IMPLICATIONAL COMPLEXITY

IN INTUITIONISTIC ARITHMETIC

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0. Introduction. In classical arithmetic, a natural measure for the complexity of relations is provided by the number of quantifier alternations in an equivalent prenex normal form. However, the proof of the Prenex Normal Form Theorem uses the following intuitionistically invalid rules for permuting quantifiers with propositional constants.

(1) $A \to \exists x Bx \quad . \leftrightarrow \quad \exists x[A \to Bx]$

(2) $\forall x Bx \to A \quad . \leftrightarrow \quad \exists x[Bx \to A]$

(3) $\forall x[A \vee Bx] \leftrightarrow . \; A \vee \forall x Bx$

Each one of these schemas, when added to Intuitionistic (Heyting's) Arithmetic IA, generates full Classical (Peano's) Arithmetic. Schema (3) is of little interest here, since one can obtain a formula intuitionstically equivalent to $A \vee \forall x Bx$, which is prenex if A and B are: $A \vee \forall x Bx \; . \leftrightarrow .$

$\exists y[(y \neq 0 \to A) \wedge (y \neq 0 \to \forall x Bx)] \; . \leftrightarrow . \; \exists y \forall x[(y=0 \to A) \wedge (y \neq 0 \to Bx)]$. For the two conjuncts on the r.h.s. (1) may be successively applied, since $y=0$ is decidable.

We shall readily verify that there is no way of similarly going around (1) or (2). This fact calls for counting implication (though not conjunction or disjunction) in measuring, in IA , the complexity of arithmetic relations. The natural implicational measure, for our purpose, is the depth of negative nestings of implication, defined as follows. $I(F) := 0$ if F is atomic; $I(F \wedge G) = I(F \vee G) := \max[I(F), I(G)]$; $I(\forall x F) = I(\exists x F) := I(F)$; $I(F \to G) := \max[I(F)+1, I(G)]$.

This measure is also the minimal one satisfying the following properties. (1) One can define I, as above, for functional types. Then, for each type τ, if $N = I(\tau)$, then τ is canonically embedded in the n'th pure type. An explicit connection between implicational complexity and functional complexity is provided by Gödel's (so-called "Dialectica") interperetation. To isolate the role of implication, consider a propositional schema F in Σ^o_1 sentences. The interpretation of F is of the form $\exists x^{\tau} \forall y^{\sigma} F_o$, where F_o is quantifier free, $I(\tau) = I(F)$ and $I(\sigma) < I(F)$. (2) Let IA_n stand for IA with induction restricted to formulas F with $I(F) \leq n$; the consistency of IA_n can be proved in IA_{n+1} (Le[75]).

Note that positive nesting of implications may be essential in other contexts. E.g., in IA + Church's Thesis there exists a truth definition for formulas with a given bound on the depth of all implication nestings. (I am indebted to the referee for this remark.)

In treating intuitionistic implicational complexity, one should avoid literal analogies with classical treatment of arithmetical complexity. For instance, an arithmetic relation R cannot be distinguished from its prenex form R_p simply by refuting $\forall \vec{x}.\ R\vec{x} \leftrightarrow R_p\vec{x}$, as the refutation is underivable in any classically sound theory. As on similar occasions, the problem is to state intuitionistic invalidity without using refutation. We confine ourselves here to the proof theoretic analogues of classical theorems. Given a theory A we say that two relations R,Q are A-equivalent (notation: $R \equiv_A Q$), if $\vdash_A \forall \vec{x}.\ R\vec{x} \leftrightarrow Q\vec{x}$. The stronger theory A is, the larger the equivalence classes.

In order to discuss syntactic forms more general than the prenex ones we refer to first-order predicate formulas (without equality), which we call frames. An arithmetic formula F is a case of the frame Φ (notation: $F \varepsilon \Phi$) if F comes from Φ by substituting atomic formulas (i.e., equations between primitive recursive terms) for the predicate letters. E.g., the Π^0_2 unary relations are exactly the cases of $\forall x \exists y\, P(x,y,z)$. For frames Φ, Ψ we write $\Phi \leq_A \Psi$, if for any $F \varepsilon \Phi$ there is a $G \varepsilon \Psi$ with $F \equiv_A G$. Also, $\Phi \equiv_A \Psi$ if both $\Phi \leq_A \Psi$ and $\Psi \leq_A \Phi$. E.g., $\neg\exists x Px \equiv_{IA} \forall x Px$. Using these notations, we may describe our subject as the study of the partial ordering $\leq_A$ on frames, for suitable (intuitionistic) arithmetic theories A.

