The workload process with a Poisson cluster input can look like a Fractional Brownian motion even in the slow growth regime

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Abstract

We show that, contrary to the common wisdom, the workload process in a fluid queue with a cluster Poisson input can converge, in the slow growth regime, to a Fractional Brownian motion, and not to a Lévy stable motion. This emphasizes lack of robustness of Lévy stable motions as "bird-eye" descriptions of the traffic in communication networks.

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1 Introduction

Understanding the effect of heavy tails on networks has been a major topic of discussion in literature. It is believed that infinite variance in the distribution of the file sizes or bandwidth requests in communication networks causes long range dependence and self-similar structure in the network (see e.g. Park and Willinger (2000)). Infinite variance in the connectivity distribution of nodes in social networks is believed to be present, and lead to scale-free networks (see e.g. Liljeros et al. (2001)). A discussion of heavy tails in neural networks is in Kosko and Mitaim (2003). In most cases heavy tails appear to cause unusual (and negative) effects.

Networks with heavy tailed inputs are also difficult to analyze, since they are not well suited to Brownian or Poisson approximations. Therefore, other approximations have been sought. In

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the influential paper of Mikosch et al. (2002) they showed that for the so-called ON/OFF model and the infinite source Poisson model, the properly compensated and normalized workload in a fluid queue looks like a Fractional Brownian motion in the *fast growth regime* and like a Lévy stable motion in the *slow growth regime*. This result was later extended to networks of fluid queues by D'Auria and Samorodnitsky (2005). A random field version of such results is in Kaj et al. (2007).

These results appeared to indicate that the deviations from the mean in a fluid queuing network with heavy tailed inputs could look, in the limit, as either a Fractional Brownian motion or a Lévy stable motion. Robustness of this conclusion was investigated in Mikosch and Samorodnitsky (2006) in a general setup, described below. Consider a stationary marked point process

$$((T_n^{(0)}, Z_n))_{n \in \mathbb{Z}},$$

where we interpret $\ldots < T_{-1}^{(0)} < T_0^{(0)} < 0 < T_1^{(0)} < \ldots$ as the arrival times of packets brought to the system and Z_n as the amount of work brought to the system at $T_n^{(0)}$. Each arrival corresponds to a "*source*", and it transmits its work at a unit rate. The number of active sources at time t is given by the process

$$M(t) = \sum_{n \in \mathbb{Z}} \mathbf{1}_{\{T_n^{(0)} \leq t < T_n^{(0)} + Z_n\}} \qquad \text{for } t \geq 0,$$

and the amount of work brought to the system in the interval $[0, t]$ is given by the stochastic process

$$A(t) = \int_0^t M(y)\, dy = \sum_{n \in \mathbb{Z}} [Z_n \wedge (t - T_n^{(0)}) - Z_n \wedge (-T_n^{(0)})].$$

Under the assumption that the marks (Z_n) have, under the Palm measure, a finite mean, $A(t)$ has a finite mean with $\mathbb{E}(A(t)) = \mu t$ for all $t > 0$, where $\mu > 0$ is the expected amount of work arriving in a time interval of unit length, i.e. $\mu = \mathbb{E}(A(1))$.

Let $(A_i)_{i \in \mathbb{N}}$ be iid copies of the process A. With n input processes and at a time scale T, the deviation of the *cumulative workload* from its mean is the stochastic process

$$D_{n,T} = \sum_{i=1}^n (A_i(tT) - \mu tT) \quad \text{for } t \geq 0. \tag{1.1}$$

One is interested in the limits of the sequence of processes $(D_{n,T})$ as n and T grow to infinity. The situation where the number of input processes is relatively large in comparison with the time scale, is referred to as the fast growth regime, while the opposite situation is referred to as the slow growth regime (boundary regimes may exist as well; see e.g. Gaigalas and Kaj (2003)). What Mikosch and Samorodnitsky (2006) discovered was that the Fractional Brownian limits of Mikosch et al. (2002) in the fast growth regime were very robust, and held under very general assumptions on the underlying stationary marked point process. On the other hand,

the Lévy stable limits turned out to be non-robust, and very special conditions were needed to ensure such limits. One of the conclusions of Mikosch and Samorodnitsky (2006) was, in certain circumstances of a very irregular arrival process, a Fractional Brownian limit was possible even in the slow growth regime. They provided a somewhat artificial example of such situation, and conjectured that the same was true in the important case of a cluster Poisson arrival process. It is the purpose of this paper to consider that case and establish the Fractional Brownian limit.

This paper is arranged as follows. The arrival cluster Poisson model we are working with is formally described in Section 2. The main result of the paper is stated and discussed in Section 3. The arguments required to prove the main result uses a number of renewal theoretical and extreme value results, some of which may be of independent interest. These appear in Section 4. Section 5 presents the proof of the main theorem. Finally, Section 6 contains additional lemmas and other technical results needed for the proof of the main theorem.

2 The cluster Poisson model

We assume that the work requirements $(Z_n)_{n \in \mathbb{Z}}$ form an iid sequence independent of the arrival process $(T_n^{(0)})_{n \in \mathbb{Z}}$. Let the number of sources arriving in the interval $(s, t]$ be described by

$$N(s, t] = \sum_{n \in \mathbb{Z}} \mathbf{1}_{\{s < T_n^{(0)} \leq t\}} \qquad \text{for } s < t.$$

Furthermore, we assume that this arrival point process is a *cluster Poisson process*. Specifically:

(i) initial cluster points, denoted by $\ldots < \Gamma_{-1} < 0 < \Gamma_1 < \Gamma_2 < \ldots$ form a homogeneous Poisson process $\widetilde{N}$ with rate λ_0;

(ii) at each initial cluster center Γ_m an independent copy of a randomly stopped renewal point process N_c starts.

A generic point process N_c has the form

$$N_c[0, t] = N_0[0, t] \wedge (K + 1),$$

where N_0 is a renewal point process with arrival times $0 = T_0 < T_1 < \ldots$, and K is a positive integer valued random variable independent of N_0. The interarrival times $X_n = T_n - T_{n-1}$ for $n \geq 1$ are iid random variables, with a common distribution F, and the cluster size K has distribution F_K. The cluster with the initial point Γ_m will have the points $\Gamma_m = \Gamma_m + T_{0,m} < \Gamma_m + T_{1,m} < \ldots < \Gamma_m + T_{K_m,m}$, where the second subscript refers to independent copies of a process.

The within-cluster interarrival times and the cluster sizes are assumed to satisfy the following conditions.

$$\text{The interarrival distribution function satisfies } \overline{F} \in \mathcal{R}_{-1/\beta} \text{ with } \beta > 1, \qquad (2.1)$$

and

the cluster size distribution function satisfies $\overline{F}_K \in \mathcal{R}_{-\alpha}$ with $1 < \alpha < 2$. $\qquad$ (2.2)

The assumption (2.1) assures that the within-cluster interarrival times have infinite mean; it also makes the arrival process sufficiently irregular for our result. The assumption (2.2) makes sure that the amount of work brought within each cluster has infinite variance. Note that the intensity of N is then

$$\lambda = \lambda_0(1 + \mathbb{E}(K)). \qquad (2.3)$$

We denote by

$$h(u) = \overline{F}^{\leftarrow}(1/u) = u^\beta l(u) \quad \text{for } u > 1 \qquad (2.4)$$

the generalized tail inverse function of the within-cluster interarrival time distribution (see Resnick (2007), Section 2.1.2). Here, l is a slowly varying function. One implication of the assumption (2.1) is the weak convergence

$$\left(T_{\lfloor nt \rfloor}/h(n)\right)_{t \geq 0} \Longrightarrow (S_{1/\beta}(t))_{t \geq 0} \qquad (2.5)$$

in $\mathcal{D}\,[0, \infty)$ as $n \to \infty$; see Kallenberg (2002), Theorem 16.14. Here $(S_{1/\beta}(t))_{t \geq 0}$ is an $1/\beta$-stable subordinator. We will use the notation

$$I(u) = \inf\{t \geq 0 : S_{1/\beta}(t) > u\} \quad \text{for} \ \ u > 0, \qquad (2.6)$$

for its inverse process.

We will continue using the notation $\Longrightarrow$ for weak convergence, $\xrightarrow{\mathbb{P}}$ for convergence in probability, $\xrightarrow{\nu}$ for vague convergence, and $\xRightarrow{\text{fidi}}$ for weak convergence of the finite dimensional distributions. For $x \in \mathbb{R}$ we write $x_+ = \max(0, x)$. For two random variables X, Y the symbol $X \overset{d}{=} Y$ means that X has the same distribution as Y.

We will also adopt the following convention. We will use the notation $\alpha_1, \alpha_2, \beta_1$ and β_2 for positive numbers satisfying $\alpha_1 < \alpha < \alpha_2$ and $\beta_1 < \beta < \beta_2$, in the sense that the statements in the text where this notation appears hold for any choice of numbers satisfying the above conditions with, perhaps, different multiplicative constants.

3   The Main Result

Below is the main result of this paper. It describes a slow growth regime under which the properly normalized deviations from the mean process (1.1) converge to a Fractional Brownian motion. For a positive sequence $\lambda_n \uparrow \infty$ serving as the time scale T for a system with n input processes we define

$$b_n = \sqrt{n\lambda_n \overline{F}(\lambda_n)^{-2} \mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \quad \text{for} \ \ n \geq 1. \qquad (3.1)$$

The sequence (b_n) turns out to be the right normalization for process (1.1).

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Theorem 3.1 *Assume that $1 < \alpha < \beta$, and that the distribution F of the within-cluster interarrival times (X_n) satisfies Assumption A below. Assume that the marks (Z_m) form, under the Palm distribution $\mathbb{P}_0$, a sequence of iid random variables, independent of the underlying point process, such that $\mathbb{E}_0|Z_m|^2 < \infty$. Let λ_n be a sequence of positive constants such that $\lambda_n \uparrow \infty$ and such that b_n in (3.1) satisfies*

$$\lim_{n \to \infty} n b_n^{-\frac{\alpha-1}{\beta}+\rho} = 0 \tag{3.2}$$

for some $\rho > 0$. Then the cumulative input process $S_n(t) = b_n^{-1} D_{n,\lambda_n}(t)$, $n \geq 1$, $t \geq 0$, satisfies

$$(S_n(t))_{t\geq 0} \overset{\text{fidi}}{\Longrightarrow} (\mathbb{E}_0(Z) B_H(t))_{t\geq 0} \quad \text{as } n \to \infty,$$

and the limiting process B_H is a Fractional Brownian motion with

$$H = \frac{2+\beta-\alpha}{2\beta} \tag{3.3}$$

and

$$\mathrm{Var}(B_H(1)) = \frac{2\lambda_0}{2+\beta-\alpha} \int_0^\infty y^{-(2+\beta-\alpha)/\beta} \mathbb{P}(S_{1/\beta}(1) \leq y)\,dy +$$

$$+\lambda_0 \int_0^\infty \mathbb{E}\left(\frac{2}{2-\alpha} I(w+1)^{2-\alpha} + \frac{2}{\alpha-1} I(w) I(w+1)^{1-\alpha} - \frac{2}{(2-\alpha)(\alpha-1)} I(w)^{2-\alpha} \right) dw.$$

Assumption A

Assume that either

1. *$\beta < 2$ and*

$$\limsup_{x \to \infty} x \frac{\overline{F}(x) - \overline{F}(x+1)}{\overline{F}(x)} < \infty, \tag{3.4}$$

   or

2. *F is arithmetic, with step size $\Delta > 0$, and*

$$\limsup_{n \geq 0} n \frac{F(\{n\Delta\})}{\overline{F}(n\Delta)} < \infty. \tag{3.5}$$

Remark 3.2 We need the technical Assumption A to obtain a local renewal theorem; see Lemma 4.3 below or Theorem 3 in Doney (1997). In fact, if the local renewal theorem is known to hold (if only in the form of an upper bound), then Assumption A is unnecessary. We conjecture that the local renewal theorem holds under (3.4) for any $\beta > 1$, regardless of whether or not F is arithmetic.

Remark 3.3 Note for any $\epsilon > 0$ there exist $C > 1$ such that

$$C^{-1} n^{\frac{1}{2}} \lambda_n^{H-\epsilon} \leq b_n \leq C n^{\frac{1}{2}} \lambda_n^{H+\epsilon} \quad \text{for } n \geq 1,$$

with H given by (3.3). Hence, a necessary and sufficient condition for (3.2) is that for some $\epsilon > 0$

$$\lambda_n \gg n^{\frac{2\beta-\alpha+1}{2H(\alpha-1)}+\epsilon},$$

i.e. $n^{\frac{2\beta-\alpha+1}{2H(\alpha-1)}+\epsilon} = o(\lambda_n)$ as $n \to \infty$. This identifies (3.2) as a slow growth condition.

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We will prove Theorem 3.1 by showing that the assumptions of Theorem 5.9 in Mikosch and Samorodnitsky (2006) are satisfied. For convenience, we state that theorem below, in a form simplified for the situation where the marks are independent of the arrival process.

Theorem 3.4 (Mikosch and Samorodnitsky (2006)) *Consider a marked stationary point process, where the marks (Z_m) are independent of the arrival process N (whose intensity is λ), and have a finite first moment. Let λ_n be a sequence of positive constants with $\lambda_n \uparrow \infty$. Suppose that there exists a sequence $b_n \uparrow \infty$ such that the following conditions are satisfied.*

(a) *Let N_i be iid copies of N. Then*

$$\left(b_n^{-1} \sum_{i=1}^{n} (N_i (0, \lambda_n t] - \lambda \lambda_n t) \right)_{t \geq 0} \overset{fidi}{\Longrightarrow} (\xi(t))_{t \geq 0},$$

where $(\xi(t))$ is some non-degenerate at zero stochastic process.

(b) *Let $(Z_m^{(i)})_{m \in \mathbb{Z}}$ for $i \in \mathbb{N}$ be iid copies of $(Z_m)_{m \in \mathbb{Z}}$. Then*

$$b_n^{-1} \sum_{i=1}^{n} \sum_{m=1}^{\lfloor \lambda_n \rfloor} (Z_m^{(i)} - \mathbb{E}(Z)) \overset{\mathbb{P}}{\longrightarrow} 0 \quad \text{for } n \to \infty.$$

(c) *Let $I_i^*(0)$ be the total amount of work, of the ith input process, in the session arriving by time 0 which are not finished by that time. Then*

$$b_n^{-1} \sum_{i=1}^{n} I_i^*(0) \overset{\mathbb{P}}{\longrightarrow} 0 \quad \text{for } n \to \infty.$$

Under these conditions the normalized process $S_n(t) = b_n^{-1} D_{n,\lambda_n}(t)$, $n \in \mathbb{N}$, $t \geq 0$, satisfies

$$(S_n(t))_{t \geq 0} \overset{fidi}{\Longrightarrow} (\mathbb{E}(Z)\xi(t))_{t \geq 0} \quad \text{for } n \to \infty.$$

As we will see, the slow growth condition (3.2) is needed only for verification of condition *(c)* in Theorem 3.4.

