lceil T \rceil b(T)^{-2\alpha} \left(\int_E \mathbf{1}\left(\frac{\sup_{0\leq t\leq 1}|f_t(x)|}{\sup_{0\leq s\leq T}|f_s(x)|} > \epsilon \right) \sup_{0\leq s\leq T} |f_s(x)|^\alpha \, m(dx) \right)^2$$

$$\leq \lceil T \rceil b(T)^{-2\alpha}\epsilon^{-2\alpha} \left(\int_E \sup_{0\leq t\leq 1} |f_1(x)|^\alpha \, m(dx) \right)^2 = \lceil T \rceil b(T)^{-2\alpha}\epsilon^{-2\alpha} b(1)^{2\alpha} \to 0$$

as $T \to \infty$ by the local boundedness of the process, and so we have (2.19).

Assuming that (2.19) holds, we immediately have

(2.36)

$$\varphi(T) := P\left(\text{for some } 0 \leq t \leq T,\ \Gamma_j^{-1/\alpha} \frac{\left|f_t\left(U_j^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left|f_s\left(U_j^{(T)}\right)\right|} > \epsilon \ \text{ for 2 different } j \right) \to 0$$

as $T \to \infty$. Since for every $0 < \delta < 1$ and $\epsilon > 0$ we have

$$P\left(b(T)^{-1}M(T) > \lambda\right) \leq P\left(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}(1-\delta)\lambda\right) + \varphi(T)$$

$$+P\Bigg(M(T) > b(T)\lambda,\, \Gamma_1^{-1/\alpha} \leq (1-\delta)\lambda, \ \text{ and for each } 0 \leq t \leq T,$$

$$\Gamma_j^{-1/\alpha} \frac{\left|f_t\left(U_j^{(T)}\right)\right|}{\sup_{0\leq s\leq T}\left|f_s\left(U_j^{(T)}\right)\right|} > \epsilon\lambda \ \text{ for at most one } j = 1, 2, \ldots \Bigg),$$

by selecting ϵ small enough relatively to δ and using (2.34) and (2.36), we obtain

$$\limsup_{T\to\infty} P\left(b(T)^{-1}M(T) > \lambda\right) \leq P\left(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}(1-\delta)\lambda\right)$$

$$= 1 - \exp\left\{-C_\alpha\, \lambda^{-\alpha}(1-\delta)^{-\alpha}\right\},$$

and by letting $\delta \to 0$ we see that

(2.37)
$$\limsup_{T\to\infty} P\left(b(T)^{-1}M(T) > \lambda\right) \leq 1 - \exp\left\{-C_\alpha\, \lambda^{-\alpha}\right\}.$$

Similarly, for every $0 < \delta < 1$ and $\epsilon > 0$ we have

$$P\left(b(T)^{-1}M(T) > \lambda\right) \geq P\left(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}(1+\delta)\lambda\right) + \varphi(T)$$

$$+P\Bigg(M(T) \leq b(T)\lambda,\, \Gamma_1^{-1/\alpha} > (1+\delta)\lambda, \ \text{ and for each } 0 \leq t \leq T,$$

$$\Gamma_j^{-1/\alpha} \frac{\left| f_t \left(U_j^{(T)} \right) \right|}{\sup_{0 \le s \le T} \left| f_s \left(U_j^{(T)} \right) \right|} > \epsilon\lambda \text{ for at most one } j = 1, 2, \dots \Bigg) .$$

Now by selecting ϵ small enough relatively to δ and using (2.35) and (2.36), we obtain

$$\begin{aligned} \liminf_{T\to\infty} P\left(b(T)^{-1}M(T) > \lambda\right) &\ge P\left(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}(1+\delta)\lambda\right) \\ &= 1 - \exp\left\{-C_\alpha \lambda^{-\alpha}(1+\delta)^{-\alpha}\right\} , \end{aligned}$$

and by letting $\delta \to 0$ we obtain a lower bound matching (2.37) and, hence, prove (2.20).

Finally, the statement of the first part of the theorem follows from Theorem 2.1 and (2.20). This concludes the proof. □

Remark 2.3. The statement of Theorem 2.2 extends easily to the complex-valued processes. Furthermore, one can also derive a "one-sided" result for the growth of $\sup_{0 \le s \le t} X(s)$. The procedure is similar to the one in the discrete time case. See Samorodnitsky (2002) for details.

