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__Abstract__

In this note we discuss the similarities and differences between Goedel's result about non-recursive shortening of proofs of formal systems by additional axioms and the corresponding results about the succinctness of different representations of languages.

Introduction

In this note we show that some recent results about the relative succinctness between different representations of languages [2,5-10,11,12] are very closely related to Goedel's result about non-recursive shortening of proofs in axiomatizable formal systems by addition of new axioms [1,4]. We also show that some other relative succinctness results about the representation of languages appear to be fundamentally different from the Goedel results and discuss the nature of these results.

In the proof of the Goedel speed-up result, one shows, after adding new axioms to a formal system, that the set of theorems provable in the new system but not in the old system is not recursively enumerable. From this it immediately follows that the new system must shorten proofs non-recursively relative to the old system [1,4]. The above proof is particularly simple as shown later, for intuitionistic logic or constructive mathematical systems which are frequently use in computer science [3,10].

Usually, the representation of languages is more complicated than proofs of theorems, since in general the equivalence between different representations of languages is not decidable, nor can the shortest representation be effectively computed. Nevertheless, the same method can be used to derive results about the relative succinctness for several different representations of languages. Under mild assumptions about decidable representations of languages, we prove in the next section, that if the set of representations of the more powerful formalism without equivalent representation in the other formalism is not r.e.

then it immediately follows that the relative succinctness between these representations is not recursively bounded.

From the proof of these results we will see that the non-recursive shortening of proofs or representations of languages is due to the power to prove additional results or represent new languages. In the case of formal systems one can easily show that without additional power to prove new results the relative shortening of proofs by changing formal systems is recursively bounded.

The situation is quite different for the representation of languages, where we can gain non-recursively bounded succinctness between different representations of the same family of languages. We will illustrate this by several representations of PTIME by different types of machines whose relative succinctness is not recursively bounded [6,7].

Goedel Speed-Up

Let F be an axiomatizable, intuitionistic or constructive mathematical system and let the theory, T_F, be the set of theorems provable in F,

$$T_F = \{\omega \mid F \vdash w\}.$$

It is known [3,10] that in such systems the disjunction property holds, that is:

$$F \vdash (\alpha \vee \beta) \text{ implies that } F \vdash (\alpha) \text{ or } F \vdash (\beta)$$

We denote by $F[\omega_0]$ the system F with the additional axiom ω_0 and let $T_{F[\omega_0]}$ be the set of theorems provable in $F[\omega_0]$.

<u>Theorem 1</u>: Let F be a formal system as defined above with T_F not recursive and ω_0 not in T_F. Then the set

$$\Delta = T_{F[\omega_0]} - T_F$$

is not r.e.

<u>Proof</u>: Since ω_0 is in $T_{F[\omega_0]}$ for all α, $\omega_0 \vee \alpha$ is in $T_{F[\omega_0]}$. Since ω_0 is not in T_F using the disjunction property, we know that

$$\omega_0 \vee \alpha \text{ is in } T_{F[\omega_0]} - T_F$$

if and only if α is not in T_F. If Δ is r.e. then T_F and $\overline{T_F}$ are r.e. and T_F must be recursive, which is a contradiction. Thus Δ is not r.e., as was to be shown.

For the sake of completeness we outline the same result for more classic logical systems, as formulated by Ehrenfeucht [1].

Let F be an axiomatizable, formal mathematical system with <u>implication</u> and <u>negation</u> ($\rightarrow$,$\neg$), satisfying the usual axioms of propositional logic.

<u>Theorem 2</u>: Let F be a formal system as defined above and ω_0 a sentence not in T_F such that $F[\neg\omega_0]$ is not decidable. Then the set

$$T_{F[\neg\omega_0]} - T_F$$

is not recursively enumerable.