4 Some Renewal and Extreme Value Theory

Our first proposition in this section deals with the tails of randomly stopped random sums when both the individual terms and the number of terms have infinite means. It complements the existing results dealing with the situations where at least one of these means is finite; see e.g. Fay et al. (2006).

Proposition 4.1 *Let (X_k) be iid random variables independent of the positive integer-valued random variable K with distribution function F_K and let $T_K = \sum_{k=1}^{K} X_k$ with distribution function F_{T_K}. Let G be the distribution function of $|X_1|$. Assume that*

$$\overline{F}_K \in \mathcal{R}_{-\kappa} \quad \text{for some } 0 < \kappa < 1 \tag{4.1}$$

and

$$\overline{G} \in \mathcal{R}_{-\gamma} \quad \text{for some } 0 < \gamma < 1, \qquad \lim_{x \to \infty} \frac{\mathbb{P}(X_1 > x)}{\overline{G}(x)} = p \in (0, 1]. \tag{4.2}$$

Then

$$\lim_{x \to \infty} \frac{\mathbb{P}(T_K > x)}{\mathbb{P}(K > \overline{G}(x)^{-1})} = \mathbb{E}\left((S_\gamma)_+^{\gamma\kappa}\right), \tag{4.3}$$

where S_γ is a strictly γ-stable random variable such that

$$\mathbb{P}(S_\gamma > x) \sim p x^{-\gamma} \quad \text{and} \quad \mathbb{P}(S_\gamma < -x) \sim (1 - p) x^{-\gamma} \quad \text{as } x \to \infty.$$

In particular, $\overline{F}_{T_K} \in \mathcal{R}_{-\kappa\gamma}$.

Proof. For $k \geq 1$, let $a_k := \overline{G}^{\leftarrow}(1/k)$, and note that

$$\frac{1}{a_k}(X_1 + \ldots + X_k) \overset{k \to \infty}{\Longrightarrow} S_\gamma \tag{4.4}$$

(cf. (2.5)). For large $M > 1$ we write

$$
\begin{aligned}
\mathbb{P}(T_K > x) &= \mathbb{P}\left(T_K > x, K > M\overline{G}(x)^{-1}\right) + \mathbb{P}\left(T_K > x, K \leq M^{-1}\overline{G}(x)^{-1}\right) \\
&\quad + \mathbb{P}\left(T_K > x, M^{-1}\overline{G}(x)^{-1} < K \leq M\overline{G}(x)^{-1}\right) \\
&=: E_{1,M}(x) + E_{2,M}(x) + E_{3,M}(x).
\end{aligned}
\tag{4.5}
$$

Note that, as $x \to \infty$,

$$E_{1,M}(x) \leq \mathbb{P}\left(K > M\overline{G}(x)^{-1}\right) \sim M^{-\kappa}\mathbb{P}\left(K > \overline{G}(x)^{-1}\right),$$

and so

$$\lim_{M \to \infty} \limsup_{x \to \infty} \frac{E_{1,M}(x)}{\mathbb{P}\left(K > \overline{G}(x)^{-1}\right)} = 0. \tag{4.6}$$

We claim, further, that for any $\kappa < \kappa_1 < 1$, for all M large enough,

$$\limsup_{x \to \infty} \frac{E_{2,M}(x)}{\mathbb{P}\left(K > \overline{G}(x)^{-1}\right)} \leq M^{-(1-\kappa_1)}. \tag{4.7}$$

Indeed, suppose that (4.7) fails for some $\kappa < \kappa_1 < 1$. Then there is a sequence $x_j \uparrow \infty$ such that $j\overline{G}(x_j) \to 0$ as $j \to \infty$ and

$$E_{2,j}(x_j) \geq \frac{1}{2}j^{-(1-\kappa_1)}\mathbb{P}\left(K > \overline{G}(x_j)^{-1}\right) \tag{4.8}$$

for $j \in \mathbb{N}$. Let $p_k := \mathbb{P}(K = k)$ for $k \geq 1$. Note that

$$E_{2,j}(x_j) = \sum_{k=1}^{\lfloor j^{-1}\overline{G}(x_j)^{-1} \rfloor} p_k \mathbb{P}(X_1 + \ldots + X_k > x_j).$$

Theorem 9.1 in Denisov et al. (2007) shows that

$$\mathbb{P}(X_1 + \ldots + X_k > x_j) \sim kp\overline{G}(x_j)$$

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as $j \to \infty$ uniformly in $k \le j^{-1}\overline{G}(x_j)^{-1}$. Therefore, for large j, by Karamata's theorem,

$$E_{2,j}(x_j) \le 2 \sum_{k=1}^{\lfloor j^{-1}\overline{G}(x_j)^{-1}\rfloor} k\, p_k p \overline{G}(x_j) \le \frac{4p}{1-\kappa} j^{-1} \mathbb{P}\big(K > j^{-1}\overline{G}(x_j)^{-1}\big),$$

and by Potters's inequalities (cf. Resnick (2007), p. 36), for any $\kappa < \kappa_2 < \kappa_1$ there is $C_1 > 0$ such that for large j

$$E_{2,j}(x_j) \le C_1 j^{-(1-\kappa_2)}\mathbb{P}\big(K > \overline{G}(x_j)^{-1}\big).$$

This, clearly, contradicts (4.8), and so (4.7) has to hold. We conclude that

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{E_{2,M}(x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} = 0. \tag{4.9}$$

We now consider the term $E_{3,M}(x)$ in (4.5). For $M^{-1}\overline{G}(x)^{-1} < k \le M\overline{G}(x)^{-1}$ we denote $r = xa_k^{-1}$. Since $\overline{G}(a_k) \sim k^{-1}$ as $k \to \infty$, we see that, for all x large enough, and $M^{-1}\overline{G}(x)^{-1} < k \le M\overline{G}(x)^{-1}$,

$$(2M)^{-1}\overline{G}(x)^{-1} \le \overline{G}(a_k)^{-1} \le 2M\overline{G}(x)^{-1},$$

which implies that for the same range of x and k, $(4M)^{-1} \le r^\gamma \le 4M$. In particular, $\mathbb{P}(S_\gamma > r)$ is bounded away from 0. Since by (4.4)

$$\mathbb{P}(X_1 + \ldots + X_k > x) = \mathbb{P}\left(\frac{1}{a_k}(X_1 + \ldots + X_k) > r\right) \longrightarrow \mathbb{P}(S_\gamma > r), \quad \text{as } k \to \infty$$

(if r is kept fixed), we conclude that

$$\lim_{x\to\infty} \sup_{M^{-1}\overline{G}(x)^{-1} < k \le M\overline{G}(x)^{-1}} \left| \frac{\mathbb{P}(X_1 + \ldots + X_k > x)}{\mathbb{P}(S_\gamma > x/a_k)} - 1 \right| = 0.$$

Therefore,

$$E_{3,M}(x) \sim \sum_{k=\lfloor M^{-1}\overline{G}(x)^{-1}\rfloor + 1}^{\lfloor M\overline{G}(x)^{-1}\rfloor} p_k \mathbb{P}\big(S_\gamma > x/a_k\big) \quad \text{as } x \to \infty.$$

If f denotes the density of S_γ, this statement translates by Fubini into

$$\begin{aligned}
E_{3,M}(x) &\sim \int_0^\infty \sum_{k=\lfloor M^{-1}\overline{G}(x)^{-1}\rfloor + 1}^{\lfloor M\overline{G}(x)^{-1}\rfloor} p_k \, \mathbf{1}_{\{a_k > x/y\}} \, f(y) \, dy \\
&\sim \int_0^\infty \sum_{k=\lfloor M^{-1}\overline{G}(x)^{-1}\rfloor + 1}^{\lfloor M\overline{G}(x)^{-1}\rfloor} p_k \, \mathbf{1}_{\{k > \overline{G}(x/y)^{-1}\}} \, f(y) \, dy \\
&= \int_0^\infty \left[\mathbb{P}\big(\max\big(M^{-1}\overline{G}(x)^{-1}, \overline{G}(x/y)^{-1}\big) < K \le M\overline{G}(x)^{-1}\big) \right] f(y) \, dy.
\end{aligned}$$

Now, for every $y > 0$, as $x \to \infty$,

$$\frac{\mathbb{P}\Big(\max\big(M^{-1}\overline{G}(x)^{-1}, \overline{G}(x/y)^{-1}\big) < K \leq M\overline{G}(x)^{-1}\Big)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} \longrightarrow \Big[\min\big(M^\kappa, y^{\gamma\kappa}\big) - M^{-\kappa}\Big]_+ ,$$

while the same ratio in the left hand side is bounded from above, for large x uniformly in $y > 0$ by

$$\frac{\mathbb{P}\big(K > M^{-1}\overline{G}(x)^{-1}\big)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} \leq 2M^\kappa .$$

Therefore, by the dominated convergence theorem,

$$\lim_{x \to \infty} \frac{E_{3,M}(x)}{\mathbb{P}\big(K > \overline{G}(x)^{-1}\big)} = \int_0^\infty \Big[\min\big(M^\kappa, y^{\gamma\kappa}\big) - M^{-\kappa}\Big]_+ f(y)\, dy. \tag{4.10}$$

As $M \to \infty$, the right hand side of (4.10) converges to $\mathbb{E}\big((S_\gamma^+)^{\gamma\kappa}\big)$, and so the statement of the proposition follows from (4.5), (4.6), (4.9) and (4.10). $\qquad\square$

The next two results are renewal theorems needed in the proof of the main theorem.

Proposition 4.2 *Let (X_k) be an iid sequence of positive random variables with distribution function F, such that $\overline{F} \in \mathcal{R}_{-1/\beta}$, $0 < 1/\beta < 1$. Let $T_n = \sum_{k=1}^n X_k$, $n \in \mathbb{N}$. Suppose that $(c(t))_{t\geq 0}$ is a non-negative eventually non-increasing function, regularly varying of index $-\eta$ in infinity, $1 < \eta < 2$. Then*

$$\sum_{j=0}^\infty c(j)\mathbb{P}(T_j > r) \sim \frac{1}{\eta-1} C_{\eta,\beta} \overline{F}(r)^{-1} c(\overline{F}(r)^{-1}) \quad \text{as } r \to \infty,$$

where $C_{\eta,\beta} = \mathbb{E}((S_{1/\beta}(1))^{(\eta-1)/\beta})$, and $S_{1/\beta}$ is the positive strictly $1/\beta$ stable stochastic process in (2.5).

Proof. Let H_β be the distribution function of $S_{1/\beta}(1)$. Then by the weak convergence in (2.5)

$$\lim_{n \to \infty} \sup_{r \in \mathbb{R}} |H_\beta(r) - \mathbb{P}(T_n \leq a_n r)| = 0,$$

where $a_n = \overline{F}^{\leftarrow}(1/n)$ (cf. Petrov (1975), Theorem 11, p.15, and Theorem 10, p. 88). Thus, there exist a positive sequence $(\epsilon_j)_{j\geq 0}$ with $\epsilon_j \downarrow 0$ as $j \to \infty$ such that for any $r > 0$

$$\mathbb{P}(T_j > r) \leq \overline{H}_\beta(a_j^{-1}r) + \epsilon_j.$$

Let $\delta_1, \delta_2 > 0$, $\delta_1 < \delta_2$ and $\delta = (\delta_1, \delta_2)$. Then

$$\sum_{j=\lceil \delta_1 \overline{F}(r)^{-1}\rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1}\rfloor} c(j)\mathbb{P}(T_j > r) \leq \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1}\rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1}\rfloor} c(j)\overline{H}_\beta(a_j^{-1}r) + \sum_{j=\lceil \delta_1 \overline{F}(r)^{-1}\rceil}^{\lfloor \delta_2 \overline{F}(r)^{-1}\rfloor} c(j)\epsilon_j =: J_1(\delta, r) + J_2(\delta, r).$$

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First, we study the first summand. Let $x_j^{(r)} := j\overline{F}(r)$ and ℓ be a slowly varying function such that $\overline{F}(x) = \ell(x)x^{-1/\beta}$. Then, as $n \to \infty$,

$$n\overline{F}(a_n) = n\ell(a_n)a_n^{-1/\beta} \to 1. \tag{4.11}$$

Since $\delta_1\overline{F}(r)^{-1} \le j \le \delta_2\overline{F}(r)^{-1}$ we have for some $C_1, C_2 > 0$, for all r large enough,

$$C_1 r \le \overline{F}^{\leftarrow}(j) = a_j \le C_2 r.$$

By Theorem 1.5.2 of Bingham et al. (1987) we obtain $\ell(a_j) \sim \ell(r)$ as $r \to \infty$ uniformly for $\delta_1\overline{F}(r)^{-1} \le j \le \delta_2\overline{F}(r)^{-1}$. Thus, (4.11) gives $\ell(r) \sim j^{-1}a_j^{1/\beta}$ as $r \to \infty$ uniformly for $\delta_1\overline{F}(r)^{-1} \le j \le \delta_2\overline{F}(r)^{-1}$, and

$$(x_j^{(r)})^{-\beta} = \left(j\ell(r)r^{-1/\beta}\right)^{-\beta} \sim \frac{r}{a_j} \quad \text{as } r \to \infty.$$

Hence, as $r \to \infty$,

$$\sum_{j=\lceil\delta_1\overline{F}(r)^{-1}\rceil}^{\lfloor\delta_2\overline{F}(r)^{-1}\rfloor} c(j)\overline{H}_\beta(r/a_j) \sim \sum_{j=\lceil\delta_1\overline{F}(r)^{-1}\rceil}^{\lfloor\delta_2\overline{F}(r)^{-1}\rfloor} c\left(x_j^{(r)}\overline{F}(r)^{-1}\right)\overline{H}_\beta((x_j^{(r)})^{-\beta})$$

$$= \overline{F}(r)^{-1} \sum_{j=\lceil\delta_1\overline{F}(r)^{-1}\rceil}^{\lfloor\delta_2\overline{F}(r)^{-1}\rfloor} (x_{j+1}^{(r)} - x_j^{(r)})c\left(x_j^{(r)}\overline{F}(r)^{-1}\right)\overline{H}_\beta((x_j^{(r)})^{-\beta}).$$

Since $c \in \mathcal{R}_{-\eta}$ we obtain by Theorem 1.5.2 of Bingham et al. (1987) as $r \to \infty$,

$$\frac{1}{\overline{F}(r)^{-1}c(\overline{F}(r)^{-1})} \sum_{j=\lceil\delta_1\overline{F}(r)^{-1}\rceil}^{\lfloor\delta_2\overline{F}(r)^{-1}\rfloor} c(j)\overline{H}_\beta(r/a_j) \sim \sum_{j=\lceil\delta_1\overline{F}(r)^{-1}\rceil}^{\lfloor\delta_2\overline{F}(r)^{-1}\rfloor} (x_{j+1}^{(r)} - x_j^{(r)})(x_j^{(r)})^{-\eta}\overline{H}_\beta((x_j^{(r)})^{-\beta})$$

$$\sim \int_{\delta_1}^{\delta_2} y^{-\eta}\overline{H}_\beta(y^{-\beta})\,dy,$$

and so

$$J_1(\delta, r) \sim \overline{F}(r)^{-1}c(\overline{F}(r)^{-1}) \int_{\delta_1}^{\delta_2} y^{-\eta}\overline{H}_\beta(y^{-\beta})\,dy \quad \text{as } r \to \infty.$$

