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SOME NEW RESULTS FOR THE MARKOV RANDOM WALK

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Some New Results For The Markov Random Walk

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Abstract

Markov random walks are discrete time versions of Markov additive processes. We present some basic theory of the Markov random walk (MRW) viewing it as a family of embedded standard random walks. We give a comprehensive discussion of the problem of degeneracy. The properties of the time-reversed MRW are studied. Using the semirecurrent sets associated with the MRW, we obtain a complete description of the fluctuation behaviour of the nondegenerate MRW. We also derive a Wiener-Hopf factorization of the measures based on these sets. The case where the increments of the MRW have finite means is then investigated. Finally, we study the problem of exit of the MRW from a bounded interval.

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1 Introduction

Markov random walks (MRW) have been studied by several authors; see in particular, Miller [7, 8], Presman [11] and Arjas and Speed [1, 2, 3]. In the earlier papers the treatment was purely analytical while the later authors adopted a probabilistic approach. The primary focus of attention of all these papers was the derivation of some sort of transform identities, rather than the study of the probabilistic structure of the MRW. Newbould [9] dealt with some aspects of 'degeneracy', but his main interest was in the fluctuation behaviour of nondegenerate MRW's with a finite state ergodic Markov chain.

In this paper, we aim to provide further insights into the probabilistic structure of the MRW. In section 2, we define an MRW $\{(S_n, J_n); n \geq 0\}$ as a Markov process with the first component S_n having an additive property, J_n being a Markov chain on a countable state space $\mathcal{E}$. We observe at the outset that the MRW is a family of embedded (standard) random walks. This enables us in section 3 to investigate the problem of degeneracy of the MRW in a broader framework and obtain more significant results than those obtained by Newbould [9]. We also study in section 2 the structure of time-reversed MRW. In particular, we establish a simple yet useful result (Lemma 1) connecting the probability measures associated with the given MRW and its time-reversed counterpart. A special case of this result (Lemma 3), recently published by Asmussen ([5], Theorem 3.1) was already known to us.

The fluctuation theory of the MRW is developed in section 4. Our approach is based on the theory of semirecurrent phenomena given by Prabhu [10]. For the MRW the semirecurrent sets correspond to the observed ranges of the Markov renewal processes given by the sequences of ascending and descending ladder points. In the case of a nondegenerate MRW with an irreducible and persistent J, it turns out that its fluctuation behaviour is exactly the same as in the standard random walk (Theorem 8). This is an extension of a result obtained by Newbould [9] for the special case of a finite state ergodic J. For a degenerate MRW the situation is very different; here the process oscillates between (finite or infinite) bounds, except in the trivial case where $S_1 = S_2 = \cdots = 0$ for any initial state $J_0 = j$ (Theorem 4). We believe that our results concerning the fluctuation behaviour of the MRW are much more comprehensive than those previously established.

Section 4 continues with a Wiener-Hopf factorization of the MRW in terms of positive Borel measures based on the associated semirecurrent sets. It turns out that this factorization is really a property of these measures. We show that the Fourier transforms of these measures, if necessary, can be derived and lead to the standard form of the factorization more familiar in the literature, expressed in terms of the transforms of ladder points. The earliest result of this type is due to Miller [7] and the one demonstrated here is due to Presman [11]. Arjas and Speed [2, 3] derived various results for such factorizations using heavily analytical tools involving operators. Our approach is probabilistic and brings out the essential simplicity of the factorization.

The fluctuation theory developed in section 4 does not assume the existence of means. In section 5 we investigate the case where the increments of the MRW have bounded means. Our approach is again based on the family of standard random walks embedded in the MRW. These results are analogous to those for bounded additive functionals defined on a Markov chain; see for example Chung ([6], part I, section 14).

In the final section 6 we study the problem of exit of the MRW from a bounded interval, the classical version of which occurs in sequential analysis. For the nondegenerate MRW with J persistent nonnull, we obtain results analogous to those in sequential analysis concerning the exit time.

Finally, we would like to point out that the idea behind the proof of Theorem 2 is due to Arnold H. Buss.

2 Definitions and Basic Properties

Suppose that we are given a probability space $(\Omega, \mathcal{F}, P)$. We denote $\mathbf{R} = (-\infty, \infty)$ and $\mathcal{E} =$ countable set.

Definition 1 *A Markov Random Walk* $(S, J) = \{(S_n, J_n), n \geq 0\}$ *is a Markov process on the state space* $\mathbf{R} \times \mathcal{E}$ *whose transition distribution measure is given by*

$$P\{(S_{m+n}, J_{m+n}) \in A \times \{k\} | (S_m, J_m) = (x, j)\} = P\{(S_{m+n} - S_m, J_{m+n}) \in (A - x) \times \{k\} | J_m = j\} \tag{1}$$

$\forall j, k \in \mathcal{E}$ *and a Borel subset* A *of* $\mathbf{R}$.

We shall only consider the time-homogeneous case where the second probability in (1) does not depend on m. We denote this probability as $Q_{jk}^{(n)}\{A - x\}$, so that

$$Q_{jk}^{(n)}\{A\} = P\{(S_n, J_n) \in A \times \{k\} | J_0 = j\}. \tag{2}$$

It is seen that the transition distribution measure Q satisfies the conditions

$$Q_{jk}^{(n)}\{A\} \geq 0, \quad \sum_{k \in \mathcal{E}} Q_{jk}^{(n)}\{\mathbf{R}\} = 1 \tag{3}$$

and the Chapman-Komolgorov equations

$$Q_{jk}^{(m+n)}\{A\} = \sum_{l \in \mathcal{E}} \int_{-\infty}^{\infty} Q_{jl}^{(m)}\{dx\} Q_{lk}^{(n)}\{A - x\}, \quad \forall\, m, n \geq 0. \tag{4}$$

where

$$\begin{aligned} Q_{jk}^{(0)}\{A\} &= 1 \quad \text{if } j = k, 0 \in A \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5}$$

The initial measure is given by

$$\begin{aligned} P\{(S_0, J_0) \in \{A\} \times \{j\}\} &= a_j \quad \text{if } 0 \in A \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{6}$$

From (2) and (6) the finite dimensional distributions of the process are given by

$$
\begin{aligned}
& P\{(S_{n_t}, J_{n_t}) \in \{A_t\} \times \{k_t\}, (1 \leq t \leq r)\} \\
= & \sum_{j \in \mathcal{E}} a_j \int_{\{x_t \in A_t, (1 \leq t \leq r)\}} Q_{jk_1}^{(n_1)}\{dx_1\} Q_{k_1 k_2}^{(n_2 - n_1)}\{dx_2 - x_1\} \cdots Q_{k_{r-1} k_r}^{(n_r - n_{r-1})}\{dx_r - x_{r-1}\} \quad (7)
\end{aligned}
$$

for any Borel sets $A_1, A_2, \ldots, A_r$ of $\mathbf{R}$.