3. An example

In this section we consider a new class of stationary SαS processes generated by conservative flows and apply Theorem 2.2 to investigate the rate of growth of the maxima of these processes.

Let $\mathbf{B}_H = (B_H(t), -\infty < t < \infty)$ be the standard Fractional Brownian motion, a centered stationary increment Gaussian process, self-similar with exponent $0 < H < 1$ and such that $EB_H(1)^2 = 1$; see Samorodnitsky and Taqqu (1994) or Embrechts and Maejima (2002) for details on this process. In particular, this process is locally Hölder continuous with a Hölder exponent $H - \epsilon$ for every $0 < \epsilon < H$.

Let $E = C(-\infty, \infty)$, and m a σ-finite cylindrical measure on E defined by

$$m(A) = \int_{-\infty}^{\infty} P(\mathbf{B}_H \in A - y)\, dy, \quad A \text{ a cylindrical set.} \tag{3.1}$$

That is, m is the (infinite) law of the Fractional Brownian motion shifted according to the Lebesgue measure on $\mathbb{R}$.

Let $\varphi : \mathbb{R} \to [0, \infty)$ be a (globally) Hölder continuous even function, with a Hölder exponent $0 < \rho \le 1$, such that

1. φ is non-increasing on $[0, \infty)$
2. $\varphi \in L^\alpha(\text{Leb})$
3. The Hölder function

$$H(x) = \sup_{x \le s < t} \frac{\varphi(s) - \varphi(t)}{(t-s)^\rho}, \quad x \ge 0 \tag{3.2}$$

also belongs to $L^\alpha(\mathrm{Leb})$.

Define

$$X(t) = \int_E \varphi(x_t)\, M(d\mathbf{x}), \quad t \in \mathbb{R}, \tag{3.3}$$

where M is a SαS random measure on E with control measure m, and $\mathbf{x} = (x_t,\, t \in \mathbb{R})$.

Clearly,

$$\varphi(x_t) = \varphi(x_0) \circ \phi_t(\mathbf{x}), \quad t \in \mathbb{R},$$

where (ϕ_t) is the flow of the left shifts on E. The stationarity of the increments of the Fractional Brownian motion immediately implies that this flow preserves the measure m.

We will see shortly that

$$\int_E \varphi(x_0)^\alpha\, m(d\mathbf{x}) < \infty. \tag{3.4}$$

Therefore, the process $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ in (3.3) is a well defined stationary SαS process. As we will see, this process is generated by a conservative flow.

We start with computing the function $b(T)$ in (2.2). The finiteness of this function will ensure, in particular, (3.4) and the fact that our process is well defined. We have for $T > 0$, using the assumption that φ is even and non-increasing on $[0, \infty)$,

$$\begin{aligned} b(T)^\alpha &= \int_E \sup_{0 \le t \le T} \varphi(x_t)^\alpha\, m(d\mathbf{x}) \\ &= \int_{-\infty}^{\infty} E \sup_{0 \le t \le T} \varphi(y + B_H(t))^\alpha\, dy = \int_{-\infty}^{\infty} E\left(\varphi\Big(\inf_{0 \le t \le T} |y + B_H(t)|\Big)\right)^\alpha dy \\ &= 2\int_0^{\infty} E\left(\varphi\Big(\inf_{0 \le t \le T} |y - B_H(t)|\Big)\right)^\alpha dy = 2\int_0^{\infty} E\left(\varphi\Big(\Big(y - \sup_{0 \le t \le T} B_H(t)\Big)_+\Big)\right)^\alpha dy \\ &= 2E\int_{-\sup_{0 \le t \le T} B_H(t)}^{\infty} \varphi(y_+)^\alpha\, dy = 2\left[\int_0^{\infty} \varphi(y)^\alpha\, dy + \varphi(0)^\alpha E \sup_{0 \le t \le T} B_H(t)\right] \\ &= \|\varphi\|_\alpha^\alpha + 2\varphi(0)^\alpha E \sup_{0 \le t \le T} B_H(t) = \|\varphi\|_\alpha^\alpha + 2\varphi(0)^\alpha T^H E \sup_{0 \le t \le 1} B_H(t), \end{aligned} \tag{3.5}$$

where in the last step we used H-self-similarity of the Fractional Brownian motion. In particular, $b(T) < \infty$.