On the other hand,

$$\begin{aligned} J_2(\delta, r) &\le \epsilon_{\delta_1\overline{F}(r)^{-1}} \sum_{j \ge \delta_1\overline{F}(r)^{-1}} c(j) \\ &\sim \epsilon_{\delta_1\overline{F}(r)^{-1}}(\eta - 1)^{-1}\delta_1\overline{F}(r)^{-1}c(\delta_1\overline{F}(r)^{-1}) \\ &\sim \epsilon_{\delta_1\overline{F}(r)^{-1}}(\eta - 1)^{-1}\delta_1^{1-\eta}\overline{F}(r)^{-1}c(\overline{F}(r)^{-1}) \quad \text{as } r \to \infty \end{aligned} \tag{4.12}$$

by Bingham et al. (1987), Proposition 1.5.10. Since δ_1 is arbitrary and $\epsilon_{\delta_1\overline{F}(r)^{-1}} \to 0$ as $r \to \infty$ we obtain

$$\lim_{\delta_2 \to \infty} \lim_{\delta_1 \downarrow 0} \lim_{r \to \infty} \overline{F}(r)c(\overline{F}(r)^{-1})^{-1}(J_1(\delta, r) + J_2(\delta, r)) = \frac{1}{\eta - 1}\mathbb{E}((S_{1/\beta}(1))^{(\eta-1)/\beta}). \tag{4.13}$$

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Next, Proposition 1.5.8 in Bingham et al. (1987) and Lemma 6.4 result in

$$
\begin{aligned}
\sum_{j \leq \delta_1 \overline{F}(r)^{-1}} c(j) \mathbb{P}(T_j > r) & \leq C_3 \sum_{j \leq \delta_1 \overline{F}(r)^{-1}} c(j) j \overline{F}(r) \\
& \sim C_4 \overline{F}(r) c(\delta_1 \overline{F}(r)^{-1}) \delta_1^2 (\overline{F}(r))^{-2} \\
& \sim C_4 \delta_1^{2-\eta} \overline{F}(r)^{-1} c(\overline{F}(r)^{-1})
\end{aligned}
$$

as $r \to \infty$ for some $C_3, C_4 > 0$. Hence,

$$
\lim_{\delta_1 \downarrow 0} \lim_{r \to \infty} \overline{F}(r) c(\overline{F}(r)^{-1})^{-1} \sum_{j \leq \delta_1 \overline{F}(r)^{-1}} c(j) \mathbb{P}(T_j > r) = 0. \tag{4.14}
$$

Also, by Bingham et al. (1987), Proposition 1.5.10,

$$
\begin{aligned}
\overline{F}(r) c(\overline{F}(r)^{-1})^{-1} \sum_{j \geq \delta_2 \overline{F}(r)^{-1}} c(j) \mathbb{P}(T_j > r) & \leq \overline{F}(r) c(\overline{F}(r)^{-1})^{-1} \sum_{j \geq \delta_2 \overline{F}(r)^{-1}} c(j) \\
& \sim \frac{1}{\eta - 1} \delta_2^{1-\eta} \overset{\delta_2 \to \infty}{\longrightarrow} 0. \tag{4.15}
\end{aligned}
$$

By (4.13), (4.14) and (4.15) the result follows. $\qquad\square$

The following result is a local renewal theorem.

Lemma 4.3 *Let the conditions of Proposition 4.2 hold, ans assume additionally Assumption A of Theorem 3.1. Then*

$$
\sum_{j=0}^{\infty} c(j) \left[\mathbb{P}(T_j > x) - \mathbb{P}(T_j > x + 1) \right] \sim \frac{1}{\beta} C_{\eta,\beta} x^{-1} \overline{F}(x)^{-1} c(\overline{F}(x)^{-1}) \quad \text{as } x \to \infty.
$$

Proof. Under the first scenario of Assumption A, the proof, using Proposition 4.2, is the same as the proof of Theorem 2 in Anderson and Athreya (1988), which in particular requires $\beta < 2$. Under the second scenario of Assumption A, the statement is Theorem 3 of Doney (1997). $\qquad\square$

5 Verification of the conditions of Theorem 3.4

The main result of this paper, Theorem 3.1, is proved in this section via verifying the conditions of Theorem 3.4.

5.1 Verification of condition (a) of Theorem 3.4

We can write

$$
\begin{aligned}
b_n^{-1} \sum_{i=1}^{n} \left[N_i\left(0, \lambda_n t\right) - \lambda \lambda_n t \right] & = b_n^{-1} \sum_{i=1}^{n} \left[N_i^{(0, \lambda_n t]}\left(0, \lambda_n t\right) - \mathbb{E}(N_i^{(0, \lambda_n t]}\left(0, \lambda_n t\right)) \right] \\
& \quad + b_n^{-1} \sum_{i=1}^{n} \left[N_i^{(-\infty, 0]}\left(0, \lambda_n t\right) - \mathbb{E}(N_i^{(-\infty, 0]}\left(0, \lambda_n t\right)) \right] \\
& =: \xi_n^+(t) + \xi_n^-(t),
\end{aligned}
$$

where $N_i^A(B)$ is the number of arrivals of packets in a measurable set B belonging to a cluster initiated in a measurable set A. We will show that for every $t > 0$

$$\xi_n^+(t) \stackrel{n\to\infty}{\Longrightarrow} B_H^+(t) \quad \text{and} \quad \xi_n^-(t) \stackrel{n\to\infty}{\Longrightarrow} B_H^-(t), \tag{5.1}$$

where $(B_H^+(t))_{t\geq 0}$ and $(B_H^-(t))_{t\geq 0}$ are independent fractional Brownian motions of index H with time 1 variances

$$\sigma_+^2 = \frac{2\lambda_0}{2+\beta-\alpha} \int_0^\infty y^{-(2+\beta-\alpha)/\beta} \mathbb{P}(S_{1/\beta}(1) \leq y)\, dy \tag{5.2}$$

and

$$\sigma_-^2 = \lambda_0 \int_0^\infty \mathbb{E}\left(\frac{2}{2-\alpha} I(w+1)^{2-\alpha} + \frac{2}{\alpha-1} I(w)I(w+1)^{1-\alpha} - \frac{2}{(2-\alpha)(\alpha-1)} I(w)^{2-\alpha} \right)\, dw. \tag{5.3}$$

By independence, (5.1) will imply that for $t > 0$

$$\xi_n(t) := \xi_n^+(t) + \xi_n^-(t) \stackrel{n\to\infty}{\Longrightarrow} B_H^+(t) + B_H^-(t) =: B_H(t), \tag{5.4}$$

where $B_H(t) \sim \mathcal{N}(0, t^{2H}\sigma^2)$ with $\sigma^2 = \sigma_+^2 + \sigma_-^2$. Applying Lemma 4.8 of Kallenberg (2002) we see that for every $k \geq 1$ and $0 \leq t_1 < t_2 < \ldots < t_k < \infty$, the family of the laws of the random vectors

$$(\xi_n(t_1), \ldots, \xi_n(t_k))_{n\in\mathbb{N}}, \tag{5.5}$$

are tight. Let $\widetilde{\mathbf{B}}_H = (\widetilde{B}_H(t_1), \ldots, \widetilde{B}_H(t_k))$ be a weak subsequential limit of this family, i.e. there exist a subsequence (n_i) such that

$$(\xi_{n_i}(t_1), \ldots, \xi_{n_i}(t_k)) \stackrel{i\to\infty}{\Longrightarrow} \widetilde{\mathbf{B}}_H.$$

On the one hand, $\widetilde{\mathbf{B}}_H$ is infinitely divisible (because of the Poisson arrivals of clusters). On the other hand, the one-dimensional marginal distributions of $\widetilde{\mathbf{B}}_H$ are Gaussian with $\widetilde{B}_H(t_i) \stackrel{d}{=} B_H(t_i)$ by (5.4). Hence, $\widetilde{\mathbf{B}}_H$ is zero mean multivariate Gaussian. We will compute now its covariance matrix. The stationarity of the N_i's and, hence, that of the ξ_n's implies by (5.4) that for $1 \leq j \leq i \leq k$,

$$\xi_n(t_i) - \xi_n(t_j) \stackrel{n\to\infty}{\Longrightarrow} B_H(t_i - t_j).$$

Thus, $\widetilde{B}_H(t_i) - \widetilde{B}_H(t_j) \stackrel{d}{=} B_H(t_i - t_j)$, and so

$$\begin{aligned}
\mathrm{Cov}(\widetilde{B}_H(t_i), \widetilde{B}_H(t_j)) &= \frac{1}{2}\left(\mathbb{E}(\widetilde{B}_H(t_i)^2) + \mathbb{E}(\widetilde{B}_H(t_j)^2) - \mathbb{E}((\widetilde{B}_H(t_i) - \widetilde{B}_H(t_j))^2) \right) \\
&= \frac{1}{2}\left(\mathbb{E}(B_H(t_i)^2) + \mathbb{E}(B_H(t_j)^2) - \mathbb{E}(B_H(t_i - t_j)^2) \right) \\
&= \frac{\sigma^2}{2}(t_i^{2H} + t_j^{2H} - (t_i - t_j)^{2H}).
\end{aligned}$$

This implies that the random vectors in (5.5) converge weakly to the corresponding finite dimensional distributions of the appropriate fractional Brownian motion, and this will verify condition (a) of Theorem 3.4.

In order to prove (5.1) we notice that $\xi_n^+(t)$ and $\xi_n^-(t)$ are infinitely divisible random variables whose characteristic function can be written in the form

$$\mathbb{E}(\exp(i\theta\xi_n^\pm(t))) = \exp\left\{\int_0^\infty \left(e^{i\theta x} - 1 - i\theta x\right)\nu_n^\pm(dx)\right\},$$

where $\nu_n^\pm$ are the corresponding Lévy measures. These can be represented in the form

$$\nu_n^\pm = n\lambda_0(\mathbb{P}_1 \times \mathrm{Leb}) \circ \zeta_\pm^{-1},$$

with the following notation. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be a probability space on which a generic cluster process $(N_c[0, u])_{u \geq 0}$ is defined. The maps ζ_+ and ζ_- are defined as follows: $\zeta_+ : \Omega_1 \times (0, \lambda_n t] \to [0, \infty)$ is given by $\zeta_+(\omega_1, u) = N_c[0, u](\omega_1)/b_n$, and $\zeta_- : \Omega_1 \times \mathbb{R}_+ \to [0, \infty)$ is given by $\zeta_-(\omega_1, u) = N_c(u, u + \lambda_n t](\omega_1)/b_n$. To see this write $N_i^{(-\infty, 0]}(0, \lambda_n t]$ and $N_i^{(-\infty, 0]}(0, \lambda_n t]$ as integrals with respect to a Poisson random measure and use, for example, Lemma 12.2 *(i)* in Kallenberg (2002) (cf. proof of Proposition 3.5 in Fay et al. (2006)). For the notational simplicity below we often drop the subscript in $\mathbb{P}_1$ and, hence, write for $A \in \mathcal{B}(\mathbb{R})$

$$\nu_n^+(A) = n\lambda_0 \int_0^{\lambda_n t} \mathbb{P}(N_c[0, u]/b_n \in A)\, du, \quad \nu_n^-(A) = n\lambda_0 \int_0^\infty \mathbb{P}(N_c(u, u + \lambda_n t]/b_n \in A)\, du.$$

Since the Lévy measures are concentrated on the positive half line, the standard results for the weak convergence of infinitely divisible distributions, see e.g. Theorem 15.14 in Kallenberg (2002), say that one needs to check that for every $\epsilon > 0$:

$$\lim_{n\to\infty} \int_{\{|x|\leq\epsilon\}} x^2 \nu_n^\pm(dx) = \sigma_\pm^2 \quad \text{and} \quad \lim_{n\to\infty} \int_{\{|x|>\epsilon\}} x\nu_n^\pm(dx) = 0. \tag{5.6}$$

Without loss of generality we will assume $\lambda_0 = 1$ in the following.

5.1.1 Convergence of $\boldsymbol{\xi_n^+}$

Let $\epsilon > 0$. Denote

$$I_1(n) := n \int_0^{\lambda_n t} \mathbb{E}\left(\frac{N_c[0, u]}{b_n} \mathbf{1}_{\{N_c[0,u]\leq\epsilon b_n\}}\right)^2 du$$

and

$$I_2(n) := n \int_0^{\lambda_n t} \mathbb{E}\left(\frac{N_c[0, u]}{b_n} \mathbf{1}_{\{N_c[0,u]>\epsilon b_n\}}\right) du.$$

The statement (5.6) for $\boldsymbol{\xi_n^+}$ reduces to

$$\lim_{n\to\infty} I_1(n) = t^{2H}\sigma_+^2 \tag{5.7}$$

and

$$\lim_{n\to\infty} I_2(n) = 0. \tag{5.8}$$

We use the decomposition

$$I_1(n) = \frac{n}{b_n^2}\mathbb{E}\left(\mathbf{1}_{\{K+1\leq\epsilon b_n\}}\int_0^{\lambda_n t} N_c[0, u]^2\, du\right) + \frac{n}{b_n^2}\mathbb{E}\left(\mathbf{1}_{\{K+1>\epsilon b_n\}}\int_0^{\lambda_n t \wedge T_{\lfloor\epsilon b_n-1\rfloor}} N_c[0, u]^2\, du\right)$$

$$=: I_{1,1}(n) + I_{1,2}(n),$$

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and

$$I_2(n) = n\epsilon \int_0^{\lambda_n t} \mathbb{P}(N_c\,[0,u] > \epsilon b_n)\,du + n \int_0^{\lambda_n t} \int_\epsilon^\infty \mathbb{P}(N_c\,[0,u] > x b_n)\,dx\,du$$
$$=: I_{2,1}(n) + I_{2,2}(n).$$

The claim (5.7) now follows from Lemma 6.5 and Lemma 6.6, while the claim (5.8) follows from Lemma 6.7 and Lemma 6.8.

5.1.2 Convergence of ξ_n^-

The argument is similar to that of the previous subsection, but somewhat more involved technically. Denote

$$I_3(n) := \frac{n\lambda_n}{b_n^2} \int_0^\infty \mathbb{E}\left(N_c\left(\lambda_n w, \lambda_n(w+t)\right)^2 \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t))] \leq \epsilon b_n\}}\right)\,dw$$

and

$$I_4(n) := \frac{n\lambda_n}{b_n} \int_0^\infty \mathbb{E}\left(N_c\left(\lambda_n w, \lambda_n(w+t)\right)\right] \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t))] > \epsilon b_n\}}\right)\,dw.$$

Then

$$\lim_{n\to\infty} I_3(n) = t^{2H}\sigma_-^2, \tag{5.9}$$

and

$$\lim_{n\to\infty} I_4(n) = 0, \tag{5.10}$$

such that (5.6) follows.