If the distribution of $\{S_n\}$ is concentrated on $[0, \infty)$, then the Markov random walk (MRW) defined above reduces to a Markov Renewal Process (MRP). On the other hand, the MRW is the discrete time analogue of Markov-additive processes (MAP). From (1) and (2) we see that the marginal porcess $J(\infty) = \{J_n,\ n \geq 0\}$ is a Markov Chain on $\mathcal{E}$ with the transition probabilities given by

$$P_{jk}^{(n)} = Q_{jk}^{(n)}\{\mathbf{R}\}. \tag{8}$$

We shall write J for $J(\infty)$. Also we shall denote the conditional probabilities and expectations given $J_0 = j$ as P_j and E_j respectively. As usual, we write Q_{jk} for $Q_{jk}^{(1)}$.

To exploit the structure of the MRW, it is useful to consider some random walks embedded in the process as follows. Denote the successive hitting times of state $j \in \mathcal{E}$ as

$$\tau_r^j = min\{n > \tau_{r-1}^j : (S_n, J_n) \in \{\mathbf{R}\} \times \{j\}\} \quad (r \geq 1), \qquad \tau_0^j = 0 \tag{9}$$

and the number of such hits (visits) as

$$N_n^j = max\{r : \tau_r^j \leq n\}. \tag{10}$$

Also on $\tau_r^j < \infty$ we define

$$S_r^j = S_{\tau_r^j} \quad (r \geq 0). \tag{11}$$

We note that although the processes $\{S_{N_n^j}^j, n \geq 0\}$ and $\{S_r^j, r \geq 0\}$ have different index sets which affects their evolution, their observed ranges remain identical. In section 5, we shall obtain the limit behaviour of these processes and relate them to that of the MRW. First we make a simple observation arising from the properties of the embedded processes.

Proposition 1 $\{(\tau_r^j, S_r^j), r \geq 0\}$ *is a random walk on the state space* $(\mathbf{R}_+ \times \mathbf{R})$.

Proof: The sequence $\{\tau_r^j\}$ constitutes an embedding renewal process, and the result follows from the strong markov property of (S, J). $\square$

We need a classification of the set $\mathcal{E}$; this is essentially based on the recurrence properties of the marginal chain J, but it seems more appropriate to consider the two dimensional random walk $\{\tau_r^j, S_r^j\}$ rather than $\{\tau_r^j\}$.

Definition 2 *$j \in \mathcal{E}$ is persistent if $\{\tau_1^j, S_1^j\}$ has a proper distribution, ie.*

$$P_j\{\tau_1^j < \infty, |S_1^j| < \infty\} = 1 \tag{12}$$

and transient otherwise.

Proposition 2 .

(a) If j is transient, then the random walk $\{\tau_r^j, S_r^j\}$ is terminating.

(b)Suppose j is persistent and $P_j\{S_1^j = 0\} < 1$. Then $\{S_r^j\}$ either (i) drifts to $+\infty$; (ii) drifts to $-\infty$; or (iii) oscillates a.s..

Proof: (a) If j is transient then $\{\tau_r^j\}$ is a terminating renewal process.

(b) It is clear that the marginal process $\{S_r^j\}$ is a random walk and the result follows from the fact that any random walk that is nondegenerate at zero belongs to one of the 3 types indicated. $\square$

Next we consider a time-reversed version of the MRW. The importance of this will be apparent in a later section. We first give the following definition.

Definition 3 *$\{(\widehat{S}_n, \widehat{J}_n), n \geq 0\}$ is defined to be a time-reversed MRW induced by (S, J) if its transition probability measure is given by*

$$\widehat{Q}_{jk}\{A\} = \frac{\pi_k}{\pi_j} Q_{kj}\{A\} \quad \forall k, j \in \mathcal{E} \tag{13}$$

where $Q_{kj}\{\cdot\}$ is the transition probability measure of (S, J), $\{\pi_i\}$ is a sequence of real numbers and A is any Borel subset of $\mathbf{R}$.

By setting $A = \mathbf{R}$ in the above equation, we see that

$$\pi_j \widehat{P}_{jk} = \pi_k P_{kj},$$

which shows that $\{\pi_i\}$ is necessarily the stationary measure of both J and $\widehat{J}$.

Henceforth, $\widehat{(\cdot)}$ will denote the counterpart of $(\cdot)$ for the time-reversed MRW.

The following result relates the given MRW to its time-reversed counterpart; in particular, it will lead us to the n^{th}-step transition probability measure for the time-reversed MRW.

Lemma 1 *We have*

$$P_k\{\mathcal{A}_n; J_n = j\} = \frac{\pi_j}{\pi_k} P_j\{\widehat{\mathcal{A}}_n; \widehat{J}_n = k\} \tag{14}$$

$\forall \mathcal{A}_n \in \sigma(X_1, \cdots, X_n)$, *where* $\widehat{\mathcal{A}}_n = \{\omega : (X_n(\omega), \cdots, X_1(\omega)) \in \mathcal{A}_n\}$; *ie.* $\widehat{\mathcal{A}}_n$ *is the corresponding set for the time-reversed MRW and* $X_r = S_r - S_{r-1}$ $(r \geq 1)$ *are the increments of the MRW.*

Proof: We first consider the special case where

$$\mathcal{A}_n = \{\omega : X_i(\omega) \in A_i, i = 1, \ldots, n\}$$

the $A_i's$ being Borel subset of $\mathbf{R}$. We have

$$\begin{aligned}
P_k\{\mathcal{A}_n; J_n = j\} &= P_k\{X_i \in A_i, i = 1, \ldots, n; J_n = j\} \\
&= \sum_{i_1 \in \mathcal{E}} \cdots \sum_{i_{n-1} \in \mathcal{E}} \int_{A_n} \cdots \int_{A_1} Q_{ki_1}\{dx_1\} \cdots Q_{i_{n-1}j}\{dx_n\} \\
&= \sum_{i_1 \in \mathcal{E}} \cdots \sum_{i_{n-1} \in \mathcal{E}} \int_{A_n} \cdots \int_{A_1} \frac{\pi_j}{\pi_{i_{n-1}}} \widehat{Q}_{ji_{n-1}}\{dx_n\} \cdots \frac{\pi_{i_1}}{\pi_k} \widehat{Q}_{i_1 k}\{dx_1\} \\
&= \frac{\pi_j}{\pi_k} \sum_{i_1 \in \mathcal{E}} \cdots \sum_{i_{n-1} \in \mathcal{E}} \int_{A_n} \cdots \int_{A_1} \widehat{Q}_{ji_{n-1}}\{dx_n\} \cdots \widehat{Q}_{i_1 k}\{dx_1\} \\
&= \frac{\pi_j}{\pi_k} P_j\{\widehat{\mathcal{A}}_n; \widehat{J}_n = k\}.
\end{aligned}$$

The collection of sets for which the above equality holds is a $\lambda-$system and we have shown (14) for the collection of $\mathcal{A}_n$ which is a $\pi-$system. The Dynkin π-λ theorem implies that the desired identity holds for all $\mathcal{A}_n \in \sigma(X_1, \cdots, X_n)$. $\square$

As a special case of Lemma 1 we obtain the n^{th}-step transition probability of the time-reversed MRW as follows. Another special case will be treated in section 4 (see Lemma 3).