We claim, further, that the process $\mathbf{X}$ defined by (3.3) is a.s. sample continuous, hence locally bounded. To see this, note that a series representation of this process is given by

$$X(t) = K \sum_{j=1}^{\infty} G_j \Gamma_j^{-1/\alpha} e^{Z_j^2/2\alpha} \varphi\Big(Z_j + B_{j,H}(t)\Big), \quad t \in \mathbb{R}, \tag{3.6}$$

where K is a finite positive constant, $G_1, G_2, \ldots$ and $Z_1, Z_2, \ldots$ are sequences of iid standard normal random variables, $\Gamma_1, \Gamma_2, \ldots$ is a sequence of the arrival times of a unit rate Poisson process on $(0,\infty)$, and $\mathbf{B}_{j,H} = (B_{j,H}(t),\, t \in I\!R)$, $j = 1,2,\ldots$ are iid copies of the Fractional Brownian motion $\mathbf{B}_H$. All four sequences are independent. See Section 3.11 in Samorodnitsky and Taqqu (1994).

With the series representation (3.6), by Fubini's theorem it is enough to prove that the process $\mathbf{X}$ is a.s. sample continuous for each fixed realization of $Z_1, Z_2, \ldots,\ \Gamma_1, \Gamma_2, \ldots$ and $\mathbf{B}_{j,H} = (B_{j,H}(t),\, t \in I\!R)$, $j = 1,2,\ldots$. Then $\mathbf{X}$ becomes, conditionally, a centered Gaussian process. Denoting by $\mathcal{H}$ the σ-field generated by the three random sequences above, we have for the incremental variance of this process

$$E\left(\left(X(t) - X(s)\right)^2 \Big| \mathcal{H}\right) = K^2 \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} e^{Z_j^2/\alpha} \left(\varphi\left(Z_j + B_{j,H}(t)\right) - \varphi\left(Z_j + B_{j,H}(s)\right)\right)^2 \tag{3.7}$$

for all $s, t \in I\!R$.

Denote by W the random variable in the right hand side of (3.7), and consider W for $s,t \in [-A, A]$ for some $A > 0$. We have

$$\begin{aligned} W &= K^2 \sum_{j=1}^{\infty} (\cdot) \mathbf{1}\left(|Z_j| \le 2 \sup_{|u|\le A} |B_{j,H}(u)|\right) + K^2 \sum_{j=1}^{\infty} (\cdot) \mathbf{1}\left(|Z_j| > 2 \sup_{|u|\le A} |B_{j,H}(u)|\right) \\ &:= W_1 + W_2. \end{aligned} \tag{3.8}$$

Let

$$\|\varphi\|_{\text{ Hölder}} = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\rho}$$

be the Hölder norm of φ. Fixing $0 < \epsilon < H$ we also denote by

$$\|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A) = \sup_{-A\le s,t\le A, s\neq t} \frac{|B_{j,H}(t) - B_{j,H}(s)|}{|t-s|^{H-\epsilon}}$$

the Hölder norm of $\mathbf{B}_{j,H}$ on $[-A, A]$, $j = 1,2,\ldots$.

Notice that

$$\begin{aligned} W_1 &\le K^2 \|\varphi\|^2_{\text{Hölder}} \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} e^{Z_j^2/\alpha} \mathbf{1}\left(|Z_j| \le 2 \sup_{|u|\le A} |B_H(u)|\right) |B_{j,H}(t) - B_{j,H}(s)|^{2\rho} \\ &\le K^2 \|\varphi\|^2_{\text{Hölder}} |t-s|^{2\rho(H-\epsilon)} \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} e^{Z_j^2/\alpha} \mathbf{1}\left(|Z_j| \le 2 \sup_{|u|\le A} |B_{j,H}(u)|\right) \|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)^{2\rho}. \end{aligned} \tag{3.9}$$

Notice that

$$\begin{aligned} &E\left(e^{Z_j^2/\alpha} \mathbf{1}\left(|Z_j| \le 2 \sup_{|u|\le A} |B_{j,H}(u)|\right) \|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)^{2\rho}\right)^{\alpha/2} \\ &= \sqrt{\frac{8}{\pi}} E\left(\sup_{|u|\le A} |B_{j,H}(u)| \, \|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)^{\rho\alpha}\right) \end{aligned}$$

$$\leq \sqrt{\frac{8}{\pi}} \left(E \left(\sup_{|u| \leq A} |B_{j,H}(u)| \right)^2 \right)^{1/2} \left(E \left(\|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)^{2\rho\alpha} \right) \right)^{1/2} < \infty$$

because both $\sup_{|u|\leq A} |B_{j,H}(u)|$ and $\|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)$ are the suprema of certain a.s. bounded centered Gaussian processes and, as such, have all finite moments (much more than that, actually; see Borell's inequality (Borell (1975)), also Theorem 2.1 in Adler (1990).)