We start with introducing some notation. Let

$$H_n^{(1)}(w) := \mathbb{E}\left(N_c\left(\lambda_n w, \lambda_n(w+t)\right)^2 \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t))] \leq \epsilon b_n\}} \mathbf{1}_{\{K > N_0(0, \lambda_n(w+t))]\}}\right),$$
$$H_n^{(2)}(w) := \mathbb{E}\left(N_c\left(\lambda_n w, \lambda_n(w+t)\right)^2 \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t))] \leq \epsilon b_n\}} \mathbf{1}_{\{N_0(0, \lambda_n w] < K \leq N_0(0, \lambda_n(w+t))]\}}\right),$$

so that

$$\mathbb{E}\left(N_c\left(\lambda_n w, \lambda_n(w+t)\right)^2 \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t))] \leq \epsilon b_n\}}\right) = H_n^{(1)}(w) + H_n^{(2)}(w).$$

By Lemma 6.9, Lemma 6.10 and Theorem 6.11 we can use the dominated convergence theorem such that

$$\lim_{n\to\infty} I_3(n) = \int_0^\infty \lim_{n\to\infty} \frac{n\lambda_n}{b_n^2}(H_n^{(1)}(w) + H_n^{(2)}(w))\,dw$$
$$= \int_0^\infty \Big[\mathbb{E}((I(w+t) - I(w))^2 I(w+t)^{-\alpha})$$
$$+ \mathbb{E}\left(\frac{\alpha}{2-\alpha}I(w+t)^{2-\alpha} + \frac{2\alpha}{\alpha-1}I(w)I(w+t)^{1-\alpha}\right)$$
$$- \mathbb{E}\left(I(w)^2 I(w+t)^{-\alpha} + \frac{2}{(2-\alpha)(\alpha-1)}I(w)^{2-\alpha}\right)\Big]\,dw.$$

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Now, by substituting w by tz and using the self-similarity of I of index $1/\beta$ (cf. Meerschaert and Scheffler (2004), Proposition 3.1) we obtain (5.9).

Therefore, it remains to check (5.10). Here we assume, once again for the ease of notation, that $\epsilon = 1$, and write for $M > 0$,

$$
\begin{aligned}
I_4(n) &= \frac{n\lambda_n}{b_n} \int_M^\infty \mathbb{E}\Big(N_c\left(\lambda_n w, \lambda_n(w+t)\right] \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t)] > b_n\}}\Big)\, dw \\
&\quad + \frac{n\lambda_n}{b_n} \int_0^M \mathbb{E}\Big(N_c\left(\lambda_n w, \lambda_n(w+t)\right] \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t)] > b_n\}}\Big)\, dw \\
&=: I_{4,1}(n) + I_{4,2}(n).
\end{aligned}
$$

Let $\theta > 0$. We have

$$
\begin{aligned}
I_{4,1}(n) &\leq b_n^{-\theta} \frac{n\lambda_n}{b_n^2} \int_M^\infty \mathbb{E}\Big(N_c\left(\lambda_n w, \lambda_n(w+t)\right]^{2+\theta} \mathbf{1}_{\{N_c(\lambda_n w, \lambda_n(w+t)] > b_n\}}\Big)\, dw \\
&\leq b_n^{-\theta} \overline{F}(\lambda_n)^{-\theta} \int_M^\infty \frac{1}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \mathbb{E}\Big(\Big(\frac{N_c\left(\lambda_n w, \lambda_n(w+t)\right]}{\overline{F}(\lambda_n)}\Big)^{2+\theta}\Big)\, dw.
\end{aligned}
$$

As in the proof of Theorem 6.11 we have that the integral is bounded above by $C_1 \int_M^\infty w^{-r}\, dw$ for some $C_1 > 0, r > 1$. Then by (6.1) we conclude that $I_{4,1}(n) \to 0$ as $n \to \infty$. For $I_{4,2}$, notice that

$$
\begin{aligned}
&\mathbb{E}\left(\frac{N_0\left(0, \lambda_n(M+t)\right]}{\overline{F}(\lambda_n)^{-1}} \mathbf{1}_{\{N_0(0, \lambda_n(M+t)] > b_n\}}\right) \\
&= b_n \overline{F}(\lambda_n) \mathbb{P}(N_0\left(0, \lambda_n(M+t)\right] > b_n) + \frac{1}{\overline{F}(\lambda_n)^{-1}} \int_{b_n}^\infty \mathbb{P}(N_0\left(0, \lambda_n(M+t)\right] > x)\, dx \\
&\leq b_n \overline{F}(\lambda_n) \mathrm{e}^{-b_n \overline{F}(\lambda_n(M+t))} + \overline{F}(\lambda_n) \int_{b_n}^\infty \mathrm{e}^{-x \overline{F}(\lambda_n(M+t))}\, dx \\
&= b_n \overline{F}(\lambda_n) \mathrm{e}^{-b_n \overline{F}(\lambda_n(M+t))} + \overline{F}(\lambda_n(M+t))^{-1} \overline{F}(\lambda_n) \mathrm{e}^{-b_n \overline{F}(\lambda_n(M+t))}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_{4,2}(n) &\leq \frac{n\lambda_n}{b_n} M \mathbb{E}(N_0\left(0, \lambda_n(M+t)\right] \mathbf{1}_{\{N_0(0, \lambda_n(M+t)] > b_n\}} \mathbf{1}_{\{K > b_n\}}) \\
&\leq M \overline{F}(\lambda_n) b_n \mathbb{E}\left(\frac{N_0\left(0, \lambda_n(M+t)\right]}{\overline{F}(\lambda_n)^{-1}} \mathbf{1}_{\{N_0(0, \lambda_n(M+t)] > b_n\}}\right) \frac{\mathbb{P}(K > b_n)}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \\
&\leq C_1 \big[(\overline{F}(\lambda_n) b_n)^2 + \overline{F}(\lambda_n(M+t))^{-1} \overline{F}(\lambda_n) \overline{F}(\lambda_n) b_n \big] \mathrm{e}^{-b_n \overline{F}(\lambda_n(M+t))} (b_n \overline{F}(\lambda_n))^{-\alpha_1} \\
&\xrightarrow{n \to \infty} 0
\end{aligned}
$$

by (6.1). Therefore, (5.10) follows. $\qquad\square$

5.2 Verification of condition (b) of Theorem 3.4

This is an immediate consequence of the Chebyshev inequality and (6.2).

5.3 Verification of condition (c) of Theorem 3.4

It is in this part of the argument that the slow growth condition (3.2) plays a role, as will be presently seen.

The initial step is to show that we may, without loss of generality, assume that the interarrival times of the cluster process N_0 are bounded from below by a positive number. To this end we modify the renewal point process N_0 into a different renewal point process, $\widetilde{N}_0$, as follows. Let $\delta > 0$ be such that $\mathbb{P}(X \geq \delta) > 0$.

Let $\widetilde{T}_1 := \min\{T_j : T_j \geq \delta\}$. Define $\widetilde{Z}_0 := Z_0 + \sum_{i=1}^{M_1-1}(Z_i+\delta)$, where $M_1 := \min\{j : T_j \geq \delta\}$. We view $\widetilde{Z}_0$ as the amount of work brought in the single arrival, at time $\widetilde{T}_0 := 0$.

In general, given $\widetilde{T}_n$ and M_n, we define the next arrival by $\widetilde{T}_{n+1} := \min\{T_j : T_j - \widetilde{T}_n \geq \delta\}$, and the amount of work brought in by the arrival at time $\widetilde{T}_n$ as $\widetilde{Z}_n := Z_{M_n} + \sum_{i=M_n+1}^{M_{n+1}-1}(Z_i + \delta)$, where $M_{n+1} := \min\{j : T_j - \widetilde{T}_n \geq \delta\}$.

Note that with this (sample path) modification, every arrival point of the original process N_0 will arrive, in the new process, not later than before (but it may be aggregated with other points of N_0 into a single new arrival), and its work will last in the new process for at least as long as in the original process. We will still take K of the new aggregated arrivals, so this modification can only increase the random variable $I^*(0)$.

For the new process the random amount of work brought in with any arrival has the representation $\widetilde{Z}_0 = Z_0 + \sum_{i=1}^{M_1-1}(Z_i + \delta)$, and since M_1 is stochastically dominated by a geometric random variable, we see that $\mathbb{E}(\widetilde{Z}_0^2) < \infty$. Furthermore, the interarrival times of the new process satisfy $\widetilde{X}_i \geq \delta$ a.s. and $\mathbb{P}(X_1 > x) \leq \mathbb{P}(\widetilde{X}_i > x) \leq \mathbb{P}(X_1 + \delta > x) \sim \mathbb{P}(X_1 > x)$ as $x \to \infty$. Hence, $\mathbb{P}(\widetilde{X}_1 > x) \sim \mathbb{P}(X_1 > x)$ as $x \to \infty$.

Therefore, for the purpose of obtaining an upper bound, we may work with the new renewal process, and we will simply assume that the original renewal process N_0 has interarrival times that are bounded from below by a positive constant.

We observe that $I^*(0)$ is an infinitely divisible random variable with Lévy measure given by

$$\mu(B) = \lambda_0 \int_0^\infty \mathbb{P}(A^{(c)}(x) \in B)\, dx \quad \text{for } B \in \mathcal{B}(\mathbb{R}),$$

where $A^{(c)}(x)$ is the total amount of work in a session belonging to a single cluster, initiated at zero, that does not finish by time $x > 0$. To see this write $I^*(0)$ with respect to a Poisson random measure and use, for example, Lemma 2.2 *(i)* in Kallenberg (2002). Without loss of generality let $\lambda_0 = 1$. We have, therefore, the decomposition

$$\mu(z, \infty) = \int_0^\infty \mathbb{P}(A^c(x) > z)\, dx \leq I_{5,0} + I_{5,1} + I_{5,2} + I_{5,3}, \tag{5.11}$$

where

$$
\begin{aligned}
I_{5,0} &= \int_0^z \mathbb{P}(A^{(c)}(x) > z)\,dx, \\
I_{5,1} &= \int_z^\infty \mathbb{P}(Z_{N_c(0,x]} > z + (x - T_{N_c(0,x]}), T_K > z)\,dx, \\
I_{5,2} &= \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \int_z^\infty \mathbb{P}\left(\left.\bigcup_{j=0}^{N_c(0,x]-1} \{Z_j > x - T_j\}\right| \mathcal{F}\right) dx\right), \\
I_{5,3} &= \mathbb{E}\left(\mathbf{1}_{\{T_K \le z\}} \int_z^\infty \mathbb{P}\left(\left.\bigcup_{j=0}^{K} \{Z_j > x - T_j\}\right| \mathcal{F}\right) dx\right),
\end{aligned}
$$

where $\mathcal{F}$ is the σ-field generated by the cluster point process N_c.

Let $z \ge 1$. Then Proposition 4.1 in Fay et al. (2006) gives us

$$
I_{5,0} \le z\mathbb{P}\left(\sum_{j=0}^{K} Z_j > z\right) \le C_0 z^{1-\alpha_1}. \tag{5.12}
$$

Next,

$$
\begin{aligned}
I_{5,1} &\le \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \sum_{k=N_0(0,z]}^{K-1} \int_{T_k}^{T_{k+1}} \mathbb{P}(Z_k > z + (x - T_k)|\mathcal{F})\,dx\right) \\
&\quad + \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \int_{T_K}^\infty \mathbb{P}(Z_K > z + (x - T_K)|\mathcal{F})\,dx\right).
\end{aligned}
$$

By Markov's inequality we obtain

$$
\begin{aligned}
I_{5,1} &\le C_1 \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \sum_{k=N_0(0,z]}^{K-1} \int_{T_k}^{T_{k+1}} (z + (x - T_k))^{-2}\,dx\right) \\
&\quad + C_2 \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \int_{T_K}^\infty (z + (x - T_K))^{-2}\,dx\right) \\
&= C_1 \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} \sum_{k=N_0(0,z]}^{K-1} [z^{-1} - (z + X_{k+1})^{-1}]\right) + C_2 \mathbb{E}\left(\mathbf{1}_{\{T_K > z\}} z^{-1}\right) \\
&\le C_3 z^{-1} \mathbb{E}(\mathbf{1}_{\{T_K > z\}} K). \tag{5.13}
\end{aligned}
$$

Note that

$$
\mathbb{E}(\mathbf{1}_{\{T_K > z\}} K) = \mathbb{E}(K) \sum_{k=1}^\infty \mathbb{P}\big(\widetilde{K} = k\big) \mathbb{P}\big(X_1 + \ldots + X_k > z\big) = \mathbb{E}(K)\mathbb{P}(T_{\widetilde{K}} > z),
$$

where $\widetilde{K}$ is a positive integer valued random variable with $\mathbb{P}\big(\widetilde{K} = k\big) = k\mathbb{P}(K = k)/\mathbb{E}(K)$, $k \in \mathbb{N}$. Further, by Karamata's Theorem

$$
\mathbb{P}\big(\widetilde{K} > n\big) \sim \frac{1}{\mathbb{E}(K)} \frac{\alpha}{\alpha - 1} n\mathbb{P}(K > n) \quad \text{as } n \to \infty.
$$

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Hence, by Proposition 4.1

$$\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K) \sim C_4\overline{F}(z)^{-1}\mathbb{P}\left(K > \overline{F}(z)^{-1}\right) \le C_5 z^{-\frac{\alpha_1-1}{\beta_2}}, \tag{5.14}$$

and thus,

$$I_{5,1} \le C_6 z^{-1} z^{-\frac{\alpha_1-1}{\beta_2}} \le C_7 z^{-\frac{\alpha_1-1}{\beta_2}}. \tag{5.15}$$

Next, we decompose $I_{5,2}$ into

$$I_{5,2} = I_{5,2,1} + I_{5,2,2}, \tag{5.16}$$

where

$$I_{5,2,1} = \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \int_z^{T_{K+1}} \mathbb{P}\left(\left.\bigcup_{j=0}^{N_0(0,x]-1} \{Z_j > x - T_j\}\right| \mathcal{F}\right) dx\right),$$

$$I_{5,2,2} = \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \int_{T_{K+1}}^{\infty} \mathbb{P}\left(\left.\bigcup_{j=0}^{K} \{Z_j > x - T_K\}\right| \mathcal{F}\right) dx\right).$$

Then

$$I_{5,2,1} = \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \int_z^{T_{K+1}} \sum_{j=0}^{N_0(0,x]-1} \mathbb{P}(Z_j > x - T_j|\mathcal{F})\,dx\right)$$

$$\le \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \sum_{k=N_0(0,z]}^{K} \int_{T_k}^{T_{k+1}} \sum_{j=0}^{k-1} \mathbb{P}(Z_j > x - T_j|\mathcal{F})\,dx\right).$$