When A is Classical (Peano's) Arithmetic CA, each frame is $\leq_A \Pi^0_n$ or $\leq_A \Sigma^0_n$ for some n. However, $\leq_A$ cannot be replaced here by $\equiv_A$. E.g., each propositional combination of Σ^0_n relations is in $\Delta^0_{n+1} := \Sigma^0_{n+1} \cap \Pi^0_{n+1}$, but not vice versa. In other words, there are interpolant frames under $\leq_{CA}$ between the frames (for) Σ^0_n and Σ^0_{n+1} (or Π^0_{n+1}). Such frames have not drawn much interest, since their position in the neat prenex hierarchy is easily identified. Also, these frames arise in connection with a "separation of variables"; e.g., $\exists x\, f(x)=0 \wedge \forall y\, g(y)=0$ has a prenex form $\exists x \forall y\, [\, f(x)=0 \wedge g(y)=0 \,]$, in the matrix of which the variables x and y appear in distinct propositional constituents.

1. Hierarchy theorems for implicational complexity

Our first aim is to show that, in Intuitionistic Arithmetic, the effect of implication goes beyond generating interpolating frames in the prenex hierarchy.

We shall make use of the propositional formulas in one letter, $N_i[p]$, characterized by Nishimura [60]: $N_0 :\equiv \bot$ (absurdity);

$N_1 :\equiv p$; $N_2 :\equiv \neg p$; $N_{2k+1} :\equiv N_{2k} \vee N_{2k-1}$ $(k \geq 1)$; $N_{2k+2} :\equiv N_{2k+1} \to N_{2k-1}$ $(k \geq 1)$.

Note that $I(N_{2k}) = I(N_{2k+1}) = k$ $(k \geq 0)$.

From the definitions, the following properties follow.

(1) $N_{2k+2} \equiv N_{2k} \to N_{2k-1}$ $(k \geq 1)$

$\equiv (N_{2k-2} \to N_{2k-3}) \to (N_{2k-2} \vee N_{2k-3})$

$(k \geq 2$, by (1))

(2) $\equiv N_{2k} \to N_{2k-3}$ $(k \geq 2)$.

(3) $N_{2k+2} \to N_{2k} \equiv N_{2k}$ $(k \geq 1)$

This is immediate for $k = 1$; for $k \geq 1$ observe that

$(N_{2k+4} \to N_{2k+2}) \to . (N_{2k+2} \to N_{2k-1}) \to N_{2k+2}$

$. \to . (N_{2k+2} \to N_{2k-1}) \to N_{2k-1}$

$. \to . ((N_{2k} \to N_{2k-1}) \to N_{2k-1}) \to N_{2k-1}$ by (1)

$. \to . (N_{2k} \to N_{2k-1}) \equiv : N_{2k+2}$.

(4) $N_{2k+4} \to N_{2k-1} . \to . (N_{2k+2} \to N_{2k-1}) \to N_{2k-1}$ by (2)

$. \to N_{2k+2}$ as above.

(5) $N_{2k+4} \to N_{2k} . \to . N_{2k-1} \to N_{2k}$ by (2)

$. \to . N_{2k-1} \to (N_{2k-1} \to N_{2k-3})$

$. \to . N_{2k-1} \to N_{2k-3}$

$\equiv : N_{2k}$.

Let A be IA augmented with transfinite induction over certain recursive well-orderings $\{R_i\}_i$. Clearly, when $\{R_i\}_i$ is an r.e. set (under some canonical indexing) then so is the theory A. When $\{R_i\}_i$ consists of all recursive well-orderings, then A, denoted by IA^* (or HA^*), is a rather strong theory; e.g., $A \subset IA^*$ as above may consist of the arithmetic theorems of the intuitionistic impredicative theory of types (Le[75a]). Also, IA^* is complete for Π^0_2

sentences, and IA^* augmented by classical logic is complete for classical truth (in the standard model) (Kr,Sh,Wa[60]).