Theorem 1 *We have*

$$\hat{Q}_{jk}^{(n)}\{A\} = \frac{\pi_k}{\pi_j} Q_{kj}^{(n)}\{A\} \quad \forall n \geq 1. \tag{15}$$

Proof: In view of the above lemma, we only need to define a correct form of $\mathcal{A}_n$ and find the corresponding $\hat{\mathcal{A}}_n$. Let

$$\mathcal{A}_n = \{\omega : X_1 + \cdots + X_n \in A\}.$$

Since $\hat{S}_n = S_n$, we have

$$\hat{\mathcal{A}}_n = \{\omega : \hat{S}_n \in A\} = \mathcal{A}_n.$$

Therefore

$$\hat{Q}_{jk}^{(n)}\{A\} = \frac{\pi_k}{\pi_j} Q_{kj}^{(n)}\{A\}.$$

□

To obtain a classification of the MRW similar to that of the standard random walk, we need to consider the case where S_n neither drifts nor oscillates. This special case seems to have been overlooked by Asmussen[4] who gave a classification of $\{S_n\}$ under the conditions $E_j(|X_1|) < \infty$ $\forall j$ with J persistent nonnull. Here is a counter example. Let

$$F_{(a)}\{A\} = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{Q}\{A\} = \begin{pmatrix} 0 & F_{(-1)}\{A\} & 0 \\ 0 & 1/2F_{(0)}\{A\} & 1/2F_{(0)}\{A\} \\ F_{(1)}\{A\} & 0 & 0 \end{pmatrix}.$$

Note that J is persistent nonnull with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then it is clear that the observed range of $\{S_n\}$ is $\{-1, 0, 1\}$. This shows that this MRW does not fluctuate as in the standard random walk.

This motivates the discussions in the next section.

3 Degenerate Markov Random Walks

In the case of a standard random walk, degeneracy is the trivial case where $P(S_1 = 0) = 1$. In the case of the MRW, the conditions for degenaracy were investigated by Presman [11] and Newbould [9] who restricted J to be ergodic and to have finite state space. Here we shall start with the following definition.

Definition 4 *The MRW is said to be degenerate at $j \in \mathcal{E}$ if $\{S_r^j\}$ is degenerate at zero, ie.*

$$P_j(S_1^j = 0) = 1. \tag{16}$$

According to this definition j is persistent whenever the MRW is degenerate at j.

We shall show that degeneracy is a class property which will lead us to a natural way of defining degeneracy for the MRW.

Theorem 2 *Suppose that the MRW is degenerate at j and $Q_{jk}^{(n)}\{\mathbf{R}\} > 0$ for some n. Then the MRW is also degenerate at $k \in \mathcal{E}$. In particular, if J is irreducible, then either all the embedding random walks are nondegenerate or all are degenerate.*

Proof: As noted above if (16) holds for $j \in \mathcal{E}$, then j is persistent. Given that $Q_{jk}^{(n)}\{\mathbf{R}\} > 0$ for some n, k is also persistent, from the theory of Markov Chains. On $\{J_0 = k\}$ let

$$\tau_i^{kj} = \min\{n > \sigma_{i-1}^{jk} : J_n = j\} \quad i \geq 1$$

$$\sigma_i^{jk} = \min\{n > \tau_i^{kj} : J_n = k\} \quad i \geq 1.$$

Then it is clear that $\{\tau_i^{kj} + \sigma_i^{jk}, i \geq 1\}$ is a renewal process with alternating components τ_i^{kj} and σ_i^{jk}. Since both j and k are persistent with $Q_{jk}^{(n)}\{\mathbf{R}\} > 0$, we have

$$\sigma_i^{jk} < \infty \;\; a.s. \quad \text{and} \quad \tau_i^{kj} < \infty \;\; a.s. \;\; \forall i \geq 1.$$

It follows from the strong Markov property that

$$S_{\sigma_i^{jk}} - S_{\tau_i^{kj}} \stackrel{d}{=} S_{\sigma_1^{jk}} - S_{\tau_1^{kj}} \quad \forall i \geq 1$$

which in turn implies that

$$S_{\sigma_i^{jk}} - S_{\sigma_{i-1}^{jk}} \stackrel{d}{=} S_{\tau_i^{kj}} - S_{\tau_{i-1}^{kj}} \quad \forall i \geq 1.$$

Therefore, if j satisfies (16), then

$$P_k\{S_{\tau_i^{kj}} - S_{\tau_{i-1}^{kj}} = 0\} = 1 \quad \forall i \geq 1$$

would imply that

$$P_k\{S_{\sigma_i^{jk}} - S_{\sigma_{i-1}^{jk}} = 0\} = 1 \quad \forall i \geq 1$$

so that

$$P_k\{S_i^k = 0\} = 1 \quad \forall i \geq 1.$$

It follows from irreducibility of the J-chain that if (16) holds for some $j \in \mathcal{E}$, it holds for all $j \in \mathcal{E}$. □

Motivated by the above theorem, it seems reasonable to define degeneracy of the MRW as follows.

Definition 5 *The MRW is degenerate if all the embedded random walks are degenerate; ie. (16) holds* $\forall j \in \mathcal{E}$.

Note that this definition would imply that for a degenerate MRW, J is a persistent chain.

The following theorem reveals the full implication of the term degeneracy, namely, that in a degenerate MRW, the increment between any transition $j \to k$ in J is deterministic, depending on j and k.

Theorem 3 *Suppose that J is irreducible. Then the MRW is degenerate iff there exist constants $b_j < \infty \quad \forall j \in \mathcal{E}$ having the property that $Q_{jk}^{(n)}\{A\}$ is concentrated on $b_k - b_j$ whenever $Q_{jk}^{(n)}\{\mathbf{R}\} > 0$.*

Proof: (i) Suppose there exist constants b_j with the stated properties. Then since $Q_{jj}^{(n)} > 0$ for $n = \tau_1^j$, we must have $S_1^j = 0$ a.s..