Again applying Markov's inequality and (5.14) lead to

$$I_{5,2,1} \le C_8 \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \sum_{k=N_0(0,z]}^{K} \sum_{j=0}^{k-1} \int_{T_k}^{T_{k+1}} (x - T_j)^{-2}\,dx\right)$$

$$= C_8 \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \sum_{j=0}^{K-1} \left[(T_{N_0(0,z]\vee(j+1)} - T_j)^{-1} - (T_{K+1} - T_j)^{-1}\right]\right)$$

$$\le C_9 \mathbb{E}\left(\mathbf{1}_{\{T_K>z\}} \sum_{j=1}^{K} X_j^{-1}\right)$$

$$\le C_9 \delta^{-1} \mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K)$$

$$\sim C_{10} z^{-\frac{\alpha_1-1}{\beta_2}} \tag{5.17}$$

as $z \to \infty$. Further, by Markov's inequality, as above and (5.14)

$$I_{5,2,2} \le \mathbb{E}(Z^2)\mathbb{E}\left(\mathbf{1}_{\{T_K>z\}}\,K \int_{T_{K+1}}^{\infty} (x - T_K)^{-2}\,dx\right) \le C_{11}\mathbb{E}(\mathbf{1}_{\{T_K>z\}}\,K) \le C_{12} z^{-\frac{\alpha_1-1}{\beta_2}} \tag{5.18}$$

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for z large enough. We decompose $I_{5,3}$ into

$$
\begin{aligned}
I_{5,3} &= \mathbb{E}\left(\mathbf{1}_{\{z/2\leq T_K\leq z\}}\int_z^{z+1}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j > x - T_j\}\middle|\mathcal{F}\right)dx\right)\\
&\quad+\mathbb{E}\left(\mathbf{1}_{\{z/2\leq T_K\leq z\}}\int_{z+1}^{\infty}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j > x - T_j\}\middle|\mathcal{F}\right)dx\right)\\
&\quad+\mathbb{E}\left(\mathbf{1}_{\{T_K<z/2\}}\int_z^{\infty}\mathbb{P}\left(\bigcup_{j=0}^{K}\{Z_j > x - T_j\}\middle|\mathcal{F}\right)dx\right)\\
&=: I_{5,3,1} + I_{5,3,2} + I_{5,3,3}.
\end{aligned}
\tag{5.19}
$$

On the one hand, by Proposition 4.1 in Fay et al. (2006),

$$
I_{5,3,1} \leq \mathbb{P}(T_K > z/2) \leq C_{13}z^{-\frac{1}{\beta_2}}.
\tag{5.20}
$$

On the other hand, by Markov's inequality and (5.14) we obtain

$$
I_{5,3,2} \leq C_{14}\mathbb{E}\left(\mathbf{1}_{\{z/2\leq T_K\leq z\}}K\int_{z+1}^{\infty}(x-T_K)^{-2}\,dx\right) \leq C_{15}\mathbb{E}(\mathbf{1}_{\{T_K>z/2\}}K) \leq C_{16}z^{-\frac{\alpha_1-1}{\beta_2}}.
\tag{5.21}
$$

Finally, another application of Markov's inequality gives us

$$
I_{5,3,3} \leq \mathbb{E}\left(\mathbf{1}_{\{T_K<z/2\}}K\int_z^{\infty}(x-T_K)^{-2}\,dx\right) \leq C_{17}\mathbb{E}(K)z^{-1}.
\tag{5.22}
$$

A conclusion of (5.11)-(5.22) is

$$
\mu(z,\infty) \leq C_{18}z^{1-\alpha_1} + C_{19}z^{-\frac{\alpha_1-1}{\beta_2}} + C_{20}z^{-\frac{1}{\beta_2}} + C_{21}z^{-1} \leq C_{22}z^{-\frac{\alpha_1-1}{\beta_2}}.
$$

Hence, a stochastic domination argument and the fact that the tail of a regularly varying Lévy measure is equivalent to the tail of its distribution function show that for large z

$$
\mathbb{P}(I^*(0) > z) \leq C_{23}z^{-\frac{\alpha_1-1}{\beta_2}}.
$$

By assumption (3.2) (the slow growth condition) we obtain $\lim_{n\to\infty}n\mathbb{P}(I^*(0)>b_n)=0$, and the result follows. $\qquad\square$

6 Auxiliary Results

A number of lemmas and other auxiliary results are collected in this section. We start with a lemma that clarifies the behavior of the normalizing sequence (b_n) in Theorem 3.1.

Lemma 6.1 *Let the assumptions of Theorem 3.1 hold. Then*

$$
\lim_{n\to\infty}(\overline{F}(\lambda_n)b_n)^{-1} = 0,
\tag{6.1}
$$

$$
\lim_{n\to\infty}n\lambda_n b_n^{-2} = 0.
\tag{6.2}
$$

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Proof. For n large we have by Potter's theorem

$$\overline{F}(\lambda_n)b_n = n^{\frac{1}{2}}\lambda_n^{\frac{1}{2}}\mathbb{P}(K > \overline{F}(\lambda_n))^{-1} \geq n^{\frac{1}{2}}\lambda_n^{\frac{1}{2}}\overline{F}(\lambda_n)^{\frac{\alpha_2}{2}} \geq n^{\frac{1}{2}}\lambda_n^{\frac{1}{2}}\lambda_n^{-\frac{\alpha_2}{2\beta_1}}.$$

Since $(\beta_1 - \alpha_2)/(2\beta_1) > 0$ and $\lambda_n \to \infty$ as $n \to \infty$ we obtain

$$(\overline{F}(\lambda_n)b_n)^{-1} \leq n^{-\frac{1}{2}}\lambda_n^{-\frac{\beta_1-\alpha_2}{\beta_1}} \overset{n\to\infty}{\longrightarrow} 0.$$

Finally, (6.2) results from

$$n\lambda_n b_n^{-2} = \overline{F}(\lambda_n)^2\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})^{-1} \leq \overline{F}(\lambda_n)^{2-\alpha_2} \overset{n\to\infty}{\longrightarrow} 0.$$

$\square$

The next result is a simple consequence of the strong Markov property which is useful in various places in our arguments.

Lemma 6.2 *Let f, g be measurable functions and f be increasing. Suppose N_0 is a renewal process. Then for $w, \delta > 0$*

$$\mathbb{E}(f(N_0(w, w + \delta])g(N_0(0, w]) \mathbf{1}_{\{N_0(0,w] \neq N_0(0,w+\delta]\}})$$
$$\leq \mathbb{E}(f(1 + N_0(0, \delta]))\mathbb{E}(g(N_0(0, w]) \mathbf{1}_{\{N_0(0,w] \neq N_0(0,w+\delta]\}}).$$

Proof. Condition on the time and the number of the first arrival after w and use the iid assumption of the interarrival times. $\square$

The next lemma gives a simple estimate on the probability of having "too many" arrivals within a time interval.

Lemma 6.3 *Let (X_k) be an iid sequence of positive random variables with distribution function F, such that $\overline{F} \in \mathcal{R}_{-1/\beta}$, $0 < 1/\beta < 1$ and let h be the generalized tail inverse function (2.4). Let $T_n = \sum_{k=1}^n X_k$, $n \in \mathbb{N}$. For any $\delta > 0$ such that $\overline{F}(\delta) > 0$ and $m \geq 1$,*

(i) we have

$$\mathbb{P}(T_m \leq \delta) \leq F(\delta)^m \leq e^{-m\overline{F}(\delta)}; \tag{6.3}$$

(ii) if $x \geq \delta/h(m)$, then for any $\beta_1 < \beta < \beta_2$ we have

$$\mathbb{P}(T_m \leq h(m)x) \leq e^{-C\min(x^{-\frac{1}{\beta_1}}, x^{-\frac{1}{\beta_2}})} \tag{6.4}$$

for some $C = C(\delta, \beta_1, \beta_2)$;

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Proof. Trivially, for $\delta > 0$,

$$\mathbb{P}\big(T_m \leq \delta\big) \leq \big[1 - \overline{F}(\delta)\big]^m .$$

Now (6.3) follows from the fact that $(1 - a^{-1})^a \leq \mathrm{e}^{-1}$ for $a \geq 1$, and Potter's bounds (cf. Resnick (2007), p. 36) give (6.4). $\qquad\square$

The following simple result on convolution tails of random variables with infinite mean is often useful.

Lemma 6.4 *Let (X_k) be an iid sequence of positive random variables with distribution function F, $\overline{F} \in \mathcal{R}_{-1/\beta}$ and $0 < 1/\beta < 1$. Then there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that for any $x > 0$ and $n \geq n_0$,*

$$\overline{F^{n*}}(x) \leq K n \overline{F}(x). \tag{6.5}$$

Proof. Suppose that the statement is not true. Then for each $j \geq 1$ there exist a $n_j \geq j$ and a $x_j > 0$ such that

$$\overline{F^{n_j *}}(x_j) \geq j n_j \overline{F}(x_j) \tag{6.6}$$

Let h be the generalized tail inverse function (2.4). Assume first that there is a sequence $j_k \uparrow \infty$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \frac{x_{j_k}}{h(n_{j_k})} = \infty.$$

This implies $\lim_{k \to \infty} n_{j_k} \overline{F}(x_{j_k}) = 0$. Therefore, by Theorem 9.1 in Denisov et al. (2007) we obtain

$$\lim_{k \to \infty} \left| \frac{\overline{F^{n_{j_k}*}}(x_{j_k})}{n_{j_k} \, \overline{F}(x_{j_k})} - 1 \right| = 0,$$

which contradicts (6.6).

Next, we suppose that there is $M > 0$ such that

$$x_j \leq M h(n_j) \quad \text{for all } j \in \mathbb{N}. \tag{6.7}$$

Then

$$n_j \overline{F}(x_j) \geq n_j \overline{F}(M h(n_j)) \overset{j \to \infty}{\longrightarrow} M^{-1/\beta}$$

by the regular variation of $\overline{F}$. Thus, (6.6) results in

$$\overline{F^{n_j *}}(x_j) \geq j n_j \overline{F}(x_j) \overset{j \to \infty}{\longrightarrow} \infty.$$

Since $\overline{F^{n_j *}}$ is bounded by 1, this is impossible. Hence, the claim follows. $\qquad\square$

6.1 Auxiliary Results for the Proofs of Subsection 5.1.1

The next series of lemmas provides estimates needed to prove the convergence of $\boldsymbol{\xi}_n^+$ in Subsection 5.1.1. We are using the same notation.

Lemma 6.5 *In the notation of Subsection 5.1.1*

$$\lim_{n\to\infty} I_{1,1}(n) = \frac{2}{2+\beta-\alpha} t^{\frac{2+\beta-\alpha}{\beta}} \int_0^\infty y^{-\frac{2+\beta-\alpha}{\beta}} \, \mathbb{P}(S_{1/\beta}(1) \leq y) \, dy.$$

Proof. We have by the independence of K and N_0

$$
\begin{aligned}
\mathbb{E}(N_c\,[0,u]^2\,\mathbf{1}_{\{K+1\leq \epsilon b_n\}}) &= \int_0^{\epsilon^2 b_n^2} \mathbb{P}((N_0\,[0,u]^2 \wedge (K+1)^2)\,\mathbf{1}_{\{K+1\leq \epsilon b_n\}} > x)\,dx \\
&= 2\int_0^{\epsilon b_n} y\mathbb{P}(N_0\,[0,u] > y)\mathbb{P}(y < K+1 \leq \epsilon b_n)\,dy.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_{1,1}(n) &= 2\frac{n}{b_n^2} \int_0^{\lambda_n t} \int_0^{\epsilon b_n} y\mathbb{P}(N_0\,[0,u] > y)\mathbb{P}(y < K+1 \leq \epsilon b_n)\,dy\,du \\
&= 2\frac{n}{b_n^2} \int_0^{\epsilon b_n} y\mathbb{P}(y < K+1 \leq \epsilon b_n) \int_0^{\lambda_n t} \mathbb{P}(T_{\lfloor y\rfloor} \leq u)\,du\,dy \\
&= 2\frac{n}{b_n^2} \int_0^{\epsilon b_n} y\mathbb{P}(y < K+1 \leq \epsilon b_n)\mathbb{E}(\lambda_n t - T_{\lfloor y\rfloor})_+\,dy \\
&= 2\frac{n}{b_n^2}\overline{F}(\lambda_n)^{-2} \int_0^{\epsilon b_n\overline{F}(\lambda_n)} z\mathbb{P}(z\overline{F}(\lambda_n)^{-1} < K+1 \leq \epsilon b_n)\mathbb{E}(\lambda_n t - T_{\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor})_+\,dz \\
&= 2\int_0^\epsilon z\frac{\mathbb{P}(z\overline{F}(\lambda_n)^{-1} < K+1 \leq \epsilon b_n)}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})}\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor}}{\lambda_n}\right)_+\,dz \\
&\quad + 2\int_\epsilon^{\epsilon b_n\overline{F}(\lambda_n)} z\frac{\mathbb{P}(z\overline{F}(\lambda_n)^{-1} < K+1 \leq \epsilon b_n)}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})}\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor}}{\lambda_n}\right)_+\,dz \\
&=: J_1(n,\epsilon) + J_2(n,\epsilon). \tag{6.8}
\end{aligned}
$$

By Karamata's theorem

$$
\begin{aligned}
J_1(n,\epsilon) &\leq \frac{2t}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \int_0^\epsilon z\mathbb{P}(K+1 > z\overline{F}(\lambda_n)^{-1})\,dz \\
&= \frac{\mathbb{P}(K+1 > \overline{F}(\lambda_n)^{-1})}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})}\frac{2t}{\overline{F}(\lambda_n)^{-2}\mathbb{P}(K+1 > \overline{F}(\lambda_n)^{-1})} \int_0^{\epsilon\overline{F}(\lambda_n)^{-1}} z\mathbb{P}(K+1 > z)\,dz \\
&\stackrel{n\to\infty}{\longrightarrow} \frac{2t\alpha}{2-\alpha}\epsilon^{2-\alpha}, \tag{6.9}
\end{aligned}
$$

and we conclude that $\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} J_1(n,\epsilon) = 0$. We estimate $J_2(n,\epsilon)$ as follows. By Potter's inequality there exists $C_1 > 0$ such that for $z \geq \epsilon$ and n large,

$$\frac{\mathbb{P}(K+1 > z\overline{F}(\lambda_n)^{-1})}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \leq C_1 z^{-\alpha_1}.$$

Similarly, Potter's inequality leads to

$$\frac{h(\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor)}{\lambda_n} \geq \frac{h(z\overline{F}(\lambda_n)^{-1} - 1)}{h(\overline{F}(\lambda_n)^{-1} + 1)} \geq C_2 z^{\beta_1} \quad \text{for } z \geq \epsilon.$$