We immediately conclude that the sum in the right hand side of (3.9) converges a.s. (see Section 1.5 in Samorodnitsky and Taqqu (1994)) and, hence,

$$W_1 \leq B_1 |t-s|^{2\rho(H-\epsilon)}, \tag{3.10}$$

where B_1 is a finite $\mathcal{H}$-measurable random variable.

Similarly, in the notation of (3.2),

$$W_2 \leq K^2 |t-s|^{2\rho(H-\epsilon)} \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} e^{Z_j^2/\alpha} H\left(|Z_j|/2\right)^2 \|\mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)^{2\rho}. \tag{3.11}$$

Observe that

$$E\left(e^{Z_j^2/\alpha} H\left(|Z_j|/2\right)^2 \mathbf{B}_{j,H}\|_{\text{ Hölder}}(A)^{2\rho} \right)^{\alpha/2} = \sqrt{\frac{8}{\pi}} \left(\int_0^{\infty} H(x)^{\alpha}\, dx \right) E\|_{\text{ Hölder}}(A)^{\rho\alpha} < \infty$$

by the assumptions. As before, we conclude that the sum in the right hand side of (3.11) converges a.s. and, hence,

$$W_2 \leq B_2 |t-s|^{2\rho(H-\epsilon)}, \tag{3.12}$$

where B_2 is a finite $\mathcal{H}$-measurable random variable.

Putting (3.7), (3.10) and (3.12) together we see that there is a finite $\mathcal{H}$-measurable random variable B such that

$$E\left(\left(X(t) - X(s) \right)^2 \middle| \mathcal{H} \right) \leq B|t-s|^{2\rho(H-\epsilon)}$$

for all $s, t \in [-A, A]$, and this is a sufficient condition for a.s. continuity of a Gaussian process, as the metric entropy condition easily checks. See Dudley (1967) or Corollary 4.15 in Adler (1990).

Therefore, the process $\mathbf{X}$ defined by (3.3) is sample continuous and, hence, locally bounded. Therefore, Theorem 2.2 applies. Using (3.5) we see that the process $\mathbf{X}$ is generated by a conservative flow, and for this process

$$\left(T^{-H/\alpha} \sup_{0\leq t\leq T} X(t),\ T>1 \right) \quad \text{is tight.} \tag{3.13}$$

Furthermore, if $H > 1/2$ then

$$T^{-H/\alpha} \sup_{0\le t\le T} X(t) \Rightarrow \left(2C_\alpha \varphi(0)^\alpha E \sup_{0\le t\le 1} B_H(t)\right)^{1/\alpha} Z_\alpha \tag{3.14}$$

weakly as $T \to \infty$.

It is interesting that the statement (3.14) holds also for $H = 1/2$, even though the sufficient condition (2.21) does not hold in that case. To see this we need to check (2.19) directly. As before, we denote the probability in the left hand side of (2.19) by $r(T)$. Notice that in our case $\mathbf{B}_{1/2} = \mathbf{B}$ is the Brownian motion and by (3.5)

$$b(T)^\alpha = c_1 + c_2 T^{1/2}$$

for some $c_1, c_2 > 0$. We have

$$r(T) = b(T)^{-2\alpha} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\Bigg[\sup_{0\le s\le T} \varphi(y+B(s))^\alpha \sup_{0\le s\le T} \varphi(z+\tilde{B}(s))^\alpha$$

$$\mathbf{1}\left(\text{for some } 0\le t\le T,\ \frac{\varphi(y+B(t))}{\sup_{0\le s\le T}\varphi(y+B(s))} > \epsilon,\ \frac{\varphi(z+\tilde{B}(t))}{\sup_{0\le s\le T}\varphi(z+\tilde{B}(s))} > \epsilon\right)\Bigg] dydz$$

$$:= \left(c_1 + c_2 T^{1/2}\right)^{-2} I(T), \tag{3.15}$$

where $\mathbf{B}$ and $\tilde{\mathbf{B}}$ are independent standard Brownian motions. By the symmetry,

$$I(T) = 4\int_0^\infty \int_0^\infty E\Bigg[\sup_{0\le s\le T} \varphi(y-B(s))^\alpha \sup_{0\le s\le T} \varphi(z-\tilde{B}(s))^\alpha$$