(ii) By the definition of degeneracy, we have $\forall r \geq 1$,

$$S_{\tau_r^j} - S_{\tau_1^j} = 0 \ \ a.s.$$

regardless of J_0. Consider the epochs m and m+n at which $J_m = j$ and $J_{m+n} = k$. At these epochs, we must have $S_m = S_{\tau_r^j} = b_j$ and $S_{m+n} = S_{\tau_s^k} = b_k$ for some $r \geq 0, s > 0$. Accordingly,

$$S_{n+m} - S_m = S_{\tau_r^j} - S_{\tau_s^k} = b_k - b_j;$$

which shows that $Q_{jk}^{(n)} > 0$ implies that $Q_{jk}^{(n)}\{A\}$ is concentrated on $b_k - b_j$. □

The fluctuation aspect of degenerate MRW can be briefly summarized as follows.

Theorem 4 *For a degenerate MRW we have the following.*

$$-\infty \leq \liminf_n S_n \leq 0 \leq \limsup_n S_n \leq +\infty \ \ a.s. \ P_j. \tag{17}$$

i.e. these limits depend on J_0. If, in addition, $\mathcal{E}$ is finite then it is uniformly bounded i.e.

$$-\infty < \min_k(b_k - b_j) = \liminf_n S_n \leq 0 \leq \limsup_n S_n = \max_k(b_k - b_j) < +\infty \ \ a.s. \ \ P_j$$

Proof: By definition of degeneracy, we have $P_j\{S_r^j = 0\} = 1 \ \ \forall r \geq 0$. As J is persistent in this case, we have $S_n = 0 \ \ i.o. \ \ a.s.$. which proves our assertion. If $\mathcal{E}$ is finite it follows from Theorem 3 that

$$\sup_n |S_n| = \max_{k,j} |b_k - b_j| < \infty.$$

□

Remark 1 *It should be noted that for a degenerate MRW the only possible limit of $\{S_n\}$, if it exists, is zero. This corresponds to the case where*

$$P_j\{S_0 = S_1 = S_2 = \cdots = 0\} = 1 \ \ \ a.s. \ \ P_j. \tag{18}$$

Otherwise, the MRW 'oscillates'.

We conclude this section by the following simple but useful result.

Lemma 2 *If an MRW with J persistent is uniformly bounded then it is degenerate.*

Proof: If the MRW is uniformly bounded, then each embedded random walk is also unifromly bounded since $\tau_r^j < \infty$ a.s.. This means that all these embedded random walks are degenerate so that the MRW is degenerate. □

In view of Theorem 4, the converse of this lemma need not be true .

4 Semirecurrent Sets, Fluctuation Theory and Wiener-Hopf Factorization

In this section we give a classification of the additive component of a nondegenerate MRW which is parallel to that of a standard random walk. The classification will be based on the maximal and minimal functionals and the corresponding semirecurrent sets. Throughout this section, we shall assume that J is irreducible and persistent. As in the classical setup, we denote

$$M_n = \max_{0 \leq k \leq n} S_k, \quad m_n = \min_{0 \leq k \leq n} S_k, \quad n \geq 0; \tag{19}$$

and

$$\zeta^+ = \{(n,j) : S_n = M_n, J_n = j\}, \;\; \zeta^- = \{(n,j) : S_n = m_n, J_n = j\}. \tag{20}$$

We note that $M_n \geq 0$ and $m_n \leq 0$. Also $\{M_n\}$ is a nondecreasing sequence while $\{m_n\}$ is a nonincreasing sequence. Therefore by monotone convergence theorem,

$$M_n \to M \leq \infty, \quad m_n \to m \geq -\infty \;\; a.s.. \tag{21}$$

The following result was proved by Prabhu [10] as an example of semirecurrent phenomena.

Theorem 5 *The sets ζ^+ and ζ^- are semirecurrent sets.*

Proof: Let

$$Z_{nl} = 1_{\{(n,l)\in\zeta^+\}}. \tag{22}$$

Then from the definition of ζ^+, it follows that $P\{Z_{0j} = 1\} = P\{J_0 = j\}$, and for $0 = n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_r \ \ (r \geq 1)$

$$P\{Z_{n_1 l_1} = Z_{n_2 l_2} = \cdots = Z_{n_r l_r} = 1 | Z_{0j} = 1\} = u_{jl_1}(n_1) u_{l_1 l_2}(n_2 - n_1) \cdots u_{l_{r-1} l_r}(n_r - n_{r-1}) \tag{23}$$

where

$$u_{jk}(n) = P_j\{Z_{nk} = 1\} = P_j\{S_r \leq S_n \ \ (0 \leq r \leq n),\ J_n = k\}. \tag{24}$$

This verifies that $\{Z_{nl}\}$ is a semirecurrent phenomenon and the set ζ^+ is a semirecurrent set. Similar remarks apply to ζ^-. □

We also observe that ζ^+ and ζ^- are sets with the following elements:

$$\zeta^+ = \{(T_0, J_0), (T_1, J_{T_1}), \ldots\},\ \ \zeta^- = \{(\bar{T}_0, J_0), (\bar{T}_1, J_{\bar{T}_1}), \ldots\} \tag{25}$$

where

$$T_k = \min\{n > T_{k-1} : S_n \geq S_{T_{k-1}}\},\ \ \bar{T}_k = \min\{n > \bar{T}_{k-1} : S_n \leq S_{\bar{T}_{k-1}}\} \tag{26}$$

with $T_0 = \bar{T}_0 = 0$. We then have the following.

Theorem 6 *The processes $\{(S_{T_k}, T_k, J_{T_k}), k \geq 1\}$ and $\{(-S_{\bar{T}_k}, \bar{T}_k, J_{\bar{T}_k}), k \geq 1\}$ are Markov Renewal Processes on the state space $\mathbf{R}_+ \times N_+ \times \mathcal{E}$.*

Proof: Since the T_k are stopping times for the MRW, we have by the strong Markov property,

$$\begin{aligned} & P\{(S_{T_k}, T_k, J_{T_k}) \in A \times \{n\} \times \{j\} | (S_{T_m}, T_m, J_{T_m}) = (x_m, n_m, j_m), 0 \leq m \leq k-1\} \\ = \ & P\{(S_{T_k} - S_{T_{k-1}}, T_k - T_{k-1}, J_{T_k}) \in \{A - x_{k-1}\} \times \{n - n_{k-1}\} \times \{j\} | J_{T_{k-1}} = j_{k-1}\} \\ = \ & P_{j_{k-1}}\{(S_{T_1}, T_1, J_{T_1}) \in \{A - x_{k-1}\} \times \{n - n_{k-1}\} \times \{j\}\} \end{aligned}$$

which shows that the process considered is a Markov Renewal Process. The proof for $\{-S_{\bar{T}_k}, \bar{T}_k, J_{\bar{T}_k}\}$ is similar. □

Before we provide a classification of the MRW, we give a result concerning the limit behaviour of S_n in relation to M and m which is identical to that of a standard random walk.