THEOREM I (implicational complexity for $\leq_A$). For each $k \geq 1$ there is a frame Φ_k with $I(\Phi_k) = k$, s.t. $\Phi_k \not\leq_A \Psi$ for any frame Ψ with $I(\Psi) < k$ (regardless of the quantificational complexity of Ψ). Explicitly, we may take $\Phi_k := N_{2k}[\exists x \forall y P(x,y)]$ for an r.e. theory A, and $\Phi_k := N_{2k}[\forall x \exists y \forall z P(x,y,z)$ for $A = IA^*$.

A weaker form of thorem I, for a measure counting all nestings of implication, can be proved by the method of D.H.J. de Jongh [73] (for an exposition, see Tr[73], 5.3.20-5.3.22).

Let Σ_n^* denote the propositional combinations of Σ_n^0 sentences, and let $\Sigma_n^{(j)} := \{ F \in \Sigma_n^* : I(F) \leq j \}$. $\Sigma_n^{(j)}$ frames are defined analogously.

THEOREM II (2-step hierarchy for Σ_1^*). Let A be r.e.. For each $k \geq 1$ there is a $\Sigma_1^{(k+1)}$ frame Φ_k s.t. $\Phi_k \not\leq_A \Psi$ for any frame Ψ with $I(\Psi) < k$.

The following theorem will be used in the proofs of theorems I and II.

Theorem III Let A be r.e. ($A=IA^*$), and let $\phi:\equiv\neg\forall x Px$ ($\neg\forall x\exists y\forall z P(x,y,z,)$, respectively). Then $\phi \not\equiv_A \psi$ for any prenex frame ψ (and hence, for any frame ψ with $I(\psi) = 0$). Explicitly: if B is any Π^0_1 (resp., Π^0_3) sentence undecidable in A, then $\neg B$ is not equivalent in A to any prenex sentence.

PROOF: Case (I) A is r.e.. Let B be a Π^0_1 A-undecidable sentence, and let

$$C \equiv \forall y_0 \exists z_0 \ldots \forall y_n \exists z_n \; C_0(\vec{y},\vec{z})$$

be a prenex sentence. Assume

(1) $\vdash_A \neg B \rightarrow C$, and

(2) $\vdash_A C \rightarrow \neg B$.

By (2) and the soundness of A , the sentence C is false. From (1) we conclude, by the closure of A under the extended Church's Rule ECR_0 (cf. Tr[73] 3.7.2), that for certain $e_0,\ldots,e_n$

(3) $$\vdash_A \bigwedge_{i\leq n} \forall a\exists b\, T(\bar{e}_i,a,b) \wedge \forall y_0\ldots\forall y_n\forall w_0\ldots\forall w_n[T(\bar{e}_0,y_0,w_0) \wedge T(\bar{e}_1,\langle y_0,y_1\rangle,w_1) \wedge \ldots \wedge T(\bar{e}_n,\langle y_0,\ldots,y_n\rangle,w_n) \wedge \neg B \rightarrow C_0(\vec{y},U(w_0),\ldots,U(w_n))]$$

But C is false, so for suitable $y_0,\ldots,y_n,w_0,,\ldots,w_n$

(4) $$T(\bar{e}_0,\bar{y}_0,\bar{w}_0) \wedge \ldots \wedge T(\bar{e}_n,\langle\bar{y}_0,\ldots,\bar{y}_n\rangle,\bar{w}_n) \wedge \neg C_0(\bar{y},U(\bar{w}_0),\ldots,U(\bar{w}_n)) \; .$$

The expression (4) is closed and quantifier-free, so it is provable in $IA \subset A$. We therefore conclude from (3) that $\vdash_A \neg\neg B$, and hence $\vdash_A B$, contradicting the choice of B .

Case (II) $A=IA^*$. We first note that there is a Π^0_3 sentence $B \equiv \forall x\exists y\forall z\; B_0(x,y,z)$ undecided in IA^* . Let B be any Π^0_3 sentence s.t. (i) B is classically true, i.e.: $B' :\equiv \forall x\neg\exists y\forall z\; B_0$ is true; (ii) no

recursive function f satisfies $\forall x \forall z\ B_o(x,f(x),z)$. Since B' is true, $\nvdash_{IA^*} \neg B$. On the other hand, if $\vdash_{IA^*} B$, then B is recursively realizable, contradicting (ii).