<u>Proof</u>: Clearly,

$$F[\neg\omega_0] \vdash \alpha \Leftrightarrow F \vdash [\neg\omega_0 \Rightarrow \alpha] \Leftrightarrow F \vdash [\neg\alpha \Rightarrow \omega_0]$$

and for any sentence α

$$F[\omega_0] \vdash [\neg\alpha \Rightarrow \omega_0].$$

Therefore

$$\neg\alpha \Rightarrow \omega_0 \text{ is in } T_{F[\omega_0]} - T_F \Leftrightarrow \alpha \text{ is not in } T_{F[\neg\omega_0]}.$$

Therefore, recursive enumerability of $F_F[\neg\omega_o] - T_F$ implies that $F[\neg\omega_0]$ is a decidable theory, contrary to our assumption.

As it will be seen from the next corollary it follows that $F[\omega_0]$ must shorten proofs of theorems in F non-recursively.

For any ω in T_F let $||\omega||$ and $||\omega||_0$ denote the length of the shortest proof in F and $F[\omega_0]$ respectively. (We assume that the length of proof measures satisfy the natural axioms that for each n there are only finitely many different possible proofs of length n, m_n, and that we can recursively compute m_n and effectively list all the possible m_n proofs of length n and decide if any of them prove a given sentence in F or $F[\omega_0]$.)

Corollary 3: Let F be an axiomatizable formal system such that $T_{F[\omega_0]} - T_F$ is not r.e. Then there is no recursive function f such that for all

$$\omega \text{ in } T_F \quad ||\omega|| \le f(||\omega||_0).$$

Proof: If such a recursive bound f exists then we can recursively enumerate

$$T_{F[\omega_0]} - T_F$$

by checking for each $\omega \in T_{F[\omega_0]}$ whether ω is not in T_F by verifying that no sequence of length $f(||\omega||_0)$ or less is a proof of ω in T_F. This contradicts the fact that

$$T_{F[\omega_0]} - T_F$$

is not recursively enumerable.

Relative Succinctness of Language Representations

Recently a number of results have been obtained about the relative succinctness of different representations of languages [2,5-9,11,12]. After a closer inspection, many of these results show a considerable similarity to the Goedel speed-up discussed in the previous section as explained below.

We first establish some notation for representations of languages.

A set of names $N = \{N_i | N_i \subseteq \Sigma^*, i \geq 1\}$ is a _representation_ of a family of languages $F = \{A_i | A_i \subseteq \Gamma^*, i \geq 1\}$ iff to every N_i there is assigned a language $L(N_i)$ and

$$\{L(N_i) | i \geq 1\} = F.$$

A representation N is _decidable_ iff there exists a recursive function D such that for all $x \in \Gamma^*$ and $i \geq 1$,

$$D(x,i) = \underline{\text{if}}\ x \in L(N_i)\ \underline{\text{then}}\ 1\ \underline{\text{else}}\ 0.$$

A representation is _r.e._ or _recursive_ if N is, respectively, r.e. or recursive. For N_i in the representation N let the _size_ of N_i, $|N_i|$, be the number of symbols in N_i. For representations K and N, $K \supseteq N$ we say that the _relative succinctness_ of the representations is _recursively_

<u>bounded</u> if there exists a recursive function B such that for any language A represented by N the minimal length representations N_i and K_j, in N and K respectively, satisfy

$$B(|K_j|) \geq |N_i|.$$

<u>Theorem 4</u>: Let K and N be decidable representations, let $N \subset K$, let K be recursively enumerable and let N be recursive. If

$$\Delta = K - \{K_i \mid (\exists N_j)[L(K_i) = L(N_j)]\}$$

is not r.e. then the relative succinctness of the representations K and N is not recursively bounded.

<u>Proof</u>: We will show that if the relative succinctness of the representations K and N is recursively bounded then Δ is r.e. Assume that B is the recursive bound of the relative succinctness. Since K is r.e. and N is recursive we can enumerate K and for each enumerated K_i we can compute all N_i in N such that

$$B(|K_i|) \geq |N_j|.$$

Since the representations are decidable we can in the dove-tail manner check for all such N_j and successive elements x of Γ^* whether such an $L(N_j)$ differs from $L(K_i)$. If $L(K_i)$ differs from all $L(N_j)$ such that

$$B(|K_i|) \geq |N_j|$$

then K_i is in Δ, and Δ is seen to be r.e. Thus if Δ is known not to be r.e. the relative succinctness of the representations is not recursively bounded.