Theorem 7 *For a nondegenerate MRW with J irreducible and persistent it is impossible for both M and m to be finite. Furthermore, we have the following.*

(i) If $M_n \to +\infty, \quad m_n \to m > -\infty$ a.s., then $S_n \to +\infty$ a.s. .

(ii) If $M_n \to M < +\infty, \quad m_n \to -\infty$ a.s., then $S_n \to -\infty$ a.s. .

(iii) If $M_n \to +\infty, \quad m_n \to -\infty$ a.s., then $-\infty = \liminf_n S_n < \limsup_n S_n = +\infty$ a.s. .

Proof: Since $\sup |S_n| = \max\{M, -m\}$, by Lemma 2 the MRW cannot be bounded. Accordingly we need consider only cases (i)-(iii).

(i)From the Markov property and from (26) that $M_{T_k} = S_{T_k}$, we have

$$\begin{aligned} P_{J_0}\{\min_{n\geq T_k} S_n - M_{T_k} \in I | S_m, J_m; 0 \leq m \leq T_k\} &= P_{J_0}\{\min_{n\geq T_k} S_n - S_{T_k} \in I | S_{T_k}, J_{T_k}\} \\ &= P_{J_{T_k}}\{\min_{n\geq T_k} S_{n-T_k} \in I\} \\ &= P_{J_{T_k}}\{m \in I\}. \end{aligned}$$

So for $a, b > 0$,

$$P_{J_0}\{\min_{n\geq T_k} S_n > a\} \geq P_{J_0}\{M_{T_k} > a+b\} P_{J_{T_k}}\{m > -b\}.$$

Since $M_n \to +\infty, \quad m_n \to m > -\infty$ *a.s.*, given a and ϵ, we can find some sufficiently large b and k so that

$$P_{J_0}\{M_{T_k} > a+b\} > 1-\epsilon, \quad \text{and} \quad P_{J_{T_k}}\{m > -b\} > 1-\epsilon.$$

Therefore

$$P_{J_0}\{\min_{n\geq T_k} S_n > a\} > (1-\epsilon)^2 > 1-2\epsilon,$$

which shows that $S_n \to +\infty$ a.s. .

(ii) This follows from (i) by symmetry.

(iii)From (26) $M_{T_k} = S_{T_k} \to +\infty$ and $m_{T_k} = S_{T_k} \to -\infty$ which give us two divergent sequences that prove our assertion. □

Theorem 8 *For a nondegenerate MRW with J irreducible and persistent it is impossible for both ζ^+ and ζ^- to be terminating. Furthermore, we have the following.*
(i) ζ^+ is nonterminating and ζ^- is terminating $\Leftrightarrow M = +\infty, m > -\infty \Leftrightarrow \lim_n S_n = \infty$;
(ii) ζ^+ is terminating and ζ^- is nonterminating $\Leftrightarrow M < +\infty, m = -\infty \Leftrightarrow \lim_n S_n = -\infty$;
(iii) ζ^+ and ζ^- are both nonterminating $\Leftrightarrow M = +\infty, m = -\infty \Leftrightarrow \limsup_n S_n = \infty, \liminf_n S_n = -\infty$.

Proof: We first show that for the given MRW
(a) ζ^+ is terminating iff $M < \infty$, and (b) ζ^- is terminating iff $m > -\infty$.
By symmetry it suffices to show (a). Note that

$$\zeta^+ \text{is terminating} \Longrightarrow \exists N < \infty : S_n \leq S_N,\ \ n \leq N \ \text{ and } \ S_n < S_N,\ \ \forall n > N$$

$$\Longrightarrow M = S_N = \sum_{i=1}^{N} X_i < \infty \quad a.s..$$

Conversely suppose $M < \infty$. By Lemma 2 the MRW is unbounded so $m = -\infty$. But from Theorem 7

$$M < \infty,\ \ m = -\infty \Rightarrow \lim_n S_n = -\infty \Rightarrow \zeta^+ \ \text{ is terminating}$$

as there is a last epoch n for which $S_n \in \mathbf{R}_+$. From this result we conclude that
(i) ζ^+ is nonterminating and ζ^- is terminating $\Leftrightarrow M = +\infty, m > -\infty$,
(ii) ζ^+ is terminating and ζ^- is nonterminating $\Leftrightarrow M < +\infty, m = -\infty$,
(iii) ζ^+ and ζ^- are both nonterminating $\Leftrightarrow M = +\infty, m = -\infty$.
The second implication concerning S_n follows from Theorem 7 for these cases. □

Next we obtain a factorization in terms of measures by considering the semirecurrent sets similar to those defined in (20). To begin with, we derive the following result which is a direct consequence of Lemma 1.

Lemma 3 *For every finite interval I we have*

$$P_k\{S_m \leq 0 \ \ (0 \leq m \leq n), S_n \in I, J_n = j\}$$

$$= \frac{\pi_j}{\pi_k} P_j\{\widehat{S}_n - \widehat{S}_{n-m} \leq 0 \ \ (0 \leq m \leq n), \widehat{S}_n \in I, \widehat{J}_n = k\}. \tag{27}$$

Proof: Let

$$\mathcal{A}_n = \{\omega : S_1 \le 0, S_2 \le 0, \ldots, S_n \le 0, S_n \in I\}.$$

For the time-reversed MRW $(\widehat{S}_n, \widehat{J}_n)$ we have

$$S_m = \widehat{S}_n - \widehat{S}_{n-m} \quad \text{for m} = 1, \ldots, n; \quad S_n = \widehat{S}_n; \quad \widehat{J}_n = k; \quad \widehat{J}_0 = j$$

so the set $\widehat{\mathcal{A}}_n$ corresponding to $\mathcal{A}_n$ in Lemma 1 is given by

$$\widehat{\mathcal{A}}_n = \{\omega : \widehat{S}_n - \widehat{S}_{n-m} \le 0 \ \ (0 \le m \le n), \widehat{S}_n \in I\}.$$

The desired result follows form Lemma 1. □

With a slight change in notation, let $\zeta^+(\widehat{\zeta}^+)$ denote the strong ascending ladder set of the MRW (time-reversed MRW); thus

$$\zeta^+ = \{(n,j) : S_n = M_n, J_n = j, \rho_n = n\} \tag{28}$$

where ρ_n is the first epoch to attain M_n.