The proof in (I) goes through now, with the aid of the following:

LEMMA 1.1: If $\vdash_{IA^*} \neg\neg B$ for a prenex sentence B , then weakly (i.e., $\rightsquigarrow$) $\vdash_{IA^*} B$

We postpone the proof of this lemma to §4.

2. Unbounded width of the implicational hierarchy.

From theorem I we easily obtain k frames incomparable under $\leq_A$, simply by taking frames with increasing I-complexity and decreasing quantificational complexity. However, the frames obtained in theorem I are loaded with the kind of separated variables mentioned above, and one might be led to see this as the reason of the incomparability (as in the classical case). In order to focus on the more interesting cases, let us call a frame unified if it contains a single predicate letter.

THEOREM IV: Let A be r.e. ($A = IA^*$, resp.). For any $k \geq 3$ there are k unified frames Φ_i $(i=1,\ldots k)$, incomparable under $\leq_A$, and with $I(\Phi_i) = 1$.

PROOF: (I) A is r.e.. Given $k \geq 3$ (or $k \geq 1$, for that matter), let

$$\Phi_i :\equiv \exists z_1 \ldots \forall z_{2i} \neg \forall u \exists w_1 \ldots Q w_{k-i}\ P(\vec{z},u,\vec{w}) ,$$

where the dots stand for alternating quantifiers. Φ_i is, classically, the same as Σ^0_{k+i+1} .

Let $j < i \leq k$, and let $A_r := A$ augmented with all classically true Π^0_r sentences. Let F be a case of Φ_i which is classically equivalent to a Σ^0_{k+j+1} (relativized) Gödel sentence G for A_{k+j}. Then G is not equivalent in A_{k+j} to any case of Φ_j, and so F is not equivalent in A to any case of Φ_j. Hence $\Phi_i \not\leq_A \Phi_j$.

For the converse, we observe that theorem III may be relativized to A_{k-j-1}, yielding a Σ^0_{k-j} formula B(x) s.t. the sentence $\neg\forall x B$ is not equivalent in A_{k-j-1} to any sentence

prenex in a classically Σ^0_{k-j} matrix. Since $\neg\forall x B$ is a case of Φ_j, and each case of Φ_i is prenex in a classically Σ^0_{k-j} matrix, we conclude that $\Phi_j \not\leq_A \Phi_i$.

(II) $A = IA^*$. Let

$$\Phi_i := Qz_1 \ldots \forall z_{7i} \neg\forall u \exists w_1 \ldots Q' w_{3k-3i}\, P(\vec{z},u,\vec{w}) \qquad (i = 1,\ldots,k)$$

and let $j < i \leq k$. Φ_i is, classically, the same as $T_i := \Sigma^0_{3k+4i+1}$ or $\Pi^0_{3k+4i+1}$ (according to the evenness of i). As at the end of the proof of theorem III, we now take a classically true T_i sentence which is not implied by any true T_j sentence. Thus $\Phi_i \not\leq_{IA^*} \Phi_j$.

For the converse, we proceed as in part (I), using the case $A = IA^*$ of theorem III. □

3. Application to truth definitions.

The classical arithmetical hierarchy is intimately related to enumeration theorems. In particular, it follows from Kleene's Hierarchy Theorem that there is

no Π^0_k or Σ^0_k relation enumerating all (unary, say) Π^0_{k+1} relations. The proof-theoretic analogue of an enumerating relation is a truth-definition. Given an arithmetic theory A, and a set δ of arithmetic sentences, a unary predicate $V(x)$ is a truth-definition (t.d.) in A for δ, if

$$\vdash_A V(\widetilde{\ulcorner F \urcorner}) \leftrightarrow F \qquad \text{for each } F \in \delta .$$

In analogy to the classical Enumeration Theorem, Troelstra ([74] 1.5.4) shows that for each frame Φ there is a case V of Φ which is a truth-definition in A for all closed cases of Φ. On the other hand, no truth definition exists in any consistent extension of IA for all sentences (Tarski's Theorem).