The previous result yields easily the following.

__Corollary 5:__ Let K and N be decidable representations, let $N \subseteq K$, K be recursive and K-N r.e. If

$$\Delta = K - \{K_i \mid (\exists N_j)[L(K_i) = L(N_j)]\}$$

is not r.e. then the relative succinctness of the representations K and N is not recursively bounded.

__Proof:__ Similar to previous proof.

The previous results have many immediate applications. We illustrate them by several examples.

1. Consider deterministic context-free languages. Let K be the set of push-down automata and N be the set of deterministic push-down automata. Clearly $N \subseteq K$ and N and K are recursive, furthermore, K and N are decidable representations. Since we can show that

$$\Delta = \{K_i \mid K_i \text{ is a pda accepting cfl which is non-deterministic}\}$$

is not r.e., Theorem 4 applies and we conclude that the relative succinctness of representing deterministic context-free languages by push-down automata and nondeterministic push-down automata is not recursively bounded [5,11,12].

2. Consider unambiguous context-free languages and let K be the set of all context-free grammars and let N be the set of unambiguous context-free grammars. Clearly K is recursive, $N \subseteq K$ (but N is not recursive) and N-K is r.e. [5]. Thus Corollary 5 applies and, since it is known that Δ is not r.e., we conclude that representing unambiguous context-free languages by ambiguous grammars must achieve recursively unbounded gains in succinctness [5,11,12].

3. Similar results can be obtained for the languages in PTIME if we assume PTIME ≠ NPTIME and represent PTIME by nondeterministic and deterministic Turing machines with standard polynomial-time clocks [6,7]. As a matter of fact

$$\text{PTIME} \neq \text{NPTIME}$$

if and only if the relative succinctness of the above representations is not recursively bounded. Therefore, one way of showing that PTIME ≠ NPTIME would be by proving that the succinctness in representation which can be gained from deterministic to nondeterministic polynomially clocked representation of sets in PTIME cannot be recursively bounded [7].

4. From Theorem 3 we can also easily derive results which show that even a slight increase in computation resources yields non-recursive gains in succinctness of representations. For example, let K be the set of Turing machines which for any input of length n first lay off in a standard way exactly n^3 tape squares and halt the computation if the machine tries to exceed this tape bound while processing the input. Let N be the corresponding set of machines which are n^2-tape bounded. In this case K and N are decidable representations and both are recursive. It is easily seen that Δ for these representations is not recursively enumerable (or else we could enumerate the set of Turing machines that do not halt on blank tape) and therefore the relative succinctness is not recursively bounded by Theorem 4. This result implies that there exist n^2-tape recognizable languages such that relatively small machines can recognize them on n^3-tape, but the n^2-tape recognizers must be immensely large, and thus not easily found and understood by us. This

phenomenon may be contributing to the well-known difficulties of finding fast algorithms or providing lower bounds for many practical problems.

The relative, non-recursive shortening of proofs, between two formal systems F_1 and F_2, as described by Corollary 3, requires that in the formal system F_1 we can prove more theorems than in F_2. Similarly, the non-recursive shortening of descriptions of languages in K was based in the fact that K described a larger family of languages than N. The above condition is required for formal systems to achieve non-recursive speed-up but not necessarily for descriptions of languages.

For the sake of completeness we give a few examples of representations of the same family of languages between whose relative succinctness is not recursively bounded.

1. Consider the languages in PTIME [6,7]. Let N be the representation of these languages by Turing machines with uniformly attached, standard polynomial-time clocks, let K be simply the Turing machines which run in polynomial-time. It can quite easily be seen that the relative succinctness between these representations of PTIME is not recursively bounded [6,7].

2. Let N be defined as above and let F be an axiomatized formal system (say, Peano Arithmetic) and let K consist of all the Turing machines for which we can prove in F that they run in polynomial time. Again it can be seen that the relative succinctness is not recursively bounded [5,6].

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