For fixed $s \in (0,1)$, we define the following matrix-valued measures μ_s^+, $\widehat{\mu}_s^-$ and μ_s as follows:

$$\begin{aligned}
\mu_s^+\{I\} &= \{(\mu_{ij}^+)_{\mathcal{E}\times\mathcal{E}} : \mu_{ij}^+\{I\} = \sum_{n=0}^{\infty} E_i(s^n; (n,j) \in \zeta^+, S_n \in I, J_n = j)\} && (29)\\
\widehat{\mu}_s^-\{I\} &= \{(\widehat{\mu}_{ij}^-)_{\mathcal{E}\times\mathcal{E}} : \widehat{\mu}_{ij}^-\{I\} = \sum_{n=0}^{\infty} E_i(s^n; (n,j) \in \widehat{\zeta}^-, \widehat{S}_n \in I, \widehat{J}_n = j)\} && (30)\\
\mu_s\{I\} &= \{(\mu_{ij})_{\mathcal{E}\times\mathcal{E}} : \mu_{ij}\{I\} = \sum_{n=0}^{\infty} s^n Q_{ij}^{(n)}\{I\}\ \} && (31)
\end{aligned}$$

for every finite interval I. Note that each of these measures is finite, a simple bound being $(\mathbf{I} - s\mathbf{P})^{-1}$, where $\mathbf{P}$ is the transition probability matrix of J. We then have the following result.

Theorem 9 *We have*

$$\mu_s = \mu_s^+ * \nu_s^- \tag{32}$$

and in particular,

$$\mathbf{U}(s)\mathbf{V}(s) = (\mathbf{I} - s\mathbf{P})^{-1} \tag{33}$$

where

$$\nu_s^- = \{(\nu_{ij}^-)_{\mathcal{E}\times\mathcal{E}} : \nu_{ji}^- = \frac{\pi_i}{\pi_j}\widehat{\mu}_{ij}^-\} \tag{34}$$

$$\mathbf{U}(s) = \mu_s^+\{(0,\infty)\}, \quad \mathbf{V}(s) = \nu_s^-\{(-\infty,0]\} \tag{35}$$

Proof: First we note that

$$\begin{aligned}
Q_{ij}^{(n)}\{I\} &= \sum_{k\in\mathcal{E}}\sum_{l=0}^{n} P_i\{\rho_n = l, J_{\rho_n} = k, S_n \in I, J_n = j\} \\
&= \sum_{k\in\mathcal{E}}\sum_{l=0}^{n}\int_0^\infty P_i\{S_m < S_l \ \ (0 \le m \le l-1), J_l = k, S_l \in dx\} \\
&\cdot \ \ P_i\{S_m \le S_l \ \ (l \le m \le n), J_n = j, S_n \in I | S_l = x, J_l = k\}
\end{aligned}$$

The second term in the last expression turns out to be

$$P_k\{S_m \le 0 \ \ (0 \le m \le n-l), J_{n-l} = j, S_{n-l} \in I - x\}$$

and by Lemma 3 this equals

$$\frac{\pi_j}{\pi_k}P_j\{\widehat{S}_{n-l} - \widehat{S}_{n-l-m} \le 0 \ \ (0 \le m \le n-l), \widehat{S}_{n-l} \in I - x, \widehat{J}_{n-l} = k\}$$

$$= \frac{\pi_j}{\pi_k}P_j\{(n-l,k) \in \widehat{\zeta}^-, \widehat{S}_{n-l} \in I, \widehat{J}_{n-l} = k\}.$$

It follows that

$$\begin{aligned}
\mu_{ij}\{I\} &= \sum_{n=0}^{\infty} s^n Q_{ij}^{(n)}\{I\} \\
&= \sum_{n=0}^{\infty} s^n \sum_{k\in\mathcal{E}}\sum_{l=0}^{n}\int_0^\infty P_i\{(l,k) \in \zeta^+, J_l = k, S_l \in dx\} \\
&\quad . \ \ \frac{\pi_j}{\pi_k}P_j\{(n-l,k) \in \widehat{\zeta}^-, \widehat{S}_{n-l} \in I - x, \widehat{J}_{n-l} = k\} \\
&= \sum_{k\in\mathcal{E}} \mu_{ik}^+ * \frac{\pi_j}{\pi_k}\widehat{\mu}_{jk}^- \ \ = \sum_{k\in\mathcal{E}} \mu_{ik}^+ * \nu_{kj}^-.
\end{aligned}$$

This establishes the main factorization (32). As a special case we have

$$\mu_s\{\mathbf{R}\} = \mu_s^+\{\mathbf{R}_+\}\nu_s^-\{\mathbf{R}_-\} = \mathbf{U}(s)\mathbf{V}(s).$$

On the other hand,

$$
\begin{aligned}
\mu_s\{\mathbf{R}\} &= \sum_{n=0}^{\infty} s^n \mathbf{Q}^{(n)}\{\mathbf{R}\} \\
&= \sum_{n=0}^{\infty} s^n \mathbf{P}^n = (\mathbf{I} - s\mathbf{P})^{-1}.
\end{aligned}
$$

□

As a direct consequence of Theorem 9 we arrive at a Wiener-Hopf factorization obtained in terms of transforms by Presman [11] who called it the basic factorization identity. Let

$$
\chi^+(s,\omega) = \{[\chi^+_{jk}(s,\omega)]_{\mathcal{E}\times\mathcal{E}} : \chi^+_{jk}(s,\omega) = E_j(s^T e^{i\omega S_T}, J_T = k)\} \tag{36}
$$

$$
\chi^-(s,\omega) = \{[\chi^-_{jk}(s,\omega)]_{\mathcal{E}\times\mathcal{E}} : \chi^-_{jk}(s,\omega) = E_j(s^{\bar{T}} e^{i\omega S_{\bar{T}}}, J_{\bar{T}} = k)\} \tag{37}
$$

where $T = T_1$ and $\bar{T} = \bar{T}_1$ and, as usual, $\widehat{\chi}^+(s,\omega)$ and $\widehat{\chi}^-(s,\omega)$ denote the counterparts for the corresponding time-reversed MRW. The following result is a consequence of Theorem 6 stating that $\{S_{T_k}, T_k, J_{T_k}\}$ and $\{-S_{\bar{T}_k}, \bar{T}_k, J_{\bar{T}_k}\}$ are Markov Renewal Processes.

Lemma 4 *We have*

$$
(i)\ \ E_j(s^{T_r} e^{i\omega S_{T_r}}; J_{T_r} = k) = [\chi^+(s,\omega)]^r_{jk} \tag{38}
$$

$$
(ii)\ \ E_j(s^{\bar{T}_r} e^{i\omega S_{\bar{T}_r}}; J_{\bar{T}_r} = k) = [\chi^-(s,\omega)]^r_{jk}. \tag{39}
$$

□

The above lemma provides us enough machinary to write down a factorization in terms of transforms. Define

$$
\Phi(\omega) = \{[\phi(\omega)]_{\mathcal{E}\times\mathcal{E}} : \phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} Q_{jk}\{dx\}\ \}.
$$

We have then the following.