If we define $m_n = \lfloor z\overline{F}(\lambda_n)^{-1}\rfloor$, then for $\delta > 0$ such that $F(\delta) < 1$,

$$
\begin{aligned}
\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor}}{\lambda_n}\right)_+ &= \mathbb{E}\left(t - \frac{T_{m_n}}{h(m_n)}\frac{h(m_n)}{\lambda_n}\right)_+ \\
&\leq \mathbb{E}\left(t - \frac{T_{m_n}}{h(m_n)}C_2 z^{\beta_1}\right)_+ \\
&= \int_0^{C_2^{-1}tz^{-\beta_1}} C_2 z^{\beta_1}\mathbb{P}\left(\frac{T_{m_n}}{h(m_n)} \leq x\right)dx \\
&= C_2 z^{\beta_1}\left[\int_0^{\delta/h(m_n)} + \int_{\delta/h(m_n)}^{C_2^{-1}tz^{-\beta_1}}\right]\mathbb{P}\left(\frac{T_{m_n}}{h(m_n)} \leq x\right)dx \\
&=: C_2 z^{\beta_1}[V_1(n,z) + V_2(n,z)].
\end{aligned}
$$

We have by (6.3) for large n,

$$V_1(n,z) \leq \left(\delta/h(m_n)\right)\mathbb{P}(T_{m_n} \leq \delta) \leq \delta\mathrm{e}^{-m_n\overline{F}(\delta)} \leq C_3^{-1}\mathrm{e}^{-C_3 z}$$

for some $C_3 > 0$, since $m_n \geq z$ for n large. Further, by (6.4)

$$V_2(n,z) \leq \int_0^{C_2^{-1}tz^{-\beta_1}} \mathrm{e}^{-C_4\min(x^{-1/\beta_1},x^{-1/\beta_2})}\,dx \leq C_5^{-1}tz^{-\beta_1}\mathrm{e}^{-C_5 z} \leq C_6^{-1}\mathrm{e}^{-C_6 z}$$

for some $C_4, C_5, C_6 > 0$. Hence, we have

$$\mathbb{E}\left(t - \frac{T_{\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor}}{\lambda_n}\right)_+ \leq C_2 z^{\beta_1}[V_1(n,z) + V_2(n,z)] \leq C_7^{-1}z^{\beta_1}\mathrm{e}^{-C_7 z},$$

and so by the dominated convergence theorem, (2.5) and the regular variation of $\overline{F}_K$,

$$\lim_{n\to\infty} J_2(n,\epsilon) = 2\int_\epsilon^\infty z^{1-\alpha}\mathbb{E}(t - z^\beta S_{1/\beta}(1))_+\,dz.$$

Therefore, by (6.8),

$$
\begin{aligned}
\lim_{n\to\infty} I_{1,1}(n) &= 2\int_0^\infty z^{1-\alpha}\,\mathbb{E}(t - z^\beta S_{1/\beta}(1))_+\,dz \\
&= \frac{2}{\beta}t^{\frac{2+\beta-\alpha}{\beta}}\int_0^\infty x^{-\frac{2+\beta-\alpha}{\beta}}\,\mathbb{E}(1 - x^{-1}S_{1/\beta}(1))_+\,dx \\
&= \frac{2}{\beta}t^{\frac{2+\beta-\alpha}{\beta}}\int_0^\infty x^{-\frac{2+\beta-\alpha}{\beta}-1}\int_0^x \mathbb{P}(S_{1/\beta}(1) \leq z)\,dz\,dx \\
&= \frac{2}{2+\beta-\alpha}t^{\frac{2+\beta-\alpha}{\beta}}\int_0^\infty z^{-\frac{2+\beta-\alpha}{\beta}}\,\mathbb{P}(S_{1/\beta}(1) \leq z)\,dz. \qquad \square
\end{aligned}
$$

Lemma 6.6 *In the notation of Subsection 5.1.1*

$$\lim_{n\to\infty} I_{1,2}(n) = 0.$$

Proof. By the independence of K and N_0 we have

$$I_{1,2}(n) \leq \frac{n}{b_n^2} \mathbb{E}\left(\mathbf{1}_{\{K+1>\epsilon b_n\}} \int_0^{\lambda_n t} N_0\,[0,u]^2\,du\right) = \frac{n}{b_n^2} \mathbb{P}(K+1>\epsilon b_n) \int_0^{\lambda_n t} \mathbb{E}(N_0\,[0,u]^2)\,du.$$

Thus,

$$
\begin{aligned}
I_{1,2}(n) &\leq \frac{n}{b_n^2} \mathbb{P}(K+1>\epsilon b_n) \int_0^{\lambda_n t} \int_0^\infty \mathbb{P}(N_0\,[0,u]^2 > x)\,dx\,du \\
&\leq 2\frac{n}{b_n^2} \lambda_n t \mathbb{P}(K+1>\epsilon b_n) \int_0^\infty z \mathbb{P}(T_{\lfloor z \rfloor} \leq \lambda_n t)\,dz.
\end{aligned}
$$

By (6.3),

$$
\begin{aligned}
I_{1,2}(n) &\leq C_1 \frac{n}{b_n^2} \lambda_n t \mathbb{P}(K+1>\epsilon b_n) \int_0^\infty z \mathrm{e}^{-\lfloor z \rfloor \overline{F}(\lambda_n t)}\,dz \\
&\leq C_2 \frac{n}{b_n^2} \lambda_n \mathbb{P}(K+1>\epsilon b_n) \overline{F}(\lambda_n)^{-2} \\
&= C_2 \frac{\mathbb{P}(K > \epsilon b_n - 1)}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

by Potter's inequality and (6.1). $\hfill\square$

Lemma 6.7 *In the notation of Subsection 5.1.1*

$$\lim_{n\to\infty} I_{2,1}(n) = 0.$$

Proof. Suppose $\epsilon = 1$. The independence of K and N_0 results in

$$I_{2,1}(n) = n\mathbb{P}(K+1>b_n) \int_0^{\lambda_n t} \mathbb{P}(N_0\,[0,u] > b_n)\,du = n\mathbb{P}(K+1>b_n) \int_0^{\lambda_n t} \mathbb{P}(T_{\lfloor b_n \rfloor} \leq u)\,du.$$

As in (6.3) we obtain

$$
\begin{aligned}
I_{2,1}(n) &\leq n\mathbb{P}(K+1>b_n) \int_0^{\lambda_n} \mathrm{e}^{-\lfloor b_n \rfloor \overline{F}(u)}\,du \\
&\leq n\mathbb{P}(K+1>b_n)\lambda_n \mathrm{e}^{-(b_n-1)\overline{F}(\lambda_n)} \\
&= \frac{\mathbb{P}(K+1>b_n)}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} (\overline{F}(\lambda_n)b_n)^2 \mathrm{e}^{-(b_n-1)\overline{F}(\lambda_n)} \overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

since $b_n \overline{F}(\lambda_n) \overset{n\to\infty}{\longrightarrow} \infty$ by Lemma 6.1. $\hfill\square$

Lemma 6.8 *In the notation of Subsection 5.1.1*

$$\lim_{n\to\infty} I_{2,2}(n) = 0.$$

Proof. Suppose $\epsilon = 1$ and $t = 1$. Then

$$
\begin{aligned}
I_{2,2}(n) &= n\int_1^\infty \mathbb{P}(K+1>b_n x) \int_0^{\lambda_n} \mathbb{P}(N_0\,[0,u] > b_n x)\,du\,dx \\
&= n\int_1^\infty \mathbb{P}(K+1>b_n x) \int_0^{\lambda_n} \mathbb{P}(T_{\lfloor b_n x \rfloor} \leq u)\,du\,dx.
\end{aligned}
$$

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Again as in (6.3) we obtain by (6.1),

$$
\begin{aligned}
I_{2,2}(n) &\leq n \int_1^\infty \mathbb{P}(K+1 > b_n x) \int_0^{\lambda_n} \mathrm{e}^{-\lfloor b_n x \rfloor \overline{F}(u)} \, du \, dx \\
&\leq n\lambda_n \int_1^\infty \mathbb{P}(K+1 > b_n x) \mathrm{e}^{-(b_n x - 1)\overline{F}(\lambda_n)} \, dx \\
&\leq \mathrm{e}^1 n\lambda_n \int_1^\infty \mathrm{e}^{-b_n x \overline{F}(\lambda_n)} \, dx \\
&\leq \mathrm{e}^1 n\lambda_n (b_n \overline{F}(\lambda_n))^{-1} \mathrm{e}^{-b_n \overline{F}(\lambda_n)} \\
&\leq \mathrm{e}^1 (n^{\frac{1}{2}}\lambda_n^{H-\frac{1}{\beta}})^{\max\{2,\frac{1}{H-1/\beta}\}} (b_n \overline{F}(\lambda_n))^{-1} \mathrm{e}^{-b_n \overline{F}(\lambda_n)} \\
&\leq C_1 (b_n \overline{F}(\lambda_n))^{\max\{2,\frac{1}{H-1/\beta}\}-1} \mathrm{e}^{-b_n \overline{F}(\lambda_n)} \xrightarrow{n\to\infty} 0,
\end{aligned}
$$

which is the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.2 Auxiliary Results for the Proofs of Subsection 5.1.2

The next several results deal with the convergence of $\boldsymbol{\xi}_n^-$ in Subsection 5.1.2.

Lemma 6.9 *In the notation of Subsection 5.1.2*

$$
\lim_{n\to\infty} \frac{n\lambda_n}{b_n^2} H_n^{(1)}(w) = \mathbb{E}((I(w+t) - I(w))^2 I(w+t)^{-\alpha}).
$$

Proof. We divide $H_n^{(1)}$ in three parts and define

$$
A_{n,w} = \{N_0\,(\lambda_n w, \lambda_n(w+t)] \leq \epsilon b_n, K > N_0\,(0, \lambda_n(w+t)]\}.
$$

For $M > 0$ let

$$
H_n^{(1,1,M)}(w) = \mathbb{E}\left(N_0\,(\lambda_n w, \lambda_n(w+t)]^2 \, \mathbf{1}_{\{M^{-1} \leq \overline{F}(\lambda_n) N_0(0,\lambda_n(w+t)] \leq M\}} \, \mathbf{1}_{A_{n,w}}\right),
$$

$$
H_n^{(1,2,M)}(w) = \mathbb{E}\left(N_0\,(\lambda_n w, \lambda_n(w+t)]^2 \, \mathbf{1}_{\{\overline{F}(\lambda_n) N_0(0,\lambda_n(w+t)] < M^{-1}\}} \, \mathbf{1}_{A_{n,w}}\right),
$$

$$
H_n^{(1,3,M)}(w) = \mathbb{E}\left(N_0\,(\lambda_n w, \lambda_n(w+t)]^2 \, \mathbf{1}_{\{\overline{F}(\lambda_n) N_0(0,\lambda_n(w+t)] > M\}} \, \mathbf{1}_{A_{n,w}}\right),
$$

so that

$$
H_n^{(1)}(w) = H_n^{(1,1,M)}(w) + H_n^{(1,2,M)}(w) + H_n^{(1,3,M)}(w). \tag{6.10}
$$

Note that

$$
\left(\frac{N_0\,(0, \lambda_n w]}{\overline{F}(\lambda_n)^{-1}}, \frac{N_0\,(0, \lambda_n(w+t)]}{\overline{F}(\lambda_n)^{-1}}\right) \xrightarrow{n\to\infty} (I(w), I(w+t))
$$

(cf. Meerschaert and Scheffler (2004), Theorem 3.2). Furthermore, regularly varying functions converge uniformly on compact sets (cf. Bingham et al. (1987), Theorem 1.5.2). Thus,

$$
\begin{aligned}
\frac{n\lambda_n}{b_n^2} H_n^{(1,1,M)}(w) &= \mathbb{E}\left(\frac{N_0\,(\lambda_n w, \lambda_n(w+t)]^2}{\overline{F}(\lambda_n)^{-2}} \mathbf{1}_{\{N_0(\lambda_n w, \lambda_n(w+t)] \leq \epsilon b_n, M^{-1} \leq \overline{F}(\lambda_n) N_0(0,\lambda_n(w+t)] \leq M\}} \right. \\
&\qquad \left. \times \frac{\mathbb{P}(K > N_0\,(0, \lambda_n(w+t)] \,|\, \mathcal{F}_0)}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})}\right) \\
&\xrightarrow{n\to\infty} \mathbb{E}((I(w+t) - I(w))^2 \, \mathbf{1}_{\{M^{-1} \leq I(w+t) \leq M\}} \, I(w+t)^{-\alpha}), \tag{6.11}
\end{aligned}
$$

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where $\mathcal{F}_0 = \sigma(N_0)$. For the second summand of $H_n^{(1)}$ we have for large n by Potter's inequality

$$\frac{n\lambda_n}{b_n^2} H_n^{(1,2,M)}(w)$$

$$= \mathbb{E}\left(\left(\frac{N_0\left(\lambda_n w, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^2 \frac{\mathbf{1}_{\{K > N_0(0,\lambda_n(w+t)]\}}}{\mathbb{P}(K > \overline{\overline{F}}(\lambda_n)^{-1})} \mathbf{1}_{\{N_0(0,\lambda_n w] \neq N_0(0,\lambda_n(w+t)] < \overline{F}(\lambda_n)^{-1}M\}}\right)$$

$$\leq C_1 \mathbb{E}\left(\left(\frac{N_0\left(0, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^{2-\alpha_2} \mathbf{1}_{\{N_0(0,\lambda_n w] \neq N_0(0,\lambda_n(w+t)] < \overline{F}(\lambda_n)^{-1}M\}}\right)$$

$$\leq C_1 M^{\alpha_2-2} \overset{M\to\infty}{\longrightarrow} 0. \tag{6.12}$$

By Potter's inequality the last term of $H_n^{(1)}$ has the upper bound

$$\frac{n\lambda_n}{b_n^2} H_n^{(1,3,M)}(w)$$

$$\leq \frac{n\lambda_n}{b_n^2} \mathbb{E}\left(N_0\left(0, \lambda_n(w+t)\right]^2 \mathbf{1}_{\{\overline{F}(\lambda_n)N_0(0,\lambda_n(w+t)] \geq M\}} \mathbb{P}(K > N_0\left(0, \lambda_n(w+t)\right] \,|\,\mathcal{F}_0)\right)$$

$$\leq C_2 \mathbb{E}\left(\left(\frac{N_0\left(0, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^{2-\alpha_1} \mathbf{1}_{\{N_0(0,\lambda_n(w+t)] \geq \overline{F}(\lambda_n)^{-1}M\}}\right)$$

$$= C_2 \int_{M^{2-\alpha_1}}^{\infty} \mathbb{P}\left(\left(\frac{N_0\left(0, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^{2-\alpha_1} > y\right) dy$$

$$+ C_2 M^{2-\alpha_1} \mathbb{P}\left(\left(\frac{N_0\left(0, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^{2-\alpha_1} > M^{2-\alpha_1}\right).$$