The rôle of implication in intuitionistic theories may now be enunciated in yet another manner by the following immediate consequence of theorems I and II and corollary I.

THEOREM V. Let A be an r.e. theory as above, and let $k \geq 0$.

(a) There is no t.d. V in A to $H^{(k)}_1$ or to $\Sigma^{(k)}_2$ with $I(V) \leq k$.

(b) There is no t.d. V in A to $\Sigma^{(k)}_1$ with $I(V) \leq k+1$. Hence:

(c) (D.H.J. de Johngh [73]) There is no t.d. in A to Σ^*_1.

(d) There is no t.d. V in IA^* to $\Pi^{(k)}_3$ with $I(V) \leq k$.

(e) There is no t.d. in IA^* to Π^*_3.

4. Proof of lemma 1.1.

In the remaining of this paper we will use the sequential calculus IA^∞ of recursively presented cut-free infinitary derivations of $Le[A]$ and the terminology and notational conventions thereof. The use of infinitary derivations is motivated mainly by the subformula property: each (positive, respectively - negative, occurence of a) formula in an infinitary (cut-free) derivation Δ for the sequent $\Gamma : F$ is a (positive, respectively - negative)

subformula of $(\Gamma) \to F$. Here (Γ) stands for the conjunction of all elements of Γ. Another important property of the calculus is this. Let A be an extension of IA as above. A sentence F is a theorem of A if there is a derivation Δ of IA^{∞} deriving F, and s.t. "bar induction over Δ" is provable in A. In particular, IA^{*} consists exactly of the theorems of IA^{∞}.

To prove lemma 1.1, let B be a prenex sentence, and let Δ be a derivation for $\neg B : \bot$. Since IA^{*} is consistent, the premiss $\neg B$ must be active at some inference in Δ, and therefore

(1) $\neg\neg \exists u$ [" $\bot \notin \Gamma^{u}$ and $\neg B$ is active at the node u of Δ, but at no node $v > u$ "].

If Δ^{u} is a sub-derivation of Δ above u, as in (1), then Δ^{u} has the form

$$\dfrac{\begin{array}{cc} \Delta^{u*\hat{0}} & \Delta^{u*\hat{1}} \\ \Gamma : B & \Gamma, \bot : F^{u} \end{array}}{\Gamma, \neg B : F^{u}} .$$

Each $G \in \Gamma$ occurs negatively in $\neg B : \bot$, so $G \equiv \bot$. Hence, by the condition on u, $\Gamma = \emptyset$, and so $\Delta^{u*\hat{0}}$ derives B. Thus (1) implies $\sim \vdash_{IA^{*}} B$, as required. □

5. A reduction lemma.

5.1 LEMMA: Let $\sigma = (\Gamma, G : H)$ be a sequent in a derivation for $\Theta : F$. If G is an active premiss of $\underline{\to R}$, then $I(G) \leq \max[I(F)-1, I(E)-2 \mid E \in \Theta]$.

PROOF: $G \to H$ is a positive subformula of $(\Theta) \to F$, so G is a negative subformula of F, or a positive, but not strictly positive, subformula of some $E \in \Theta$. The inequality follows. □

Let B be a prenex sentence. We write B_i for $N_i[B]$.

5.2. REDUCTION LEMMA. Let $n \geq 1$, and let P be a sentence with $I(P) \leq n+1$. The following is provable in IA: given a derivation Δ for $P \leftrightarrow B_{2n+4}$, we have $Q \leftrightarrow B_{2n+2}$ for some Q with $I(Q) \leq n$.

5.3. SUBLEMMA: If Δ is a derivation for $P, B_{2n+2} : B_{2n-1}$, then there is a sentence G with $I(G) \leq n-1$, satisfying

(1) $\quad P \wedge G \to B_{2n}$

(2) $\quad (P \wedge . G \to B_{2n-1}) \to B_{2n-1}$

5.4. The sublemma implies the reduction lemma. If Δ derives $P : B_{2n+4}$, i.e., $P : B_{2n+2} \to B_{2n-1}$, then a variant Δ' of Δ derives $P, B_{2n+2} : B_{2n-1}$. Apply now 5.3, and let $Q :\equiv G \to B_{2n-1}$.