$$\mathbf{1}\left(\text{for some } 0\le t\le T,\ \frac{\varphi(y-B(t))}{\sup_{0\le s\le T}\varphi(y-B(s))} > \epsilon,\ \frac{\varphi(z-\tilde{B}(t))}{\sup_{0\le s\le T}\varphi(z-\tilde{B}(s))} > \epsilon\right)\Bigg] dydz$$

$$= 4E\int_0^\infty \int_0^\infty \left(\varphi(y - \sup_{0\le s\le T} B(s))_+\right)^\alpha \left(\varphi(z - \sup_{0\le s\le T} \tilde{B}(s))_+\right)^\alpha$$

$$\mathbf{1}\Bigg(\text{for some } 0\le t\le T,\ \varphi(y-B(t)) > \epsilon\varphi(y - \sup_{0\le s\le T} B(s))_+,$$

$$\varphi(z-\tilde{B}(t)) > \epsilon\varphi(z - \sup_{0\le s\le T}\tilde{B}(s))\Bigg) dydz \tag{3.16}$$

$$= 4E\int_{-\sup_{0\le s\le T} B(s)}^\infty \int_{-\sup_{0\le s\le T}\tilde{B}(s)}^\infty \varphi(y_+)^\alpha \varphi(z_+)^\alpha$$

$$\mathbf{1}\Bigg(\text{for some } 0\le t\le T,\ \varphi(y + \sup_{0\le s\le T} B(s) - B(t)) > \epsilon\varphi(y_+),$$

$$\varphi(z + \sup_{0\le s\le T} \tilde{B}(s) - \tilde{B}(t)) > \epsilon\varphi(z_+)\Big)\, dydz$$

$$= 4E\int_0^\infty \int_0^\infty \cdot + 8E\int_0^\infty \int_{-\sup_{0\le s\le T}\tilde{B}(s)}^0 \cdot + 4E\int_{-\sup_{0\le s\le T}B(s)}^0 \int_{-\sup_{0\le s\le T}\tilde{B}(s)}^0 \cdot$$

$$:= I^{(1)}(T) + I^{(2)}(T) + I^{(3)}(T).$$

We have

$$I^{(1)}(T) \le 4\|\varphi\|_\alpha^{2\alpha}. \tag{3.17}$$

Furthermore,

$$I^{(2)}(T) \le 8\|\varphi\|_\alpha^{\alpha}\varphi(0)^\alpha E\sup_{0\le s\le T} B(s) = c_3 T^{1/2} \tag{3.18}$$

for some $c_3 > 0$.

Finally,

$$I^{(3)}(T) \le 4\varphi(0)^{2\alpha} E\int_0^{\sup_{0\le s\le T}B(s)} \int_0^{\sup_{0\le s\le T}\tilde{B}(s)}$$

$$\mathbf{1}\Big(\text{for some } 0\le t\le T,\ |y - B(t)| \le 2h\left(\epsilon\varphi(0)\right),\ |z-\tilde{B}(t)| \le 2h\left(\epsilon\varphi(0)\right)\Big)\, dydz,$$

where for $0<\delta\le\varphi(0)$, we define

$$h(\delta) = \sup\{x>0:\ \varphi(x)\ge\delta\}.$$

We have by the self-similarity of the Brownian motion

$$I^{(3)}(T) \le 4\varphi(0)^{2\alpha} E\int_0^\infty \int_0^\infty \tag{3.19}$$

$$\mathbf{1}\left(\text{for some } 0\le t\le T,\ \left\|\left(B(t),\tilde{B}(t)\right) - (y,z)\right\|_2 \le 2\sqrt{2}h\left(\epsilon\varphi(0)\right)\right) dydz$$

$$= 4\varphi(0)^{2\alpha}T\, E\int_0^\infty \int_0^\infty$$

$$\mathbf{1}\left(\text{for some } 0\le t\le 1,\ \left\|\left(B(t),\tilde{B}(t)\right) - (y,z)\right\|_2 \le \left(2\sqrt{2}h\left(\epsilon\varphi(0)\right)\right)T^{-1/2}\right) dydz$$

$$:= 4\varphi(0)^{2\alpha}T\theta(T),$$

where $\theta(T)\to 0$ as $T\to\infty$.

Now (2.19) follows from (3.15)-(3.19).

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