The first term in the right hand side above can be bounded as follows. For some constant $C_2 > 0$ we obtain as in (6.3) that

$$\int_{M^{2-\alpha_1}}^{\infty} \mathbb{P}\left(\left(\frac{N_0\left(0, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^{2-\alpha_1} > y\right) dy$$

$$= (2-\alpha_1) \int_{M}^{\infty} z^{1-\alpha_1} \mathbb{P}(N_0\left(0, \lambda_n(w+t)\right] > z\overline{F}(\lambda_n)^{-1}) \, dz$$

$$\leq (2-\alpha_1) \int_{M}^{\infty} z^{1-\alpha_1} \mathbb{P}(T_{\lfloor z\overline{F}(\lambda_n)^{-1}\rfloor+1} \leq \lambda_n(w+t)) \, dz$$

$$\leq (2-\alpha_1) \int_{M}^{\infty} z^{1-\alpha_1} \exp(-z\overline{F}(\lambda_n)^{-1}\overline{F}(\lambda_n(w+t))) \, dz$$

$$\leq C_3^{-1} \int_{M}^{\infty} z^{1-\alpha_1} e^{-C_3(\omega+t)^{-\frac{1}{\beta_1}} z} \, dz \longrightarrow 0 \quad \text{as } M \to \infty.$$

Similarly,

$$M^{2-\alpha_1} \mathbb{P}\left(\left(\frac{N_0\left(0, \lambda_n(w+t)\right]}{\overline{\overline{F}}(\lambda_n)^{-1}}\right)^{2-\alpha_1} > M^{2-\alpha_1}\right) \longrightarrow 0 \quad \text{as } M \to \infty.$$

Hence, the result follows. $\qquad\square$

Lemma 6.10 *In the notation of Subsection 5.1.2*

$$\lim_{n\to\infty} \frac{n\lambda_n}{b_n^2} H_n^{(2)}(w) = \mathbb{E}\left(\frac{\alpha}{2-\alpha}I(w+t)^{2-\alpha} + \frac{2\alpha}{\alpha-1}I(w)I(w+t)^{1-\alpha}\right)$$
$$-\mathbb{E}\left(I(w)^2 I(w+t)^{-\alpha} + \frac{2}{(2-\alpha)(\alpha-1)}I(w)^{2-\alpha}\right).$$

Proof. We define

$$A_{n,M} = \{M^{-1} \le \overline{F}(\lambda_n)N_0\,(0,\lambda_n w] \le \overline{F}(\lambda_n)N_0\,(0,\lambda_n(w+t)] \le M, N_0\,(\lambda_n w, \lambda_n(w+t)] \le \epsilon b_n\}$$

and

$$A_M = \{M^{-1} \le I(w) \le I(w+t) \le M\}.$$

By Karamata's theorem and the uniform convergence of regularly varying functions on compact sets we have

$$\mathbb{E}\left(K^2 \frac{\mathbf{1}_{\{K \le N_0(0,\lambda_n(w+t)]\}} - \mathbf{1}_{\{K \le N_0(0,\lambda_n w]\}}}{\overline{F}(\lambda_n)^{-2}\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \mathbf{1}_{A_{n,M}}\right)$$
$$\xrightarrow{n\to\infty} \frac{\alpha}{2-\alpha}\mathbb{E}\left((I(w+t)^{2-\alpha} - I(w)^{2-\alpha})\mathbf{1}_{A_M}\right), \tag{6.13}$$

and

$$\mathbb{E}\left(\frac{N_0\,(0,\lambda_n w]}{\overline{F}(\lambda_n)^{-1}}K\frac{\mathbf{1}_{\{K > N_0(0,\lambda_n w]\}} - \mathbf{1}_{\{K > N_0(0,\lambda_n(w+t)]\}}}{\overline{F}(\lambda_n)^{-1}\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \mathbf{1}_{A_{n,M}}\right)$$
$$\xrightarrow{n\to\infty} \frac{\alpha}{\alpha-1}\mathbb{E}\left(I(w)(I(w)^{1-\alpha} - I(w+t)^{1-\alpha})\mathbf{1}_{A_M}\right). \tag{6.14}$$

Further,

$$\mathbb{E}\left(\frac{N_0\,(0,\lambda_n w]^2}{\overline{F}(\lambda_n)^{-2}}\frac{\mathbf{1}_{\{K > N_0(0,\lambda_n w]\}} - \mathbf{1}_{\{K > N_0(0,\lambda_n(w+t)]\}}}{\mathbb{P}(K > \overline{F}(\lambda_n)^{-1})} \mathbf{1}_{A_{n,M}}\right)$$
$$\xrightarrow{n\to\infty} \mathbb{E}\left(I(w)^2(I(w)^{-\alpha} - I(w+t)^{-\alpha})\mathbf{1}_{A_M}\right). \tag{6.15}$$

Thus, (6.13)-(6.15) give us

$$\lim_{M\to\infty}\lim_{n\to\infty} \frac{n\lambda_n}{b_n^2}\mathbb{E}\left(N_c\,(\lambda_n w, \lambda_n(w+t)]^2\,\mathbf{1}_{\{N_0(0,\lambda_n w] < K \le N_0(0,\lambda_n(w+t)]\}}\mathbf{1}_{A_{n,M}}\right)$$
$$= \frac{\alpha}{2-\alpha}\mathbb{E}\left(I(w+t)^{2-\alpha} - I(w)^{2-\alpha}\right) - 2\frac{\alpha}{\alpha-1}\mathbb{E}\left(I(w)(I(w)^{1-\alpha} - I(w+t)^{1-\alpha})\right)$$
$$+ \mathbb{E}\left(I(w)^2(I(w)^{-\alpha} - I(w+t)^{-\alpha})\right).$$

The integral over the complement of the event $A_{n,M}$ vanishes in the limit, as $M \to \infty$, in the same way as in Lemma 6.9. $\qquad\square$

The following theorem is the last major piece needed to establish the convergence of $\boldsymbol{\xi_n^-}$.

Theorem 6.11 *In the notation of Subsection 5.1.2, there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty g(w)\,dw < \infty$ and for every $n \in \mathbb{N}$*

$$\frac{n\lambda_n}{b_n^2} \mathbb{E}(N_c\,(\lambda_n w, \lambda_n(w+1)]^2) \le g(w) \quad \forall\, w > 0.$$

Proof. The existence of the required function on the interval $(0, M]$ for an arbitrary $M > 0$ follows from Lemma 6.15 below, so we only need to construct a required function on the interval (M, ∞). We define

$$
\begin{aligned}
A_{n,w} &= \{N_0\,(0, \lambda_n w] \ne N_0\,(0, \lambda_n(w+1)]\}, \\
B_{n,w} &= \{N_0\,(0, \lambda_n] = N_0\,(0, \lambda_n w]\} \cap A_{n,w}, \\
C_{n,w} &= \{N_0\,(0, \lambda_n] \ne N_0\,(0, \lambda_n w]\} \cap A_{n,w}.
\end{aligned}
$$

We have for $w > M$

$$
\begin{aligned}
\frac{n\lambda_n}{b_n^2} \mathbb{E}(N_c\,(\lambda_n w, \lambda_n(w+1)]^2) &\le \frac{n\lambda_n}{b_n^2} \mathbb{E}(N_c\,(\lambda_n w, \lambda_n(w+1)]^2\, \mathbf{1}_{B_{n,w}}) \\
&\quad + \frac{n\lambda_n}{b_n^2} \mathbb{E}(N_0\,(\lambda_n w, \lambda_n(w+1)]^2\, \mathbf{1}_{\{K > N_0(0,\lambda_n w] > 0\}}\, \mathbf{1}_{C_{n,w}}) \\
&=:\ J_{2,1}(n, w) + J_{2,2}(n, w).
\end{aligned} \tag{6.16}
$$

Potter's inequality and Lemma 6.2 result in

$$
\begin{aligned}
J_{2,2}&(n, w) \\
&\le \mathbb{E}\left(\frac{N_0\,(\lambda_n w, \lambda_n(w+1)]^2}{\overline{F}(\lambda_n)^{-2}} \left[C_1 \left(\frac{N_0\,(0, \lambda_n w]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\alpha_1} + C_2 \left(\frac{N_0\,(0, \lambda_n w]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\alpha_2} \right] \mathbf{1}_{C_{n,w}} \right) \\
&\le \mathbb{E}\left(\frac{(N_0\,(0, \lambda_n] + 1)^2}{\overline{F}(\lambda_n)^{-2}} \right) \mathbb{E}\left(\left[C_1 \left(\frac{N_0\,(0, \lambda_n w]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\alpha_1} + C_2 \left(\frac{N_0\,(0, \lambda_n w]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\alpha_2} \right] \mathbf{1}_{C_{n,w}} \right).
\end{aligned}
$$

By (6.3) we have for large n

$$
\begin{aligned}
\mathbb{E}\left(\frac{N_0\,(0, \lambda_n]^2}{\overline{F}(\lambda_n)^{-2}} \right) &= \frac{1}{\overline{F}(\lambda_n)^{-2}} \int_0^\infty \mathbb{P}(N_0\,(0, \lambda_n]^2 > x)\,dx \\
&\le \frac{2}{\overline{F}(\lambda_n)^{-2}} \int_0^\infty y\,\mathbb{P}(T_{\lfloor y \rfloor + 1} \le \lambda_n)\,dy \\
&\le \frac{2}{\overline{F}(\lambda_n)^{-2}} \int_0^\infty y\,\mathrm{e}^{-y\overline{F}(\lambda_n)}\,dy \\
&= 2 \int_0^\infty z\,\mathrm{e}^{-z}\,dz < \infty.
\end{aligned} \tag{6.17}
$$

Hence, (6.16), (6.17) and Proposition 6.12 below show that $J_{2,2}(n, w)$ is uniformly in n bounded from above by an integrable on $[M, \infty)$ function. The fact that the same is true for $J_{2,1}(n, w)$ follows from Lemma 6.16 below. $\qquad \square$

6.3 Auxiliary Results for the Proof of Theorem 6.11

The following proposition is the first ingredient in the proof of Theorem 6.11.

Proposition 6.12 *Let $\eta > 1$ and $M > 1$, and suppose that the assumptions of Theorem 3.1 hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\, dw < \infty$ and for every $n \in \mathbb{N}$*

$$\mathbb{E}\left(\left(\frac{N_0\left(0, \lambda_n w\right]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\eta} \mathbf{1}_{\{N_0(0,\lambda_n] \neq N_0(0,\lambda_n w] \neq N_0(0,\lambda_n(w+1)]\}} \right) \leq g(w) \quad \forall\, w \geq M.$$

The statement follows from Lemma 6.13 and Lemma 6.14 below.

Lemma 6.13 *Let $\eta > 1$ and $M > 1$, and suppose that the assumptions of Theorem 3.1 hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\, dw < \infty$ and for every $n \in \mathbb{N}$*

$$\mathbb{E}\left(\left(\frac{N_0\left(0, \lambda_n w\right]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\eta} \mathbf{1}_{\{N_0(0,\lambda_n(w-1)] \neq N_0(0,\lambda_n w]\}} \right) \leq g(w) \quad \forall\, w \geq M.$$

Proof. Let $w \geq M$ and n so large such that $\lambda_n^{-1} \leq 2^{-1}$. We have

$$
\begin{aligned}
J_1(n,w) &:= \mathbb{E}\left(\left(\frac{N_0\left(0, \lambda_n w\right]}{\overline{F}(\lambda_n)^{-1}} \right)^{-\eta} \mathbf{1}_{\{N_0(0,\lambda_n(w-1)] \neq N_0(0,\lambda_n w]\}} \right) \\
&= \int_{\lambda_n(w-1)}^{\lambda_n w} \overline{F}(\lambda_n w - y) \sum_{j=1}^\infty \left(\frac{j}{\overline{F}(\lambda_n)^{-1}} \right)^{-\eta} \mathbb{P}(T_j \in dy) \\
&= \left[\int_{\lambda_n(w-1)}^{\lambda_n w - 2} + \int_{\lambda_n w - 2}^{\lambda_n w} \right] \overline{F}(\lambda_n w - y) \sum_{j=1}^\infty \left(\frac{j}{\overline{F}(\lambda_n)^{-1}} \right)^{-\eta} \mathbb{P}(T_j \in dy) \\
&=: J_{1,1}(n,\omega) + J_{1,2}(n,\omega).
\end{aligned}
$$

Now,

$$J_{1,1}(n,\omega) \leq \sum_{k=\lfloor\lambda_n(w-1)\rfloor-1}^{\lceil\lambda_n w - 2\rceil-1} \overline{F}\left(\lambda_n \left(w - \frac{k+1}{\lambda_n} \right) \right) \sum_{j=1}^\infty \left(\frac{j}{\overline{F}(\lambda_n)^{-1}} \right)^{-\eta} \left[\mathbb{P}(T_j \leq k+1) - \mathbb{P}(T_j \leq k) \right].$$

Since $\overline{F}$ is regularly varying of index $-1/\beta$, by Potter's inequality there exists a constant $0 \leq C_1 < \infty$ such that

$$J_{1,1}(n,\omega) \leq C_1 \sum_{k=\lfloor\lambda_n(w-1)\rfloor}^{\lceil\lambda_n w - 2\rceil-1} \left(w - \frac{k+1}{\lambda_n} \right)^{-\frac{1}{\beta_1}} \frac{\overline{F}(\lambda_n)^{1-\eta}}{k\overline{F}(k)^{1-\eta}} k\overline{F}(k)^{1-\eta} \sum_{j=1}^\infty j^{-\eta} \left[\mathbb{P}(T_j \leq k+1) - \mathbb{P}(T_j \leq k) \right].$$

Using Lemma 4.3, we obtain

$$J_{1,1}(n,\omega) \leq C_2 \sum_{k=\lfloor\lambda_n(w-1)\rfloor}^{\lceil\lambda_n w - 2\rceil-1} \left(w - \frac{k+1}{\lambda_n} \right)^{-\frac{1}{\beta_1}} \frac{\overline{F}(\lambda_n)^{1-\eta}}{k\overline{F}(k)^{1-\eta}}.$$

Taking again the regular variation of $\overline{F}$ and Potter's Theorem into account yields

$$
\begin{aligned}
J_{1,1}(n,\omega) &\leq C_3 \sum_{k=\lfloor \lambda_n(w-1)\rfloor}^{\lceil \lambda_n w-2\rceil-1} \left(w-\frac{k+1}{\lambda_n}\right)^{-\frac{1}{\beta_1}} \frac{1}{k}\left(\frac{k}{\lambda_n}\right)^{\frac{1-\eta}{\beta_1}} \\
&= C_3 \sum_{k=\lfloor \lambda_n(w-1)\rfloor}^{\lceil \lambda_n w-2\rceil-1} \frac{1}{\lambda_n}\left(w-\frac{k+1}{\lambda_n}\right)^{-\frac{1}{\beta_1}}\left(\frac{k}{\lambda_n}\right)^{\frac{1-\eta}{\beta_1}-1} \\
&\leq C_4 \int_{w-1}^{w}(w-z)^{-\frac{1}{\beta_1}} z^{\frac{1-\eta}{\beta_1}-1}\,dz \\
&\leq C_5 w^{\frac{1-\eta}{\beta_1}-1},
\end{aligned}
\tag{6.18}
$$

which is an integrable function on $[M,\infty)$ since $\eta > 1$.