From (2) and $B_{2n+4} \to P$ we get $(G \to B_{2n-1}) \to (B_{2n+4} \to B_{2n-1})$. But, by 1(4), $(N_{2n+4} \to N_{2n-1}) \to N_{2n+2}$, so $Q \to B_{2n+2}$.

To prove the converse observe that, from (1) and $B_{2n+4} \to P$, we have $G \to (B_{2n+4} \to B_{2n})$. But $(N_{2n+4} \to N_{2n}) \to N_{2n}$ by 1(5), so $G \to B_{2n}$, and thus $(B_{2n} \to B_{2n-1}) \to (G \to B_{2n-1})$. Since $N_{2n+2} \equiv N_{2n} \to N_{2n-1}$, by 1(1), this yields $B_{2n+2} \to Q$.

□

5.5. Proof of 5.3. Let Δ derive $P, B_{2n+2} : B_{2n-1}$. For a node u in Δ, let $\Gamma_o^u := \Gamma^u \setminus \{B_{2n+2}\}$, and $\Gamma_1^u := \{E \in \Gamma^u \mid I(E) \geq n\}$. Let $V(x)$ be a truth-definition for finite conjunctions of subformulas H of the derived sequent, with $I(H) < n$. It is easily seen that V may be taken so that $I(V) = n-1$.

We show that there is a recursive function f (depending on Δ), with the following property. Let T,U be Kleene's computation relation and result-extracting function, respectively. Define

$$G(u) :\equiv \exists x \exists y[T(f(u),x,y) \wedge V(U(y))]$$

$$\equiv \exists z \varepsilon R_u \; V(z) \qquad \text{(where } R_u \text{ is the range of } \varphi_{f(u)}\text{)}.$$

Then, for each node u in Δ,

(3) $\quad (\Gamma_1^u, G(u)) \to B_{2n}$

(4) $\quad (\Gamma_0^u, G(u) \to B_{2n-1}) \to F^u$.

Also, $I(G) = I(V) = n-1$. Taking, in particular, $u = \langle\rangle$, the root of Δ, we have $\Gamma^u = (P, B_{2n+2})$ and $F^u \equiv B_{2n-1}$. Letting $G :\equiv G(\overline{\langle\rangle})$, we get then from (3) $P \wedge G \to (B_{2n+2} \to B_{2n})$; but $(N_{2n+2} \to N_{2n}) \equiv N_{2n}$ by 1(3), so $P \wedge G \to B_{2n}$. On the other hand, we get from (4) $(P, G \to B_{2n-1}) \to B_{2n-1}$, as required.

The function f is defined by giving recursive instructions for obtaining the r.e. set R_u from $\{R_{u*\hat{m}}\}_m$. I.e., $G(u)$ is defined recursively from $G(u*\hat{m})$, $m \geq 0$, thus yielding a Herbrand-Gödel algorithm for f . Simultaneously with the definition we also verify that (3) and (4) are satisfied, by "bar induction on Δ".

Case 1 (Basis): u is a top-node in Δ . B_{2n+2} cannot occur positively in the derived sequent $P, B_{2n+2} : B_{2n-1}$; hence $F^u \not\equiv B_{2n+2}$, and so both (3) and (4) hold for $G(u) :\equiv \bot$ (i.e., $R_u = \{\ulcorner \bot \urcorner\}$) .

Case 2 (Induction Step): u is not a top-node. We consider cases for the inference rule of Δ at u . The only non-trivial cases are the following:

$\wedge R$, say $\dfrac{\Gamma^u : F_o \quad \Gamma^u : F_1}{\Gamma^u : F_o \wedge F_1}$

By ind. ass., $F(u*\hat{0})$ and $f(u*\hat{1})$ are defined so that $(\Gamma_1^{u*\hat{j}}, G(u*\hat{j})) \to B_{2n}$ and $(\Gamma_o^{u*\hat{j}}, G(u*\hat{j}) \to B_{2n-1}) \to F_j$, $j = 0,1$. Define $f(u)$ so that

$R_u = R_{u*\hat{0}} \cup R_{u*\hat{1}}$; i.e., $G(u) \leftrightarrow G(u*\hat{0}) \vee G(u*\hat{1})$. We have $\Gamma_i^{u*\hat{0}} = \Gamma_i^{u*\hat{1}} = \Gamma_i^u$ $(i=0,1)$, so $(\Gamma_1^u, G(u)) \to B_{2n}$ and $(\Gamma_0^u, G(u) \to B_{2n-1}) \to F_0 \wedge F_1$.