Theorem 10 *For the MRW with J having a stationary probability distribution $\{\pi_i\}$ we have*

$$
\mathbf{I} - s\Phi(\omega) = \mathbf{D}_\pi^{-1}[\mathbf{I} - \widehat{\chi}^-(s,\omega)]'\mathbf{D}_\pi[\mathbf{I} - \chi^+(s,\omega)] \tag{40}
$$

where $\mathbf{D}_\pi = diag(\pi)$ and the prime denotes the transpose.

Proof: We have

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{i\omega x} s^n \mathbf{Q}^{(n)}\{dx\} = \sum_{n=0}^{\infty} s^n [\Phi(\omega)]^n = [\mathbf{I} - s\Phi(\omega)]^{-1}$$

From Lemma 4, the transform of μ_{jk}^{+} is

$$\int_{-\infty}^{\infty} e^{i\omega x} \mu_{jk}^{+}\{dx\} = \sum_{r=0}^{\infty} E_j(s^{T_r} e^{i\omega S_{T_r}}; J_{T_r} = k) = \sum_{r=0}^{\infty} [\chi^{+}(s,\omega)]_{jk}^{r}$$

so that

$$\int_{-\infty}^{\infty} e^{i\omega x} \mu_s^{+}\{dx\} = [\mathbf{I} - \chi^{+}(s,\omega)]^{-1}.$$

Similarly the transform of $\widehat{\mu}_s^{-}$ is $[\mathbf{I} - \widehat{\chi}^{-}(s,\omega)]^{-1}$.

From (34), an easy manipulation gives the transform of ν_s^{-} as $[\mathbf{D}_\pi^{-1}[\mathbf{I} - \widehat{\chi}^{-}(s,\omega)]^T \mathbf{D}_\pi]^{-1}$. The desired result follows by trivial algebraic rearrangement. □

5 The Case Where Means Exist

In this section we consider the case where

$$\sup_{j\in\mathcal{E}} E_j|X_1| < \infty \tag{41}$$

where $X_k = S_k - S_{k-1}$ $(k \geq 1)$ are the increments of the MRW. With this assumption, we obtain some elementary results concerning the Cesáro limits of the additive component of a MRW and its embedded random walks. We then give a classification of the MRW based on its mean. First, we have

Lemma 5 *If the condition (41) holds, then*

$$E_j(S_1^j) = \sum_{k\in\mathcal{E}} {}^{j}P_{jk}^{*} E_k(X_1) \quad \forall j \in \mathcal{E} \tag{42}$$

where ${}^{j}P_{jk}^{*} = \sum_{m=0}^{\infty} P_j(J_m = k, \tau_1^j > m)$. *In the case where the stationary probability measure* $\{\pi_k, k \in \mathcal{E}\}$ *of the marginal chain* J *exists, (42) reduces to*

$$E_j(S_1^j) = \sum_{k\in\mathcal{E}} \frac{\pi_k E_k(X_1)}{\pi_j}. \tag{43}$$

Proof:

$$\begin{aligned}
E_j(S_1^j) &= \sum_{m=1}^{\infty} E_j(X_m)\mathbf{1}_{(\tau_1^j \geq m)} \\
&= \sum_{m=1}^{\infty} \sum_{k\in\mathcal{E}} E_j(X_m | J_{m-1} = k) P_j(J_{m-1} = k)\mathbf{1}_{(\tau_1^j > m-1)} \\
&= \sum_{k\in\mathcal{E}} E_k(X_1) \sum_{m=0}^{\infty} P_j(J_m = k; \tau_1^j > m) \\
&= \sum_{k\in\mathcal{E}} {}^jP_{jk}^* E_k(X_1).
\end{aligned}$$

□

Theorem 11 *Let $j \in \mathcal{E}$ be persistent. If $E_j(|S_1^j|) < \infty$ then*

$$\lim_{n\to\infty} \frac{S_{N_n^j}^j}{n} = \frac{E_j(S_1^j)}{E_j(\tau_1^j)} < \infty \quad a.s.\,. \tag{44}$$

If the stationary probability measure $\{\pi_k, k \in \mathcal{E}\}$ exists, (44) reduces to

$$\lim_{n\to\infty} \frac{S_{N_n^j}^j}{n} = \sum_{k\in\mathcal{E}} \pi_k E_k(X_1). \tag{45}$$

Proof: From the elementary renewal theorem we have

$$\lim_{n\to\infty} \frac{N_n^j}{n} = \frac{1}{E_j(\tau_1^j)} \quad a.s.$$

where $E_j(\tau_1^j) \leq \infty$. Also, since j is persisent, $N_n^j \to \infty$ a.s. as $n \to \infty$, and so it follows from the Strong law of large numbers that $S_r^j/r \longrightarrow E_j(S_1^j)$. Therefore,

$$\frac{S_{N_n^j}^j}{N_n^j} \longrightarrow E_j(S_1^j) \quad a.s.$$

and

$$\frac{S_{N_n^j}^j}{n} = \frac{S_{N_n^j}^j}{N_n^j} \cdot \frac{N_n^j}{n} \longrightarrow \frac{E_j(S_1^j)}{E_j(\tau_1^j)} \quad a.s.$$

□

With these preliminary results we are now ready to consider the additive component $\{S_n\}$.

Lemma 6 *We have*

$$E_j[S_n; J_n = k] = \sum_{m=0}^{n-1} \sum_{l,l' \in \mathcal{E}} P_{jl}^{(m)} E_l[X_1; J_1 = l'] P_{l'k}^{(n-1-m)} \quad \forall j \tag{46}$$

in the sense that both sides of the equation are finite or infinite together.

Proof: We have

$$\begin{aligned} E_j[S_n; J_n = k] &= \sum_{m=0}^{n-1} E_j[X_{m+1}; J_n = k] \\ &= \sum_{m=0}^{n-1} \sum_{l,l' \in \mathcal{E}} E_j[X_{m+1}; J_m = l, J_{m+1} = l'] E_{l'} 1_{\{J_n = k\}} \\ &= \sum_{m=0}^{n-1} \sum_{l,l' \in \mathcal{E}} P_{jl}^{(m)} E_l[X_1; J_1 = l'] P_{l'k}^{(n-1-m)}. \end{aligned}$$

□

Theorem 12 *Assume that the condition (41) holds and the stationary probability distribution $\{\pi_k, k \in \mathcal{E}\}$ exists. Then S_n/n converges a.s., and also in $\mathcal{L}^1$. Thus*

$$(i) \lim_{n \to \infty} \frac{S_n}{n} = \sum_{k \in \mathcal{E}} \pi_k E_k(X_1) \quad a.s. \tag{47}$$

and

$$(ii) \lim_{n \to \infty} E_j(\frac{S_n}{n}) = \sum_{k \in \mathcal{E}} \pi_k E_k(X_1). \tag{48}$$

Proof: (i) Since for each fixed $j \in \mathcal{E}$, the $\{\tau_r^j\}$ are stopping times for $\{(S_n, J_n), n \geq 0\}$ they form a convenient choice of an embedding renewal process. Therefore $\{S_n\}$ is a regenerative process. Also since the stationary distribution of J exists, $E_j(\tau_1^j) = \pi_j^{-1} < \infty \;\; \forall j$. In view of (41) we find from Lemma 5 that $E_j(|S_1^j|) < \infty$. Using the regenerative property of the $\{S_n\}$ we find that

$$\begin{aligned} \lim_{n \to \infty} \frac{S_n}{n} &= \frac{E(\text{value in a cycle})}{E(\text{cycle length})} \\ &= \frac{E_j(S_1^j)}{E_j(\tau_1^j)} = \pi_j E_j(S_1^j) \end{aligned}$$

and the desired result follows from (43).