Finally, using, once again, Lemma 4.3, we obtain

$$
\begin{aligned}
J_{1,2}(n,\omega) &\leq \overline{F}(\lambda_n)^{-\eta}\sum_{j=1}^{\infty} j^{-\eta}\left[\mathbb{P}(T_j \leq \lambda_n w)-\mathbb{P}(T_j \leq \lambda_n w-2)\right] \\
&\leq C_6 \overline{F}(\lambda_n)^{-\eta}\frac{1}{\lambda_n w}\overline{F}(\lambda_n w)^{\eta-1} \\
&\leq C_7 \frac{1}{\lambda_n \overline{F}(\lambda_n)} w^{\frac{1-\eta}{\beta_1}-1},
\end{aligned}
$$

which is uniformly bounded by an integrable function. $\qquad\square$

Lemma 6.14 Let $\eta > 1$ and $M > 1$, and suppose that the assumptions of Theorem 3.1 hold. Then there exists a non-negative measurable function $g:\mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\,dw < \infty$ and for every $n \in \mathbb{N}$

$$
\mathbb{E}\left(\left(\frac{N_0\,(0,\lambda_n w]}{\overline{F}(\lambda_n)^{-1}}\right)^{-\eta}\mathbf{1}_{\{N_0(0,\lambda_n]\neq N_0(0,\lambda_n(w-1)]=N_0(0,\lambda_n w]\neq N_0(0,\lambda_n(w+1)]\}}\right) \leq g(w) \quad \forall w \geq M.
$$

Proof. As in the previous lemma,

$$
\begin{aligned}
J_1(n,w) &:= \mathbb{E}\left(\left(\frac{N_0\,(0,\lambda_n w]}{\overline{F}(\lambda_n)^{-1}}\right)^{-\eta}\mathbf{1}_{\{N_0(0,\lambda_n]\neq N_0(0,\lambda_n(w-1)]=N_0(0,\lambda_n w]\neq N_0(0,\lambda_n(w+1)]\}}\right) \\
&= \int_1^{w-1}[\overline{F}(\lambda_n w-y)-\overline{F}(\lambda_n(w+1)-y)]\sum_{j=1}^{\infty}\left(\frac{j}{\overline{F}(\lambda_n)^{-1}}\right)^{-\eta}\mathbb{P}(T_j \in \lambda_n\,dy) \\
&\leq \sum_{k=\lfloor\lambda_n\rfloor}^{\lceil\lambda_n(w-1)\rceil-1}\left[\overline{F}(\lambda_n w-k-1)-\overline{F}(\lambda_n w+\lambda_n-k)\right] \\
&\quad\times\sum_{j=1}^{\infty}\left(\frac{j}{\overline{F}(\lambda_n)^{-1}}\right)^{-\eta}\left[\mathbb{P}(T_j \leq k+1)-\mathbb{P}(T_j \leq k)\right].
\end{aligned}
$$

By Lemma 4.3 we have for n large

$$
J_1(n,w) \leq C_1 \sum_{k=\lfloor\lambda_n\rfloor}^{\lceil\lambda_n(w-1)\rceil-1}\frac{\overline{F}(\lambda_n w-k-1)-\overline{F}(\lambda_n w+\lambda_n-k)}{\overline{F}(\lambda_n)}\frac{\overline{F}(\lambda_n)^{1-\eta}}{k\overline{F}(k)^{1-\eta}}.
$$

Note that for every k in the above sum by Assumption A

$$
\begin{aligned}
\overline{F}(\lambda_n w - k - 1) - \overline{F}(\lambda_n w + \lambda_n - k) &\leq \sum_{j=-1}^{\lceil \lambda_n \rceil - 1} \left[\overline{F}(\lambda_n w - k + j) - \overline{F}(\lambda_n w - k + j + 1) \right] \\
&\leq C_2 \sum_{j=-1}^{\lceil \lambda_n \rceil - 1} \frac{\overline{F}(\lambda_n w - k + j)}{\lambda_n w - k + j} \leq C_3 \lambda_n \frac{\overline{F}(\lambda_n w - k - 1)}{\lambda_n w - k - 1}.
\end{aligned}
$$

We conclude by Potter's Theorem that for large n and all k as above

$$
\frac{\overline{F}(\lambda_n w - k - 1) - \overline{F}(\lambda_n w + \lambda_n - k)}{\overline{F}(\lambda_n)} \leq C_4 \left(w - \frac{k+1}{\lambda_n} \right)^{-\frac{1}{\beta_2} - 1}.
$$

Hence, we obtain

$$
J_1(n, w) \leq C_4 \sum_{k = \lambda_n}^{\lambda_n (w-1) - 1} \left(w - \frac{k+1}{\lambda_n} \right)^{-\frac{1}{\beta_2} - 1} \frac{\overline{F}(\lambda_n)^{1-\eta}}{k \overline{F}(k)^{1-\eta}}.
$$

Similar calculations as in (6.18) result in

$$
J_1(n, w) \leq C_5 \int_1^{w-1} (w - z)^{-\frac{1}{\beta_2} - 1} z^{\frac{1-\eta}{\beta_2} - 1} \, dz \leq C_6 w^{-\frac{\eta - 1}{\beta_2} - 1},
$$

as an easy computation shows. This is an integrable on $[M, \infty)$ function. $\qquad \square$

The final two lemmas needed for the proof of Theorem 6.11 follow.

Lemma 6.15 *Let $M > 0$, and suppose that the assumptions of Theorem 3.1 hold. Then there exists a positive constant $C < \infty$ such that for every $n \in \mathbb{N}$*

$$
\frac{n \lambda_n}{b_n^2} \mathbb{E}(N_c \left(\lambda_n w, \lambda_n(w + 1) \right]^2) \leq C \quad \forall \, w \leq M.
$$

Proof. It is clearly enough to establish the required bound for n large enough. By Potter's inequality and Karamata's theorem, we obtain for all n large enough and $0 < w \leq M$

$$
\begin{aligned}
&\frac{n \lambda_n}{b_n^2} \mathbb{E}(N_c \left(\lambda_n w, \lambda_n(w + 1) \right]^2) \\
&\leq \frac{n \lambda_n}{b_n^2} \mathbb{E}(N_c \left(0, \lambda_n(M + 1) \right]^2) \\
&= \frac{n \lambda_n}{b_n^2} \mathbb{E} \left(N_0 \left(0, \lambda_n(M + 1) \right]^2 \mathbf{1}_{\{K > N_0(0, \lambda_n(M+1)] > 0\}} \right) + \frac{n \lambda_n}{b_n^2} \mathbb{E} \left(K^2 \mathbf{1}_{\{K \leq N_0(0, \lambda_n(M+1)]\}} \right) \\
&\leq C_1 \mathbb{E} \left(\left(\frac{N_0 \left(0, \lambda_n(M + 1) \right]}{\overline{F}(\lambda_n)^{-1}} \right)^{2 - \alpha_1} \right) + C_2 \mathbb{E} \left(\left(\frac{N_0 \left(0, \lambda_n(M + 1) \right]}{\overline{F}(\lambda_n)^{-1}} \right)^{2 - \alpha_2} \right).
\end{aligned}
$$

The right hand side is bounded for n large enough by computations similar to (6.17). $\qquad \square$

Lemma 6.16 *Let $M > 1$ and suppose that the assumptions of Theorem 3.1 hold. Then there exists a non-negative measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_M^\infty g(w)\,dw < \infty$ and for every $n \in \mathbb{N}$*

$$\frac{n\lambda_n}{b_n^2}\mathbb{E}\left(N_c\left(\lambda_n w, \lambda_n(w+1)\right]^2 \, \mathbf{1}_{\{N_0(0,\lambda_n]=N_0(0,\lambda_n w]\neq N_0(0,\lambda_n(w+1)]\}}\right) \leq g(w) \quad \forall\, w \geq M.$$

Proof. We define $B_{n,w} := \{N_0\left(0, \lambda_n\right] = N_0\left(0, \lambda_n w\right] \neq N_0\left(0, \lambda_n(w+1)\right]\}$ for $w \geq M$. Notice that by Lemma 6.2,

$$\frac{n\lambda_n}{b_n^2}\mathbb{E}(N_c\left(\lambda_n w, \lambda_n(w+1)\right]^2 \, \mathbf{1}_{B_{n,w}}) \leq \frac{n\lambda_n}{b_n^2}\mathbb{E}\left(\min\left(N_0\left(0, \lambda_n\right], K\right)^2\right)\mathbb{P}(B_{n,w}). \qquad (6.19)$$

Note that

$$\mathbb{E}\left(\min\left(N_0\left(0, \lambda_n\right], K\right)^2\right) = 2\int_0^\infty t\,\mathbb{P}\left(N_0\left(0, \lambda_n\right] > t\right)\mathbb{P}(K > t)\,dt$$

$$= 2\Big[\int_0^{\overline{F}(\lambda_n)^{-1}} + \int_{\overline{F}(\lambda_n)^{-1}}^\infty\Big]t\,\mathbb{P}\left(N_0\left(0, \lambda_n\right] > t\right)\mathbb{P}(K > t)\,dt.$$

Since by Karamata's theorem, as $n \to \infty$,

$$\int_0^{\overline{F}(\lambda_n)^{-1}} t\,\mathbb{P}\left(N_0\left(0, \lambda_n\right] > t\right)\mathbb{P}(K > t)\,dt \leq \int_0^{\overline{F}(\lambda_n)^{-1}} t\mathbb{P}(K > t)\,dt$$

$$\sim C_1\left(\overline{F}(\lambda_n)^{-1}\right)^2\mathbb{P}\left(K > \overline{F}(\lambda_n)^{-1}\right)$$

and by (6.17)

$$\int_{\overline{F}(\lambda_n)^{-1}}^\infty t\,\mathbb{P}\left(N_0\left(0, \lambda_n\right] > t\right)\mathbb{P}(K > t)\,dt \leq C_2\mathbb{P}\left(K > \overline{F}(\lambda_n)^{-1}\right)\mathbb{E}\left(N_0\left(0, \lambda_n\right]^2\right)$$

$$\leq C_3\left(\overline{F}(\lambda_n)^{-1}\right)^2\mathbb{P}\left(K > \overline{F}(\lambda_n)^{-1}\right),$$

we have the bound

$$\mathbb{E}\left(\min\left(N_0\left(0, \lambda_n\right], K\right)^2\right) \leq C_4\left(\overline{F}(\lambda_n)^{-1}\right)^2\mathbb{P}\left(K > \overline{F}(\lambda_n)^{-1}\right).$$

On the other hand, by Assumption A and the same arguments as in (6.17),

$$\mathbb{P}(B_{n,w}) = \mathbb{P}\left(N_0\left(0, \lambda_n\right] = N_0\left(0, \lambda_n w\right] \neq N_0\left(0, \lambda_n(w+1)\right]\right)$$

$$= \sum_{j=0}^\infty \int_0^{\lambda_n}\left[\overline{F}(\lambda_n w - y) - \overline{F}(\lambda_n w + \lambda_n - y)\right]\mathbb{P}(T_j \in dy)$$

$$\leq \left[\overline{F}(\lambda_n w - \lambda_n) - \overline{F}(\lambda_n w + \lambda_n)\right]\mathbb{E}(N_0\left(0, \lambda_n\right])$$

$$\leq \sum_{k=0}^{\lfloor\lambda_n\rfloor}\left[\overline{F}(\lambda_n w - \lambda_n + k) - \overline{F}(\lambda_n w + k + 1)\right]\left[C_5\overline{F}(\lambda_n)^{-1}\right]$$

$$\leq C_6\overline{F}(\lambda_n)^{-1}\sum_{k=0}^{\lfloor\lambda_n\rfloor}\frac{\overline{F}(\lambda_n w - \lambda_n + k)}{\lambda_n w - \lambda_n + k}$$

$$\leq C_7\overline{F}(\lambda_n)^{-1}\lambda_n\frac{\overline{F}(\lambda_n w - \lambda_n)}{\lambda_n w - \lambda_n}$$

$$\leq C_8 w^{-1}\frac{\overline{F}(\lambda_n w)}{\overline{F}(\lambda_n)}. \qquad (6.20)$$

We conclude that

$$\frac{n\lambda_n}{b_n^2}\mathbb{E}(N_c\left(\lambda_n w, \lambda_n(w+1)\right]^2 \mathbf{1}_{B_{n,w}}) \leq C_9 w^{-1}\frac{\overline{F}(\lambda_n w)}{\overline{F}(\lambda_n)} \leq C_{10}w^{-1-1/\beta_2},$$

which is an integrable function on $[M, \infty)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

K. K. ANDERSON and K. B. ATHREYA (1988): A strong renewal theorem for generalized renewal functions in the infinite mean case. *Probab. Th. Rel. Fields* 77:471–479.

N. H. BINGHAM, C. M. GOLDIE and J. L. TEUGELS (1987): *Regular Variation*. Cambridge University Press, Cambridge.

B. D'AURIA and G. SAMORODNITSKY (2005): Limit behavior of fluid queues and networks. *Operations Research* 53:933–945.

D. DENISOV, A. DIEKER and V. SHNEER (2007): Large deviations for random walks under subexponentiality: the big jump domain. *Annals of Probability*, to appear.

R. DONEY (1997): One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* 107:451–465.

G. FAY, B. GONZÁLEZ-ARÉVALO, T. MIKOSCH and G. SAMORODNITSKY (2006): Modeling teletraffic arrivals by a Poisson cluster process. *QESTA* 54:121–140.

R. GAIGALAS and I. KAJ (2003): Convergence of scaled renewal processes and a packet arrival model. *Bernoulli* 9:671–703.

I. KAJ, L. LESKELÄ, I. NORROS and V. SCHMIDT (2007): Scaling limits for random fields with long-range dependence. *Annals of Probability* 35:528–550.

O. KALLENBERG (2002): *Foundations of Modern Probability*. Springer, New York, 2nd edition.

B. KOSKO and S. MITAIM (2003): Stochastic resonance in noisy threshold neurons. *Neural Networks* 16:755–761.

F. LILJEROS, C. EDLING, L. AMARAL, H. STANLEY and Y. ÅBERG (2001): The web of human sexual contacts. *Nature* 411:907–908.

M. MEERSCHAERT and H.-P. SCHEFFLER (2004): Limit theorems for continuous-time random walks with infinite mean waiting time. *J. Appl. Prob,* 41:623–638.

T. MIKOSCH, S. RESNICK, H. ROOTZÉN and A. STEGEMAN (2002): Is network traffic approximated by stable Lévy motion or Fractional Brownian motion? *Annals of Applied Probability* 12:23–68.

T. Mikosch and G. Samorodnitsky (2006): Scaling limits for cumulative input process. *Mathematics of Operations Research*, 32:890-919.

K. Park and W. Willinger, editors (2000): *Self–similar Network Traffic and Performance Evaluation*, New York. John Wiley and Sons, Inc.

V. Petrov (1975): *Sums of Independent Random Variables*. Springer-Verlag, New York.

S. I. Resnick (2007): *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer, New York.