$\to R$, say $\dfrac{\Gamma^u, F_0 : F_1}{\Gamma^u : F_0 \to F_1}$.

By lemma 5.1 $F_0 \notin \Gamma_1^{u*\hat{0}}$, so $\Gamma_1^u = \Gamma_1^{u*\hat{0}}$, while $\Gamma_0^u = \Gamma_0^{u*\hat{0}}$ since $F_0 \neq B_{2n+2}$ (by the subformula property). Set $f(u) = f(u*\hat{0})$, i.e. $G(u) \leftrightarrow G(u*\hat{0})$.

$\to L$, say $\dfrac{\Gamma^u : H_0 \quad \Gamma^u, H_1 : F^u}{\Gamma^u, H_0 \to H_1 : F^u}$.

Subcase (a): $(H_0 \to H_1) \neq B_{2n+2}$. Define $f(u)$ so that $G(u) \leftrightarrow G(u*\hat{0}) \vee G(u*\hat{1}) \wedge H_0$. Note that $I(H_0) < I(H_0 \to H_1) \leq n+1$.

Subcase (b): $(H_0 \to H_1) \equiv B_{2n+2} \equiv (B_{2n} \to B_{2n-1})$. This is the core of the proof. We have $\dfrac{\Gamma^u : B_{2n} \quad \Gamma^u, B_{2n-1} : F^u}{\Gamma^u, B_{2n+2} : F^u}$

Let $G(u)$ be the conjunction of all formulas in $\Gamma^u \setminus \Gamma_1^u$, i.e., of all $E \in \Gamma^u$ with $I(E) < n$. From the l.h.s. premiss we trivially have $(\Gamma_1^u, G(u)) \to B_{2n}$. From the r.h.s. premiss and ind.ass. $(\Gamma_0^u, G(u*\hat{1}) \to B_{2n-1}, B_{2n-1}) \to F^u$. But, by the definition of $G(u)$, $(\Gamma_0^u) \to G(u)$, so $(\Gamma_0^u, G(u) \to B_{2n-1}) \to F^u$.

All other cases for propositional rules and for $\forall L$ and $\exists R$ are treated like the case for $\wedge R$.

$\forall R$, say $\dfrac{\{\Gamma^u : F(\bar{m})\}_m}{\Gamma^u : \forall x F(x)}$.

We let $f(u)$ be s.t. $R_u = \bigcup_m R_{u*\hat{m}}$, i.e., $G(u) \leftrightarrow \exists m G(u*\hat{m})$.

$\exists L$ - similar to $\forall R$. The presence of these infinitary rules is the reason for defining $G(u)$ via $\vee$.

□

6. Proof of theorem II. For $k = 1$, let $\Phi_k :\equiv \neg\neg\exists x Px$, and apply theorem III. For $k \geq 2$, let $\Phi_k :\equiv N_{2k+4}[\exists x P(x)]$, so $\Phi_k \in \Sigma_1^{(k)}$, and B_{2k+4} is a case of Φ_k. Using induction on k, we prove that for a suitable Σ_1^0 sentence B $\vdash\!\!\!/_A\ B_{2k+4} \leftrightarrow C$ whenever C is a sentence with $I(C) < k$. Basis: By theorem III there is a Σ_1^0 sentence B s.t. $\vdash\!\!\!/_A \neg\neg B \leftrightarrow C$ whenever $I(C) = 0$. Induction step: this is the reduction lemma 5.2. □

7. Proof of theorem I.

7.1 LEMMA: Assume that B and C are sentences with $I(B) = 0$, $I(C) = 1$. The following is provable in IA: if Δ is a derivation for $\neg\neg B: C$, then there is a sentence G, with $I(G) = 0$, s.t. $G \to \neg B$ and $\neg G \to C$.