(ii) Using Lemma 6 we obtain

$$E_j[S_n] = \sum_{m=0}^{n-1} \sum_{l\in\mathcal{E}} P_{jl}^{(m)} E_l[X_1].$$

Using (41) and applying dominated convergence theorem we find that

$$\lim_{n\to\infty} \frac{1}{n} E_j(S_n) = \sum_{l\in\mathcal{E}} \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} P_{jl}^{(m)} E_l[X_1] = \sum_{k\in\mathcal{E}} \pi_k E_k(X_1).$$

□

The above results yields the following.

Theorem 13 *Suppose that stationary distribution of J exists and (41) holds. Then $\{S_n\}$ either (i) drifts to ∞; (ii) drifts to $-\infty$ or; (iii) osicillates in the sense of (17) according as $\sum_{k\in\mathcal{E}} \pi_k E_k(X_1)$ $(i) > 0; (ii) < 0; (iii) = 0$.*

Proof: The results (i) and (ii) follow directly from Theorem 12, while (iii) corresponds to the case where the limit is zero which is always the case for degenerate MRW (From (43)). Hence by Theorem 4 the result follows for degenerate MRW. In the case of nondegenerate MRW, at least one of the embedded random walks is nondegenerate and the result follows from the fact that

$$\limsup_n S_n \geq \limsup_r S_r^j = \infty \quad \text{and} \quad \liminf_n S_n \leq \liminf_r S_r^j = -\infty.$$

□

Remark 2 *It can be seen that the above result is almost a complete analogue to that of the standard random walk except that in the case of degenerate MRW, $\limsup_n S_n$, $\liminf_n S_n$ may be finite.*

6 First Exit Time from an Interval

We have seen that the fluatuation behaviour of a non-degenerate MRW with J persistent is similar to that of a standard random walk. It is then interseting to investigate the corresponding results for the first exit time from a finite interval. Let $I = (-a, b)$ with $a, b > 0$ and

$$N = \min\{n : S_n \in I^c\}$$

so that N is the first time that the additive component of the MRW exits the interval I. In the case of a standard random walk, it is known that (i)$P\{N > n\}$ is geometrically bounded and hence $P\{N = \infty\} = 0$ and (ii) the moment generating function of N exists. We derive these analogous results for the MRW.

Lemma 7 *Suppose that the MRW is nondegenerate with J persistent. Then there exists a decreasing sequence $\{B_n, n > 0\}$ where $B_n \in [0, 1]$ such that*

$$P\{N > n\} \leq B_n \quad n > 0 \tag{49}$$

regardless of J_0.

Proof:

$$\begin{aligned} P_j\{N > n\} &= P_j\{S_i \in I, i = 1, \dots, n\} \\ &\leq P_j\{S_r^j \in I, r = 1, \dots, N_n^j\} \end{aligned}$$

where N_n^j is the number of visits to j up to time n. Since S_r^j is an nonterminating standard random walk for each j, we have by Stein's lemma

$$E_j[1_{\{N>n\}} | N_n^j = m_j] \leq A_j \delta_j^{m_j+1} \tag{50}$$

where $A_j > 0$ and $0 < \delta_j < 1$. Taking expectation on both sides of (50) we find that

$$P_j\{N > n\} \leq A_j E[\delta_j^{N_n^j+1}].$$

Since N_n^j increases with n, it is clear that $E[\delta_j^{N_n^j+1}]$ decreases with n.

Now let $B_n = \inf_j A_j E[\delta_j^{N_n^j+1}]$. The desired result follows from the fact that $B_n \leq B_{n-1}$. □

Theorem 14 *Suppose that the MRW is nondegenerate with J persistent. Then we have*
(i) $N < \infty$ *a.s.;*
(ii)if in addition the stationary distribution of J exists, $E_j[N^\alpha] < \infty$ *for* $\alpha > 0$; $\forall j \in \mathcal{E}$*;*
i.e. N has a proper distribution with finite moments of all orders.

Proof: (i) We only need to show that $B_n \to 0$ as $n \to \infty$. Since J is persistent, $N_n^j \to \infty$ as $n \to \infty$ $\forall j \in \mathcal{E}$. This proves our assertion in view of Lemma 7.
(ii) We have

$$\begin{aligned} E_j[N^\alpha] &= \sum n^\alpha P_j\{N = n\} \leq \sum n^\alpha P_j\{N \geq n\} \\ &\leq \sum n^\alpha A_j E_j[\delta_j^{N_n^j}] \leq \sum A_j E_j[e^{tn}\delta_j^{N_n^j}]. \end{aligned}$$

We may choose $t > 0$ such that $e^{t/\pi_j}\delta_j < 1$ which is possible since $\pi_j > 0$ $\forall j \in \mathcal{E}$. With this choice of t, for sufficiently large n, the n^{th} term is less than $A_j\delta_j^{n\epsilon}$ for some $\epsilon > 0$. The result follows as the sum is convergence since $\delta_j^\epsilon < 1$. Formally, for each $j \in \mathcal{E}$ choose

$$t \leq -(\pi_j - 2\epsilon)\log\delta_j.$$

Since $N_n^j/n \to \pi_j$ *a.s.*, $\exists M_j$ such that $\forall n > M_j$,

$$\left|\frac{N_n^j}{n} - \pi_j\right| < \epsilon.$$

So each term in the sum becomes

$$\begin{aligned} E_j[e^{tn}\delta_j^{N_n^j}] &\leq E_j[(1/\delta_j)^{n(\pi_j-2\epsilon)}\delta_j^{N_n^j}] \\ &= E_j[\delta_j^{-n\pi_j+2n\epsilon+N_n^j}] < E_j[\delta_j^{n\pi_j+2n\epsilon+n(\pi_j-\epsilon)}] = \delta_j^{n\epsilon}. \end{aligned}$$

□

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