PROOF: For a node u in Δ, let $\Gamma_0^u := \Gamma^u \setminus \{\neg\neg B\}$, $\Gamma_1^u := \{E \in \Gamma^u \mid I(E) > 0\}$. Exactly as in 5.5, we prove that for a suitable formula $G(u)$, with $I(G) = 0$, the following holds for any node u in Δ: $(\Gamma_0^u, G(u)) \to \neg B$ and $(\Gamma_1^u, G(u) \to \bot) \to F^u$. For the case $\to L$, subcase (b), note that $(\neg\neg B \to \neg B) \leftrightarrow \neg B$, in analogy to the equivalence $(B_{2n+2} \to B_{2n}) \leftrightarrow B_{2n}$ used in 5.5. The lemma follows by taking $u = \langle\rangle$. □

7.2. LEMMA: Let B be a prenex sentence, s.t. $\neg\neg B \to B$ is not provable in $A_1 :=$ A augmented with all true Π_1^0 sentences. Then $\vdash\!\!\!/_A \neg\neg B \leftrightarrow C$ for any sentence C with $I(C) \leq 1$.. (Note that if $A = IA^*$ then $A_1 = A$).

PROOF: If $\vdash_A \neg\neg B \to C$, then there is a derivation, probably correct in A , for $\neg\neg B : C$. By 7.1 this implies, in IA and hence in A , that for some implication-free sentence G , both $G \to \neg B$ and $\neg G \to C$. Since we also assume $\vdash_A C \to \neg\neg B$, this yields $\vdash_A \neg G \leftrightarrow \neg\neg B$.

Let Δ be a derivation for $\neg G, \neg B : \bot$ (provably correct in A). Consider the Π^0_1 statement

$S :\equiv$ " $\neg B$ is not an active conclusion at any node u in Δ where $\bot \notin \Gamma^u$ " .

If S is true, then $\neg G \to \bot$, so $\neg B$, and therefore $\neg\neg B \to B$. If not, then there (weakly) exists a node u for which the following Π^0_1 sentence is true: B is active at u but at no node $v \succ u$, and $\bot \notin \Gamma^u$. Say, we have

$$\frac{\Theta, \neg G : B \qquad \Theta, \neg G, \bot : F^u}{\Theta, \neg B, \neg G : F^u}$$

where we may assume w.l.g. that $\neg B \notin \Theta$, since $\neg B$ is not active at any node $v \succ u$. Each sentence $H \in \Theta = \Gamma^u \setminus \{\neg B, \neg G\}$ occurs negatively in the derived sequent $\neg B, \neg G : \bot$. Since $\bot \notin \Gamma^u$, we must have $\Theta = \emptyset$, and so the subderivation of Δ above $u * \hat{0}$ is known in A_1 to derive $\neg G \to B$.

We have shown that $\sim\sim \vdash_{A_1} \neg\neg B \to B$, contradicting the choice of B . Thus $\vdash_A \neg\neg B \leftrightarrow C$ is impossible. □

(Note that 7.2 implies lemma 1.1.)

7.3 Proof of theorem I - concluded.

Case (I): A is r.e.. If $k = 1$, let ϕ_k be $\neg\forall x Px$. Then, by theorem III, $\phi_1 \not\vdash_A \Psi$ for any frame Ψ with $I(\Psi) = 0$. For $k \geqslant 2$ we proceed by induction. Basis: Fix a false Σ^0_2 sentence B consistent with A_1. Then, in a manner similar to the proof of theorem III, one concludes that $\Psi_{A_1} \neg\neg B \to B$ but $\phi_2 := \neg\exists x\forall y P(x,y)$, the same corresponding to $\neg\neg B$. Then, by 7.2, $\phi_2 \not\vdash_A \Psi$ whenever $I(\Psi) = 0$. Induction step: this is the reduction lemma 5.2.

Case (II): $A = IA^*$. For $k = 1$ proceed as in (I). For $k \geqslant 2$ proceed, again, by induction. Basis: Let B be a Π^0_3 sentence which is true, but underivable in IA^* (cf. the proof of theorem III). Then, as in the proof of theorem III, $\Psi_{IA^*} \neg\neg B \to B$. The statement of the theorem follows now, as in (I), from 7.2. Induction step: again, this is the reduction lemma 5.2. □

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